ISSN 0029-3865

CBPF-NF-047/89

THE STABILITY OF THE CLASSICAL SKYRME MODEL SU(2) HEDGEHOG SOLITON

by

Juan A. MIGNACO and Stenio WULCK*

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*Instituto de Física
Universidade Federal do Rio de Janeiro
Caixa Postal 68528
21944 - Rio de Janeiro, RJ - Brasil

We show the exact power series solution at the origin for the classical SU(2) Skyrme model lagrangean with a hedgehog ansatz. We consider also the analogous solution at infinity, and exhibit the dependence of the chiral angle on two dimensionless variables (a consequence of having two completely arbitrary parameters) The classical Skyrme model soliton turns out to be as unstable as the pure non-linear sigma model. When the Skyrme parameter is fixed, breaking the scale invariance on both variables, the mass of the soliton has a stable minimum.

PACS numbers: 11.10 Lm, 11.30 Rd, 11.40 Fy

Key-words: Chiral soliton; Skyrme model; Non-linear sigma model; Nucleon.

The pioneering work by Skyrme[1] on fermionic solutions for a classical non-linear bosonic lagrangean led in this decade to interesting developments[2,3,4]. As is known, it is argued that a pure non-linear sigma model lagrangean has no stable classical solution whereas the addition of a Skyrme term quadrilinear in the unitary field (essentially, the communator of left-invariant Maurer-Cartan forms) brings stability for the solutions, while preserving the essential properties a chiral lagrangean should have in agreement with current algebra[2]. Using a "hedgehog" ansatz in SU(2) it was suggested that the classical stable solitons of such a model could provide an approximate description of the lowest baryon states[5] and in a sense establishing a link with QCD[4]. A lot of applications of this idea, more or less succesful, followed.

Recently[6] we found the exact classical solution as a power series at the origin for the classical SU(2) non linear sigma model with the "hedgehog" for the unitary field. Starting from

$$L_\sigma = -\frac{1}{2} f_\pi^2 \int d^3x' \; \mathrm{Tr}\left[(\partial_\mu U^\dagger)(\partial^\mu U)\right] \quad . \tag{1}$$

$$U = \exp\left[i\vec{n}\cdot\vec{\tau}\, F(r)\right] \tag{2}$$

with:

$r = (x^2 + y^2 + z^2)^{1/2}$

$\vec{n} = \vec{r}/r$

$\vec{\tau}$: the three Pauli spin matrices

f_π : the pion decay constant

we found that near the origin, the chiral angle is given by a power series in terms of a dimensionsless variable[7,8]:

$$F(r) = n_0\pi + \frac{1}{4} s\, X(s) \qquad (n_0 \in \mathbb{Z}) \tag{3}$$

$$s = \frac{1}{2} r\, X''(0) \tag{4}$$

The dimensional parameter $\chi''(0)$ is completely undetermined from the Euler-Lagrange equation resulting from (1) and (2):

$$\frac{d^2\chi(x)}{dx^2} = \frac{2}{x}\sin\left(\frac{\chi(x)}{x}\right) \tag{5}$$

$$x = \frac{1}{2}r \quad , \quad \chi(r) = rF(r) \tag{6}$$

The singularity at the origin determines the form of Eq. (3) and the undeterminacy of $\chi''(0)$. The function X(s) can be written as a power series with known coefficients[8]

$$\chi(s) = \sum_{n=0}^{\infty} c_n s^{2n} \tag{7}$$

The solution of Eq. (5) at infinity has been also established[6,8] and as it should be depends also on a dimensional undetermined parameter.

The problem of stability for the solution of the non-linear sigma model with an SU(2) hedgehog can be written in terms any of these parameters[8]; for instance:

$$M_\sigma = 2\pi f_\pi^2 \frac{1}{\chi''(0)} a \tag{8}$$

$$a = \int_0^\infty ds' \left\{ s'^2\left(\frac{d}{ds'}\left[s'\chi(s')\right]\right)^2 + 8\sin^2\left(\frac{s'}{4}\chi(s')\right)\right\} \tag{9}$$

This results from the scale invariance of the solution, a property emphasized originally by Carlson[9] for the complete Skyrme model. As another byproduct, we found that after quantization through collective coordinates, the theory with only a non-linear sigma model term has a minimum in the energy in terms of $\chi''(0)$[7,8].

We have recently studied the case of the SU(2) hedgehog for the complete Skyrme model[8]. This amounts to consider the

Lagrangean:

$$L_{SK} = L_G - \frac{1}{32\pi e^2}\int d^3x' \sum_{k,l=1}^{3} \mathrm{Tr}\left[U^\dagger(\partial_k U), U^\dagger(\partial_l U)\right] \tag{10}$$

where e is a pure number (the "Skyrme parameter").

