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CRITERIA TO FIX THE DIMENSIONALITY CORRESPONDING TO SOME HIGHER DERIVATIVE LAGRANGIANS

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Abstract

Newtonian potential and Huygens' principle can be simultaneously satisfied by means of a D'Alambertian (Laplacian) to a certain power for one specific dimension of the space. This can be of relevance for some physical higher derivative Lagrangians.

Key-words: Field theory; Gravitation; Huygen's principle; Green functions.

1. Introduction

When studying wave propagation there is a general principle, which plays a leading role in its physical interpretation. It is the Huygens' principle (HP)[1]. It implies a separation between those spaces where a clear cut message can be sent (only the velocity of light is present) and those where it is not (all velocities up to c are present).

This separation can be obtained by analysing the Green function corresponding to the wave equation. For the usual D'Alambertian equation, this has been done quite extensively.

However, sometimes physics put forward equations with iterated D'Alambertians, for instance, generalizations of the theory of gravity with a Lagrangian containing the square of the curvature tensor and its contractions[2]. Also in the bosonization in 2+1 dimensions[3] where $\Box^{\frac{1}{2}}$ seems to play an essential role.

It seems then that the power of the D'Alambertian is related to some specific number of dimensions in order to gain reasonable physical consequences. For instance, HP.

On the other hand, in studying gravitation the Newtonian potential NP (r^{-1}) is a necessary condition to be satisfied in the appropriate limit. For any number of dimensions, it always exists a certain power of the Laplacian for which the Green function is always r^{-1}, as will be shown below.

Thinking on both problems simultaneously i.e. when discussing gravitational waves in any number of dimensions and also the Newtonian limit r^{-1}, it comes up in a natural way to try to see, which is the power of the operator which can satisfy both conditions?

This is what we shall intend to do in the next paragraphs. In section 2 we find the power of the Laplacian operator which has for a specific number of dimensions, the

Green function equal to r^{-1}. In section 3 we see, for a given number of dimensions, which powers of the D'Alambertian satisfies HP. In section 4 we look for the conditions to satisfy Huygens' and Newtonian dynamics. In Section 5 based on the well known generalizations of the theory of gravity containing quadratic terms in the curvature tensor, and its contractions, we discuss the specific example.

$$L = \sqrt{-g} R_{\mu\nu} R^{\mu\nu} \tag{1.1}$$

and show that HP and NP are both satisfed only in six dimensions (5+1).

2. Newtonian Potential.

We define

$$R = \sum_{i=1}^{d} x_i^2 \qquad n = d+1 \ , \tag{2.1}$$

where n is the space-time dimension and d is the number of space dimensions. We are looking for the solution of an equation of the form

$$\overline{\Delta}^{\alpha} * G^{(\alpha)} = \delta, \tag{2.2a}$$

a non local operator which for integer α has to reduce to

$$\Delta^k G^{(k)} = \delta \qquad\qquad \overline{\Delta}^k = \Delta^k \delta(x) \tag{2.2b}$$

or, in the Fourier transforms:

$$\tilde{\overline{\Delta}}^{\alpha} \tilde{G}^{(\alpha)} = 1 \quad i.e. \tag{2.2c}$$

$$\tilde{G}^{(\alpha)} = \frac{1}{(k^2)^{\alpha}} \ . \tag{2.2d}$$

We shall use[4]

$$F(R^{\lambda}) = \frac{2^{2\lambda+d} \pi^{\frac{d}{2}} \Gamma(\lambda + \frac{d}{2})}{\Gamma(-\lambda)} (k^2)^{-(\frac{2\lambda+d}{2})} \ . \tag{2.3}$$

From (2.2a), (2.2b) and (2.3), it follows that

$$G^{(\alpha)} = \frac{\Gamma(\frac{d}{2} - \alpha) R^{\alpha - \frac{d}{2}}}{2^{2\alpha} \pi^{\frac{d}{2}} \Gamma(\alpha)} \ . \tag{2.4}$$

