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ON THE GEOMETRICAL INTERPRETATION OF FRACTON AND FRACTAL DIMENSIONALITIES

by

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ABSTRACT

We suggest that the Fourier-analysis of the elastic vibrations in fractals can be very enlightening for understanding the vibrational underlying fractal geometry.

Key-words: 4/3 conjecture; Fractons; Fractal dimensionalities.

The elastic dynamics of random systems like percolation clusters, polymers, rubbers, gels has been considered by Aharony et al[1]. These systems behave like continuous media for length scales $L >> \xi \equiv$ correlation length (we shall refer to this as the euclidean regime), and present a fractal geometry for $L < \xi$ (non euclidean regime). Consistently, the density $N(\omega)$ of low frequency vibrational states (where low frequency refers to $\omega << \omega_{Debye}$) crosses over from $N(\omega) \sim \omega^{d-1}$ for extremely low frequencies to $N(\omega) \sim \omega^{\tilde{d}-1}$ for the rest of the low frequency region. For future purposes, let us recall that, for the systems under analysis, it is quite generically $\tilde{d} \equiv$ fracton dimensionality $< D \equiv$ fractal dimensionality $< d \equiv$ euclidean dimensionality (mass $\sim L^D$ for $L < \xi$).

Aharony et al argue that if only one characteristic length (namely the fracton localization length λ_{fr}) describes the fracton regime, then $\tilde{d} = 4/3$ (Alexander-Orbach conjecture). However, although the numerical usefulness of this conjecture is well known for a large class of systems, its strict validity breaks down. Indeed, Aharony et al mention some counterexamples (6 - ε expansion, lattice animals). We shall include in the present analysis non-random fractals like the Sierpinski gaskets, carpets, etc. Indeed, these systems can be herein treated on equal footing with the random ones focused by Aharony et al, by just considering the particular case $\xi \to \infty$ (their euclidean regime is reduced to the $\omega = 0$ mode, which corresponds to uniform translation). The standard Sierpinski gasket yields[2] $\tilde{d} = \ln 9/\ln 5 \simeq 1.36$ (and $D = \ln 3/\ln 2 \simeq 1.58$) and, in the present context, it can also be considered as a counterexample to the $\tilde{d} = 4/3$ value. Since $\tilde{d}$ is in general not strictly 4/3 , and assuming the validity of the connection established by Aharony et al, there must be more than one characteristic length in the non-euclidean regime. We present in this Comment a plausible picture for understanding (essentially on multifractal terms) this plurality of characteristic lengths.

Let us first consider a periodic d-dimensional Bravais lattice (strictly invariant under crystalline translation) with N sites. The wave vector $\vec{k}$ is a good labeling of the dynamic states and yields, in the first Brillouin zone (whose (reciprocal) volume is given by $(1/a)^d$, a being the microscopic length of the lattice), a set of uniformly spaced points. In the thermodynamic limit ($N \to \infty$), this set becomes uniformly dense, thus providing , as the support of the dynamic states of the system, an euclidean space whose dimensionality is d (consistently, the density $\rho(\vec{k})$ of states in the

$\vec{k}$-space is constant; we recall that $N(\omega)\, d\omega = \rho\,(\vec{k})\, d^d k$, hence $N(\omega) \sim \omega^{d-1}$ in the $\omega \to 0$ limit as already mentioned).

If we consider now a d-dimensional amorphous substance like glass (statistically invariant under translation), $\vec{k}$ no more can label the modes. However, in some sense (which can be precised in terms of the dynamical structure function S (k, ω)) this quantity still can be useful, and the image of the uniformly spaced points in the "Brillouin zone" can be retained by attributing an appropriate width (in $\vec{k}$ - space) to each of these points. Oncemore, in the $N \to \infty$ limit, we obtain as support of the states a d-dimensional euclidean space (hence $\rho\,(\vec{k})$ = constant).