The Euler-Lagrange equation is [5,8]:

$$\left(\frac{1}{4}r^2 + \frac{2}{e^2 f_\pi^2}\sin^2 F(r)\right)\frac{d^2F(r)}{dr^2} + \frac{1}{2}r\,\frac{dF(r)}{dr} + \frac{1}{e^2 f_\pi^2}\sin 2F(r)\left(\frac{dF(r)}{dr}\right)^2$$
$$- \frac{1}{4}\sin 2F(r) - \frac{1}{e^2 f_\pi^2}\sin^2 F(r)\sin 2F(r) = 0 \tag{11}$$

Using a Frobenius-like procedure, we have that a solution may be expressed as a power series near the origin:

$$F(r) = F_0 + F_1 r + \frac{1}{2!}F_2 r^2 + \frac{1}{3!}F_3 r^3 + \dots \tag{12}$$

with the coefficients being obtained, order by order:

$$F_0 = n_0\pi \qquad (n_0 \in \mathbb{Z})$$

$$0 = \frac{1}{2}F_1 - \frac{1}{4}2F_1 + \frac{1}{e^2 f_\pi^2}\left(F_1^2\, 2F_1 - 2F_1^3\right)$$

$$F_{2n} = 0 \qquad (n \in \mathbb{Z})$$

$$F_3 = -\frac{4}{5}F_1^3\,\frac{1+2\phi^2}{1+8\phi^2}$$

$$F_5 = \frac{24}{7}F_1^5\,\frac{1+\frac{32}{5}\phi^2+\frac{88}{5}\phi^4+\frac{448}{5}\phi^6}{1+24\phi^2+192\phi^4+512\phi^6}$$

......

$$\phi = \frac{F_1}{e f_\pi}$$

Notice the cancellation intervening in the second expression, separately for the terms coming from each term of Eq.(10). That means that, as for the non-linear sigma model, the dimensional coefficient F_1 is not determined. The solution may be written as Eq. (3), but now the coefficients in the series will depend on the parameter ϕ (or, in other terms, on F_1/e).

The limit $e \longrightarrow \infty$ reproduces, of course, the case we considered above.

In Eq. (12), it is not more useful the dimensionless variable s (Eq. (4)), instead it seems natrural to use:

$$\eta = F_1 r \tag{13}$$

such that:

$$F(r) = F(\eta,\phi) = n_0\pi + \eta G(\eta,\phi) \tag{14}$$

As before, the solution turns out to be scale invariant[8,9].

At infinity, the solution of Eq. (11) can be written as a power series in the inverse of r, $\rho = 1/r$. The series is

$$F(\rho) = \sum_{j=1}^{\infty} \frac{1}{(2j)!} K_{2j}\, \rho^{2j} \tag{15}$$

with:

K_2 : undetermined

$$K_4 = 0$$

$$K_6 = -\frac{30}{7} K_2^3$$

$$K_8 = -6720 \frac{K_2^3}{F_1^2} \phi^2$$

$$K_{10} = \frac{6480}{11} K_2^5$$

$$K_{12} = \frac{96\,940\,800}{13} \frac{K_2^5}{F_1^4} \phi^4$$

$$K_{14} = -449\,280 \left(K_2^7 - 42\,040 \frac{K_2^5}{F_1^4} \phi^4 \right)$$

....

That is, we find the same features as in the sigma model: a dimensional undetermined coefficient, and a ρ^2 behaviour. The difference is that new derivatives (proportional to powers of

0) appear here.

Coming back now to the problem of stability for the mass of the soliton, the expression is now.

$$M_{SK} = 4\pi f_\pi^2 \int_0^\infty \frac{d\eta'}{F_1} \left\{ \eta'^2 \left(\frac{dF(\eta',\phi)}{d\eta'}\right)^2 + 2\sin^2 F(\eta',\phi) + \right.$$

$$\left. + 8\phi^2 \left[\sin^2 F(\eta',\phi) \left(\frac{dF(\eta',\phi)}{d\eta'}\right)^2 + \frac{1}{2}\frac{\sin^4 F(\eta',\phi)}{\eta'^2} \right] \right\} \tag{16}$$

$$M_{SK} = \frac{1}{2} 2\pi f_\pi^2 \frac{1}{F_1} a(\phi) + 2\pi f_\pi^2 \frac{1}{F_1}\phi^2 c(\phi) \tag{17}$$

Notice that $a(\phi)$ and $c(\phi)$ are positive definite numbers. A minimum for the mass of the classical soliton for the Skyrme model using the hedgehog for the unitary SU(2) field should be obtained from:

$$\left.\frac{\partial M_{SK}}{\partial F_1}\right|_{F_1=F_1^0,\, e=e^0} = 0 \tag{18a}$$

$$\left.\frac{\partial M_{SK}}{\partial e}\right|_{F_1=F_1^0,\, e=e^0} = 0 \tag{18b}$$

where F_1^0 and e^0 are the values of F_1 and e at the minimum.

