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AN ALTERNATIVE METHOD TO SOLVE THE HADRONIC COSMIC-RAY DIFFUSION EQUATIONS. THE MUON AND NEUTRINO FLUXES

by

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ABSTRACT

The integro-differential equations which describe the diffusion of the hadronic component in the atmosphere are solved exactly using the successive approximation method and we derived the numbers of the produced muons and neutrinos from the hadron fluxes obtained as the solution of the above equations. The primary cosmic ray spectrum used in our calculation is presented in a general form G(E).

Key-words: Successive approximation method; Lepton fluxes; Difusion equations.

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INTRODUCTION

Several authors have studied analitically the diffusion equations of hadrons in the atmosphere. A. Ohsawa[1] solved these equations applying a Laplace Transformation for the depth and a Mellin Transformation for the energy K. Mackewon, J. Sidhanta and others authors[2] solved these equations applying only the Mellin Transformation on the variable E (energy).

If we use the method of Mellin's Transform we obtain a real solution represented by a contour integral in the complex domain and, only in few particular cases this integral can be evaluated exactly; in the general cases, however, we must use some approximate method for estimate it, as for example the saddle point method.

In this paper we used the successive approximation method to solve these diffusion equations with a boundary condition, $F_N(0,E) = G(E)$, where G(E) is a continuous, positive and limited function and it represents the primary cosmic ray enegy spectrum.

We obtain the differential fluxes of hadrons, muons and neutrinos in a exact and compact form and for the particular case, $F_N(0,t) = N_0E^{-(\gamma+1)}$, our solutions result in the generally used expressions.

1 - Nucleon Diffusion Equation in the atmosphere

The diffusion of the nucleons in the atmosphere can be represented by the one-dimensional integro-differential equations,

$$\frac{\partial F_N(x,E)}{\partial x} = -\frac{F_N(x,E)}{\lambda_N} + \int_E^\infty \frac{F_N(x,E')}{\lambda_N} f_{NN}(E,E')dE' \qquad (1.1)$$

where

$F_N(x,E)$ is the differential nucleon flux at depth x and at energies between E and E + dE.

λ_N is the nucleon interaction mean free path in the atmosphere.

$f_{NN}(E,E')$ are the energy distributions of the secondary nucleons

and

E',E are respectively the primary and secondary nucleon energy.

If $f_{NN}(E,E')$ are homogeneous functions of the variables E,E',

$$f_{NN}(E,E')dE' = f_{NN}(\eta = E/E')\frac{dE'}{E'} \qquad (1.2)$$

the equations (1.1) take the form

$$\frac{\partial F_N(x,E)}{\partial x} = -\frac{F_N(x,E)}{\lambda_N} + \int_0^1 \frac{F_N(x,E/\eta)}{\lambda_N} f_{NN}(\eta)d\eta/\eta \qquad (1.3)$$

whith the limit condition

$$F_N(0,E) = G(E)$$

where G(E)dE is the differential energy spectrum of the nucleons at the top of the atmosphere. This function is supposed to be continuous, positive and bounded $(G(E) \leq M)$ in the interval $0 < E_{min} \leq E < \infty$. The existence of the integral $\int_E^\infty G(E)dE$, for $E \geq E_{min}$, must also be stated because it represents the primary integral spectrum.

To simplify the work we put $F_N(x,E) = e^{-x/\lambda_N} y_N(x,E)$. So that the equation (1.3) and the respective initial condition become.

$$\frac{\partial y_N(x,E)}{\partial x} = \frac{1}{\lambda_N}\int_0^1 y_N(x,E/\eta) f_{NN}(\eta) d\eta/\eta \tag{1.4}$$

with

$$y_N(0,E) = G(E)$$

Now we make the followings successive approximations

$$y_{N_0}(x,E) = G(E)$$

$$y_{N_n}(x,E) = G(E) + \frac{1}{\lambda_N}\int_0^x dt \int_0^1 y_{N_{n-1}}(x,E/\eta) f_{NN}(\eta)\frac{d\eta}{\eta} \tag{1.5}$$

So, we obtain successively

$$y_{N_1}(x,E) = G(E) + \chi/\lambda_N \int_0^1 G(E/\eta_1) f_{NN}(\eta_1) d\eta_1/\eta_1$$

$$y_{N_n}(x,E) = G(E) + \sum_{i=1}^{n} \frac{(x/\lambda_N)^i}{i!} \int_0^1 \cdots \int_0^1 G\left(\frac{E}{\eta_1 \cdots \eta_i}\right) \cdot \frac{f_{NN}(\eta_1)}{\eta_1}$$

$$\frac{f_{NN}(\eta_i)}{\eta_i} d\eta_1 \ldots d\eta_i \tag{1.6}$$

The convergence of the solution (1.6) is showed in the apendice A. So, the nucleon flux at the depth x, and the energy in the interval E and E + dE is

$$F_N(x,E) = e^{-x/\lambda_N}\Bigg\{G(E) + \sum_{n=1}^{\infty} \frac{(x/\lambda_N)^n}{n!} \int_0^1 \cdots \int_0^1 G\left(\frac{E}{\eta_1 \cdots \eta_n}\right) \cdot$$

$$\cdot \frac{f_{NN}(\eta_1)}{\eta_1} \cdots \frac{f_{NN}(\eta_n)}{\eta_n} d\eta_1 \ldots d\eta_n\Bigg\} \tag{1.7}$$

2 - Diffusion equation of secondary hadrons in the atmosphere.