But if we consider now a fractal structure (either strictly or statistically scale invariant for $N^{1/D} \geq L >> a$) it seems reasonable to us that the "uniformly spaced" image will break down, and we will have, as support of the dynamical states, an hierarchical (in both the location and the width of the points) set in the "Brillouin zone". This set would become, in the $N \to \infty$ limit, a kind of multifractal exhibiting a highly non uniform density. We believe that the standard relation between $N(\omega)$ and $\rho\,(\vec{k})$ would be replaced by something of the type $N(\omega) d\omega = \rho\,(\vec{k})\, d^D k$ with $\rho\,(\vec{k}) \sim k^{d_f - D}$ $(0<k \lesssim 1/a)$, d_f being a k-dependent fractal dimensionality like that indicated in Fig. 1. Let us stress that, as presented, these relations are implicitely assuming that in some course grain sense we can still speak of a dispersion relation $\omega = \omega(k)$ (it seems reasonable to consider such an approximation at least in the case where D is very close to d in the sense that we specify later on). Notice also that, if we assume in this same sense that $\omega \propto k$ $(\omega << \omega_{Debye})$, then we recover, for k satisfying $1/\xi < k \ll k^* < 1/a$, $N(\omega) \sim \omega^{\tilde{d}-1}$ as desired. We see in Fig. 1 that the non-euclidean regime would be composed by two distinct regions, the well known fracton regime $(1/\xi < k << k^*)$ and the pseudo-independent oscillator regime $(k^* << k \lesssim 1/a)$, named so because, for translationally invariant systems, that region corresponds to almost vanishing group velocities. The quantity k^* reflects the existence of at least one additional relevant characteristic length (k^* would not necessarily scale as λ_{fr}^{-1} of Aharony et al), and might be of the order of $1/a$ (e.g., we expect for a translationally invariant system $1/\xi \simeq k^* \simeq 1/a$, the fracton region thus shrinking to zero) or much smaller (we expect, for systems exhibiting fair amounts of localization due to fractality, $k^* << 1/a$ thus being consistent with the intuition that localization can be seen as an enlargement of the number of degrees of freedom essentially associated with almost vanishing group velocities). The assumption d_f ($k \simeq 1/a$)=D can be understood from the fact that, in that region, the only thing we expect to matter is just the number of degrees of freedom (we recall mass ~ L^D).

Let us now address a crucial question; is it possible to conciliate a continuous dependency of d_f on $\vec{k}$ (like that of Fig. 1) with the known[2] existence of important gaps in the ω -spectrum of fractal systems? The answer is yes if important gaps can be present in the $\vec{k}$ -support of that spectrum. Let us illustrate this possibility with a d=1 mathematical construction which generalizes the standard Cantor set.First we define a covering ratio r ($0 \leq r \leq 1$) which gives the proportion (symetrically distributed on both sides of the center of the interval) to be covered through successive iterations:see Fig. 2. Whatever be the definition we adopt for the fractal dimensionality d_f, it will have to monotonously increase from 0 to 1 when r increases from 0 to 1, and also satisfy $d_f(r=2/3) = \ln 2/\ln 3$. We now allow r to depend (quite arbitrarily) on ka and construct the generalized Cantor set as indicated in Fig. 3 : we pass from generation (0) to (1) by using the covering ratio r(1/2); from (1) to (2) by using the covering ratio r(r(1/2)/4) for the left segment and r(1-r(1/2)/4) for the right segment, and so on. We see that, after infinite iterations, we shall obtain a multifractal dust whose density will be non uniform (in Fig. 3, d_f would monotonously increase with ka) and which will exhibit the desired gaps ! Of course, when r does not depend on ka we recover the type of cases presented in Fig. 2 .

Let us finally consider fractals which are increasingly "closer" to a Bravais lattice. For example, instead of the standard Sierpinski gasket in which we iteratively eliminate 1 triangle out of 4, we might eliminate 4 (central) triangles out of 16 (hence $D = \ln 12/\ln 4 \simeq 1.79$), or 1 (central) triangle out of 16 (hence $D = \ln 15/\ln 4 \simeq 1.95$) and so on, the last element of such a family being the triangular lattice (D=d=2). Within such a family we expect d_f of Fig. 1 to gradually approach d for all values of k (hence both D and $\tilde{d}<D$ would tend to d). It is in this manner that the crossover from scale invariant systems to translationally invariant ones would appear in the present picture.

The detailed study of the Fourier transforms of the amplitudes associated to the normal dynamical modes of fractals should clarify whether the present ideas are not only plausible, but correct. It would not be the least benefit the fact that we would then gain a direct and purely geometrical interpretation of $\tilde{d}$!

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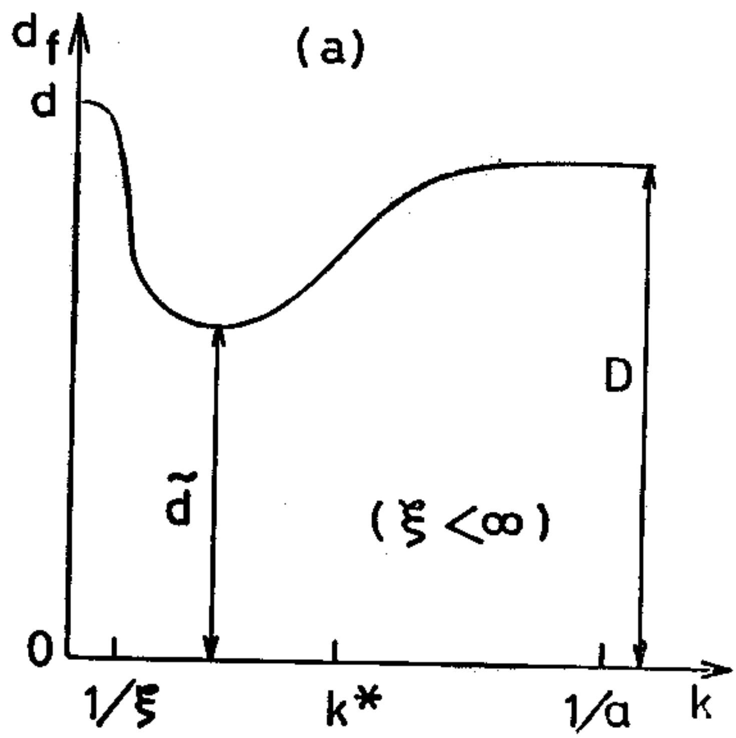
(a)
d_f
d
$\tilde{d}$
D
($\xi < \infty$)
0
$1/\xi$
k^*
$1/a$
k