From both equations, we get:

$$\frac{1}{F_1^{0^2} e^{0^2}} \frac{d}{d\phi}\left[\frac{1}{2}a(\phi) + \phi^{0^2} c(\phi^0)\right] = 0 \tag{19a}$$

$$\frac{1}{F_1^{0^2}}\left[\frac{1}{2}a(\phi^0) + \phi^{0^2} c(\phi^0)\right] = 0 \tag{19b}$$

The only possible solution for Eq. (19b) is $F_1^0 \to \infty$, which simultaneously satisfy Eq. (19a). Notice that, again, the result for the pure non-linear sigma model comes from taking $e \rightarrow \infty$ first.

This results tells us that the complete Skyrme model, when the hedgehog is introduced for the unitary SU(2) field, has no lowest finite bound for the classical mass of the soliton, as the non linear sigma model. This may be the consequence of the scale invariance property of the solution of the Euler-Lagrange equations in both cases (Eqs. (11) e (5) respectively) for the hedgehog SU(2) ansatz. This features may have been inferred from the fact that the second functional derivative of the action is not positive definite, as shown in Ref. (8).

In practice, however, the Skyrme term is not a dynamical addition, its function being only to provide stabilization for the chiral soliton. That is, e is not a free parameter, but one chosen to set some kind of equilibrium condition. In our solution, this means that the scale invariance in is necessarily broken, keeping only F_1 as a freely varying parameter.

Coming back to the above Equations (18a), (18b) the argument is equivalent to discard Eq. (18b). Then, a true minimun for F_1 appears, since Eq.(18a) is now:

$$-\frac{1}{\bar{F}_1^2}\left[\frac{1}{2}a(\bar{\phi})+\bar{\phi}^2c(\bar{\phi})\right]+\frac{1}{ef_\pi\bar{F}_1}\frac{d}{d\phi}\left[\frac{1}{2}a(\bar{\phi})+\bar{\phi}^2c(\bar{\phi})\right]=0 \quad (20)$$

and a minimum will appear for

$$\frac{\bar{F}_1}{ef_\pi}=\bar{\phi}=\frac{\left[\frac{1}{2}a(\bar{\phi})+\bar{\phi}^2c(\bar{\phi})\right]}{\frac{d}{d\phi}\left[\frac{1}{2}a(\phi)+\phi^2c(\phi)\right]\Big|_{\phi=\bar{\phi}}} \quad (21)$$

The particular value of e at which it is interesting to locate the minimum F_1 should be found from some outside criterium. It is commonly used[5] to fix the mass of the soliton, after quantization, as the proton mass (for angular momentum 1/2). There is, however, no care in general to check whether this happens at a minimum for F_1.

This brings us to the problem of quantization. For the pure non-linearsigma model, the scale invariance develops a minimum for the quantized energy. It is likely that the same happens for the complete Skyrme model, and we are currently exploring this aspect.

To conclude, it may be worth to reanalyze the previous work on applications of the Skyrme model at the light of the developments exposed here.

We wish to acknowledge Prof. F.R.A. Simão for valuable conversations on this subject. One of us (JAM) acknowledges partial support from the CNPq (Conselho Nacional de Desenvolvimento Cientifico e Tecnológico), Brasil, during this work.

REFERENCES

1. T.H.R. Skyrme, Proc. Roy. Soc. London A260, 127 (1961); Nucl. Phys. 31, 556 (1962).

2. N.K. Pak and H.C. Tze, Ann. Phys. (N.Y.) 117, 164 (1979).

3. A.P. Balachandran, V.P. Nair, S.G. Rajeev and A. Stern, Phys. Rev. Lett. 49, 1124 (1982) and ibid 50, 1630(E) (1983).

4. E. Witten, Nucl. Phys. B223, 422, 433 (1983).

5. G.S. Adkins, C.R. Nappi and E. Witten, Nucl. Phys. B228, 552 (1983).

6. "Some questions regarding chiral solitons as baryons", J.A. Mignaco and S. Wulck, Leite Lopes Festschrift, N. Fleury et al., Editors; World Scientific Singapore (1988); p. 531.

7. J.A. Mignaco and S. Wulck, Phys. Rev. Letters 62, 1449 (1989).

8. J.A. Mignaco and S. Wulck, "Physical properties of the chiral quantum baryon", preprint CBPF (1989, to appear), submitted to Phys. Rev. D.

9. J.W. Carlson, Nuclear Physics B253, 149 (1985).