Changing the sign of α, we have

$$\overline{\Delta}^{\alpha} = \frac{\Gamma(\frac{d}{2}+\alpha)R^{-\alpha-\frac{d}{2}}}{2^{-2\alpha}\pi^{\frac{d}{2}}\Gamma(-\alpha)} \quad . \tag{2.5}$$

According to Ref. 4, p. 74, R^{λ} is an analytical distribution in λ with residues

$$Re \;\; R^{\lambda}|_{\lambda=-\frac{d}{2}-k} = \frac{(-1)^{k}\pi^{\frac{d}{2}}}{4^{k}k!\Gamma(\frac{d}{2}+k)}\Delta^{k}\delta(k) \quad . \tag{2.6}$$

So that, in particular, when in (2.5) α goes to a positive integer k,

$$\overline{\Delta}^{\alpha} * u \rightarrow \Delta^{k}\delta(x) * u = \Delta^{k}u \quad . \tag{2.7}$$

In these cases it is a differential operator while for $\alpha \neq k$ it is an integral operator.

Now, we want to answer the following question: which is the operator whose Green function is proportional to R^{s}?, defined according to (2.4) and (2.5). In particular, for the Newtonian potential $s = -\frac{1}{2}$ (a similar analysis can be performed for s=0, which corresponds to the logaritmic potential. There are some differences due to the poles of the Γ functions) we have

$$\alpha = \frac{d}{2} - \frac{1}{2} = \frac{n}{2} - 1 \quad , \tag{2.8}$$

which leads to the operator

$$\overline{\Delta}^{\frac{d-1}{2}} = \frac{\Gamma(d-\frac{1}{2})R^{\frac{1}{2}-d}}{2^{(1-d)}\pi^{\frac{d}{2}}\Gamma(\frac{1-d}{2})} \quad . \tag{2.9}$$

We see that d=odd=2m+1 according to (2.6) we have poles and these poles lead to

$$\overline{\Delta}^{\frac{d-1}{2}} \cong \Delta^{m}\delta(x) \quad .$$

So, for

$$\begin{array}{lll} d=3 & m=1 & \Delta \\ d=5 & m=2 & \Delta\Delta \\ d=7 & m=3 & \Delta\Delta\Delta, \;\; \text{etc.} \end{array} \tag{2.10}$$

For d = even $\frac{d-1}{2}$ = half integer and we get integral operator

$$\begin{array}{ll} d=2 & \Delta^{\frac{1}{2}} \\ d=4 & \Delta^{\frac{3}{2}} \\ d=6 & \Delta^{\frac{5}{2}}, \quad \text{etc.} \end{array} \tag{2.11}$$

3. Huygens' Principle

An elegant discussion of HP for the wave equation can be given using M.Riesz method of analytic continuation.[1,5]

Following a parallel idea to the Liouville expression which generalizes the concept of derivation and iterative integration, Riesz was able to give the solution of the wave equation

$$\Box u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} -, \ \dots, - \frac{\partial^2 u}{\partial x_d^2} = f(x_1, \dots, x_d, t) \tag{3.1}$$

(with adequate boundary conditions) The solution will be

$$u = G^{(\alpha)}_{Rz} * f|_{\alpha=1} \quad , \tag{3.2}$$

where

$$G^{(\alpha)}_{Rz} = \frac{2Q_+^{\alpha-\frac{n}{2}}\theta(-t)}{4^\alpha \pi^{\frac{n}{2}-1}\Gamma(1+\alpha-\frac{n}{2})\Gamma(\alpha)} \tag{3.3}$$

with

$$\begin{aligned} Q_+ &= Q^\lambda \quad Q = t^2 - R \quad \text{if} \quad t^2 > R \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{3.4}$$

$$\theta(x) = 0 \quad \text{if} \quad x < 0 \qquad \theta(x) = 1 \quad \text{if} \quad x > 0 \ .$$