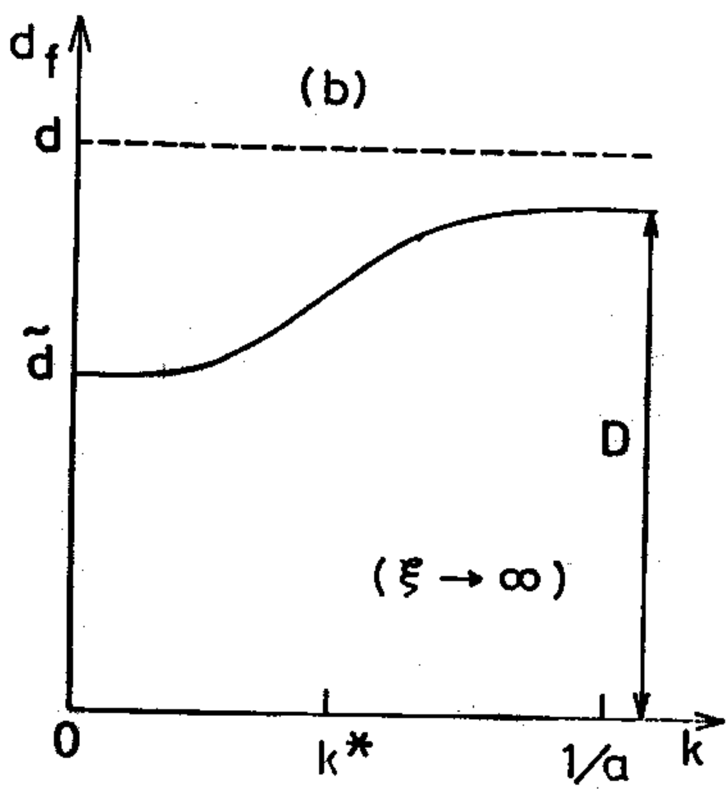
(b)
d_f
d
$\tilde{d}$
D
($\xi \to \infty$)
0
k^*
$1/a$
k