For later use we shall define

$$\begin{aligned} Q_-^\lambda &= (-Q)^\lambda \quad \text{if} \quad R > t^2 \\ &= 0 \quad \text{otherwise} \ . \end{aligned} \tag{3.5}$$

According to Riesz

$$G^{(\alpha)}_{Rz} * G^{(\beta)}_{Rz} = G^{(\alpha+\beta)}_{Rz} \tag{3.6}$$

which means that

$$\Box^{\alpha}_{RSZ} = G^{(-\alpha)}_{RSZ}$$
$$\Box^{\alpha}_{RSZ} * G^{(\alpha)}_{RSZ} = \delta \quad , \tag{3.7}$$

where we have suppressed the bar for simplicity reasons.

From (3.3) we see that if the argument of each one or of both of the Γ functions are not zero or negative integers, HP is not fulfilled as $Q_+ \neq 0$ inside the cone and all velocities are present. However, when the argument of any or both of the Γ functions is zero or a negative integer, the only possibility for $G^{(\alpha)}$ of being different from zero lies on $Q = 0$, the light cone. Only one velocity (that of light) is present.

To be more precise, we rely on the following results of Ref. 4

a) The distribution Q^{λ}_+ has, for n odd, simple poles for $\lambda = -1, -2, ..., -k$ and $\lambda = -\frac{n}{2}, -\frac{n}{2} - 1, ...$ and the respective residues are

$$Re \quad Q^{\lambda}_+|_{\lambda=-k} = \frac{(-1)^{k-1}}{\Gamma(k)} \delta^{(k-1)}(Q) \quad , \tag{3.8}$$

where $\delta(Q)$ means

$$\delta(Q) = \frac{\delta(t+r) + \delta(t-r)}{r} \quad ,$$

$$Re \quad Q^{\lambda}_+|_{\lambda=-\frac{n}{2}-k} = \frac{(-1)^{\frac{d}{2}} \pi^{\frac{n}{2}} \Box^k \delta(x)}{\Gamma(k+1)\Gamma(k+\frac{n}{2})} \quad . \tag{3.9}$$

Observe that in (3.8) the residue has support in the whole cone, while in (3.9) the residue lies at the vertex of the cone. The first ones are important in classical theories, while the second ones in quantum theories.

b) If the number of dimensions is even Q^{λ}_+ has simple poles for $\lambda = -1, -2, ..., -(\frac{n}{2} - 1)$ and double poles for $\lambda = -\frac{n}{2}, -\frac{n}{2} - 1, ...$. The residues at the simple poles coincide with (3.8). Those for the double poles will not be needed here. Now let us look at (3.3) again and ask for which values of α -integer- HP is valid? For any n and α=k (3.3) is the Green function of $\Box^k$. For odd n, HP is never satisfied (k=integer). For even n and $-1 \leq \alpha \leq \frac{n}{2} - 1$ we have a pole in the denominator and HP is valid. According to (3.8) and (b) Q^{λ}_+ has also simple poles, with residues proportional to $\delta^{(k-1)}(Q)$ which is the expression, up to a constant, of the Green function.

A slightly more general question can be answered: For which values of α (not necessarily integers) does (3.3) satisfy HP?[6] If n is odd, this happens according to a) for

$$\alpha = \frac{n}{2} - k \qquad\qquad k = 1, 2, ..., \text{etc.} \tag{3.10}$$

Observe that α is half an integer.

The residue is proportional to

$$\delta^{(k-1)}(Q) \;\simeq\; \text{Re} \;\; Q_+^{\lambda} \;\; \text{in} \;\; \lambda = -k \; . \tag{3.11}$$

For even n, according to b) we have poles whose residues obey HP for

$$\lambda = -1, -2, ..., -(\frac{n}{2} - 1), \;\; \text{which corresponds to} \;\; \alpha = \frac{n}{2} - 1, \frac{n}{2} - 2, ..., 2, 1 \tag{3.12}$$

with the same residues as (3.11). Only a finite number of positive α, obey HP. For n odd and $\alpha > 0$ also only a finite number of half integer powers obey HP.

4. HP and NP

To satisfy simultaneously HP and NP we must look for those values of α, which satisfy both requirements.

In order to satisfy NP, according to formula (2.8) $\alpha = \frac{n}{2} - 1$.

On the other hand, from (3.10) as a requirement for HP $\alpha = \frac{n}{2} - k$ for n=odd and $\frac{n}{2} - 1, \frac{n}{2} - 2, ..., 2, 1$ for n = even.

We see that only k=1 satisfies both conditions. Observe that for n even the values of α are integers (iterated D'Alambertians) while for n odd, they are half-integers.

For n=4 only α=1 (ordinary D'Alambertian) obeys HP and NP. In four dimensions if we have $\Box^2$ neither HP nor NP are satisfied . We showed that one can not have clear cut signals in such theories.

Instead, for n=6 the only value allowed is α=2 and we can have clear messages and Newtonian dynamics.

5. $R_{\mu\nu}R^{\mu\nu}$ Theory.

Higher derivative generalizations of the theory of gravity are of interest. First, they appear as effective theories coming from strings[7], second they seem to be of relevance in connection with inflation[8] and third the chances of renormalizability are higher[9]. From these classes of alternative theories it is usual to choose those where the Lagrangian is quadratic in the curvature tensor and its ordinary contractions.

In the following we want to apply the weak field approximation to the theories just mentioned and illustrate under which conditions i.e. number of dimensions it will be possible to satisfy HP and to get as a Green function for the static case the Newtonian potential. Instead of considering a linear combination of the mentioned theories, as is usually done, and in order to simplify the example, we will choose

$$L = \sqrt{-g}R_{\mu\nu}R^{\mu\nu}, \tag{5.1}$$

where $R_{\mu\nu}$ is the usual Ricci tensor. We will show that because the weak field approximation of this theory gives essentially a double D'Alambertian, HP and NP are only simultaneously satisfied in this limit in 5+1 dimensions and definitely not in four, according to (2.8).

The dynamical equations corresponding to (5.1) are

$$G_{\mu\nu} = R_{;\mu\nu} + 2R_{\mu\theta\alpha\nu}R^{\theta\alpha} - \Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Box R - \frac{1}{2}R^{\rho\theta}R_{\rho\theta}g_{\mu\nu} = -kT_{\mu\nu} \ , \tag{5.2}$$

following the usual procedure to obtain the weak field aproximations we write for $g_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.3}$$

where $\eta_{\mu\nu} = diag(1,-1,-1,-1,-1,-1)$, $|h_{\mu\nu}| < 1$. The already known[2] linearized equations are

$$\frac{1}{2}\Box\,[(\partial_\mu\partial\nu - \frac{1}{2}\eta_{\mu\nu}\Box\,)h - \Box\,h_{\mu\nu}] = -kT_{\mu\nu} \tag{5.4}$$

whose trace is

$$\Box\Box\, h = \frac{2}{3} kT \quad . \tag{5.5}$$

The $T_{\mu\nu}$ we are considering is that of a point particle located at the origin

$$T^{\mu\nu} = \delta_0^\mu \delta_0^\nu M \delta^5(x) \quad . \tag{5.6}$$

In four dimensions the Green function of the corresponding static equation to the bi D'Alambertian (5.5) is

(5.7) $$h \sim r$$

which is physically unacceptable, since the potential is not of the Newtonian type (in this case h_{00} satisfies also a bi Laplacian equation and $h_{00} \sim r$). The Green function corresponding to the scalar eq.(5.5) does not obey HP in four dimensions as has been discussed in sections 3 and 4 .

Instead , in 5+1 dimensions the eq. (5.5) obey NP and HP and has only one velocity of propagation. The condition to satisfy both HP and NP, $\alpha = \frac{n}{2} - 1 = 2$ is fulfilled in this case.

In this way, we see that six dimensions seems to be the appropiate number of dimensions for the considered class of theories (5.1).

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