FIG. 1

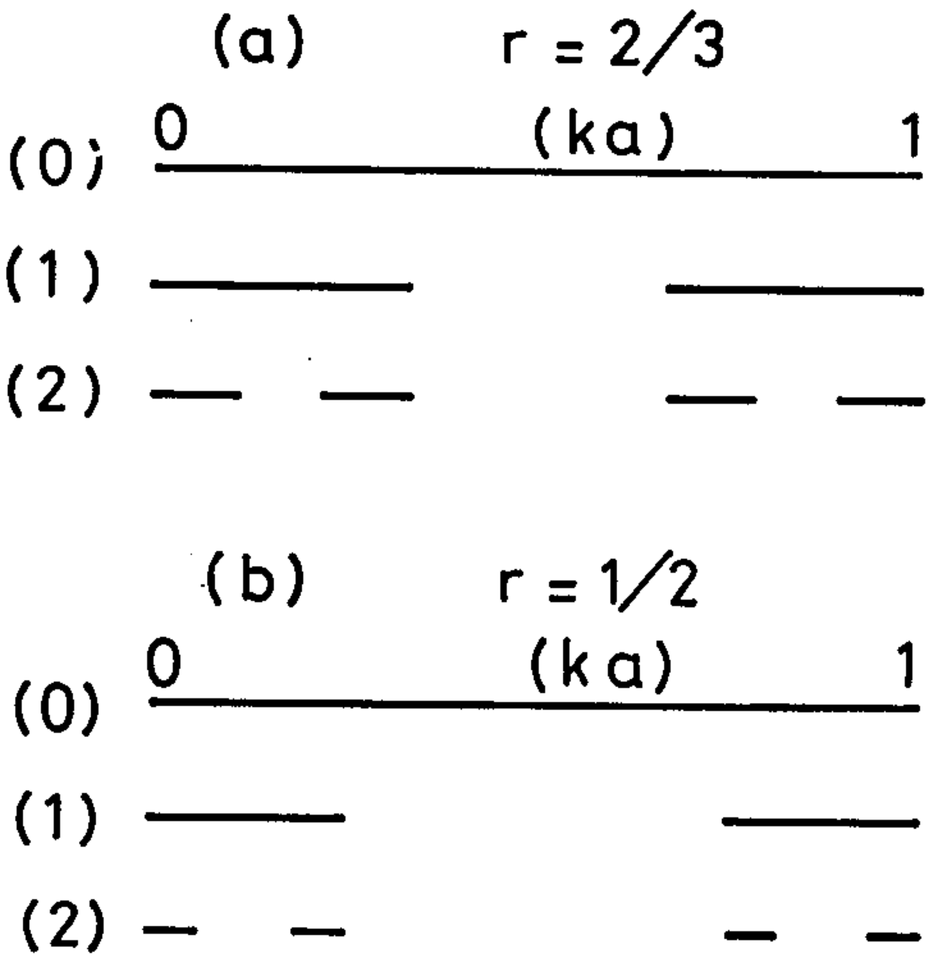

FIG. 2

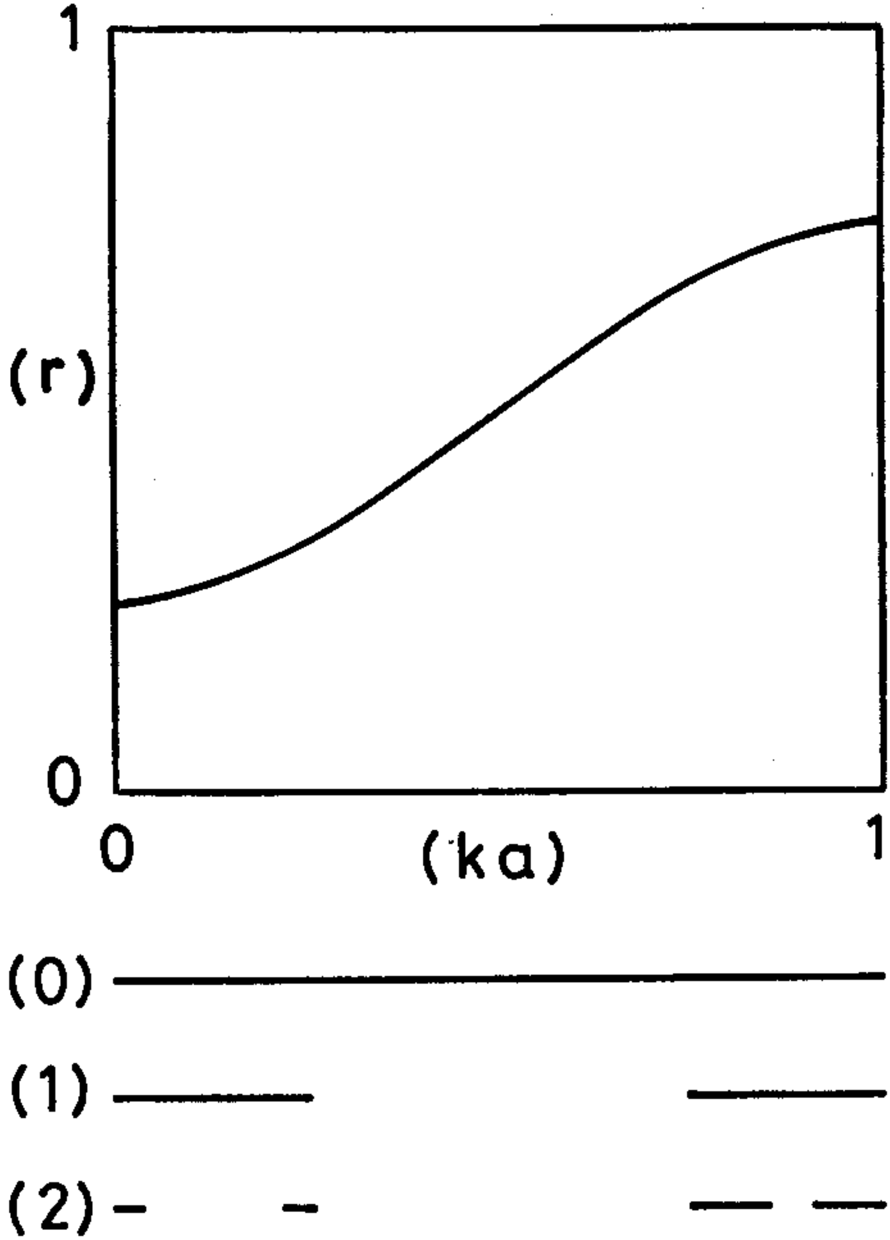

FIG.3

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(No captions for figures).