The diffusion of the secondary particles a (where a may represent $\pi^{\pm}$, $K^{\pm}$, D^0 etc) in the atmosphere can be determined by the one-dimensional differential equation

$$\frac{\partial F_a(x,E)}{\partial x} = - F_a(x,E)\left(\frac{1}{\lambda_a} + \frac{b_a}{Ex}\right) + P_a^{NA}(x,E) + P_a^{aA}(x,E) \tag{2.1}$$

with the limit condition

$$F_a(0,E) = 0$$

$P_a^{NA}(x,E)$ is the rate of production of secondary particles, a originated by the nucleon-air nuclei interactions, with energy between E and E + dE at the depth x.

$P_a^{aA}(x,E)$ is the rate of production of secondary particles, a originated by a-air nulcei interactions, with energy between E and E + dE at the depth x.

and are given by the expressions

$$P_a^{NA}(x,E) = \int_{E'_{min}}^{E'_{max}} \frac{F_N(x,E')}{\lambda_N} f_{Na}(E,E')dE' \tag{2.2}$$

$$P_a^{aA}(x,E) = \int_{E'_{min}}^{E'_{max}} \frac{F_a(x,E')}{\lambda_a} f_{aa}(E,E')dE'$$

with

λ_a = interaction mean free path of the hadron, a, in the atmosphere.

$f_{NN}(E,E')$ and $f_{aa}(E,E')$ are the energy distributions of the secondary particles, a, originated by the interactions N-air nuclei, and a-air nuclei, respectively.

b_a is the decay constant of the secondary particle, a, in the atmosphere.

If $f_{Na}(E,E')$ and $f_{aa}(E,E')$ are homogeneous functions of the variables E and E' the equation (2.1), with the expressions (2.2), takes the form

$$\frac{\partial F_a(x,E)}{\partial x} = - F_a(x,E)\left(\frac{1}{\lambda_a} + \frac{b_a}{Ex}\right) + \int_0^1 \frac{F_N(x,E/\eta)}{\lambda_N} f_{Na}(\eta)\frac{d\eta}{\eta} +$$

$$+ \int_0^1 \frac{F_a(x,E/\eta)}{\lambda a} f_{aa}(\eta)d\eta/\eta \tag{2.3}$$

To simplify the work, we put

$$F_a(x,E) = x^{-b_a/E} y_a(x,E) \tag{2.4}$$

and define the operator $\hat{A}$;

$$\hat{A}F_a(x,E) = \int_0^1 F_a(x,E/\eta)f_{aa}(\eta)d\eta/\eta \tag{2.5}$$

So, the quation (2.3) and the respective initial condition become

$$\frac{\partial y_a(x,E)}{\partial x} = - (1 - \hat{A}) \frac{y_a(x,E)}{\lambda_a} + x^{ba/E} \int_0^1 F_N(x,E/\eta)f_{Na}(\eta)\frac{d\eta}{\eta} \tag{2.6}$$

or

$$\frac{\partial y_a(x,E)}{\partial x} = - (1 - \hat{A}) \frac{y_a(x,E)}{\lambda_a} + x^{ba/E}P_a^{NA}(x,E) \tag{2.6'}$$

The term $\hat{A}y_a(x,E)$ is unknwon, and to solve the equation (2.6) we will make the followings successive approximations. Initially we obtain the approximation of zero order, $Fa_0(x,E)$, where we don't include the second generation of secondary particles a. After this we put, in the equation

(2.6), the term $\hat{A}Fa_0(x,E)$ in the place of the exact term $\hat{A}F_a(x,E)$. We obtain, then, the first estimative of the contribuition of second generation of the a-particles to the total flux.

After this, we make the successive avaliation of the contribution of the 3^{th}, 4^{th} ... n^{th} generations to the total flux.

This procedure is represented by the following recurrence equations

$$\frac{\partial y_{a_0}}{\partial x} = - \frac{y_{a_0}(x,E)}{\lambda_a} + P_{a_0}(x,E)$$

$$\frac{\partial y_{a_1}}{\partial x} = - \frac{y_{a_1}(x,E)}{\lambda_a} + P_{a_1}(x,E) \tag{2.7}$$

$$\frac{\partial y_{a_n}}{\partial x} = - \frac{y_{a_n}(x,E)}{\lambda_a} + P_{a_n}(x,E)$$

where

$$P_{a_0}(x,E) = x^{ba/E} P_a^{NA}(x,E)$$

$$P_{a_n}(x,E) = x^{ba/E} P_a^{NA}(x,E) + \frac{\hat{A}y_{a_{n-1}}(x,E)}{\lambda_a} \tag{2.8}$$

The solutions of the linear equations system must satisfy the following boundary condition

$$y_{a_n}(x,E) = 0 \quad ; \quad n = 0,1,2$$

The functions $P_{a_n}(x,E)$ and $F_{a_n}(x,E)$, $(n = 0,1,2...)$ must be continuous in the domain $\xi = [0 \le x \le x;\ E_{min} \le E \le E_{max}]$, with $E_{min} > 0$ $E_{max} > E_{min}$ and $x > 0$. This is satisfied when;

a) G(E) continuous non-negative and limited function in the interval $I = [E_{min},\infty)$, $E_{min} > 0$.

b) $f_{Na}(\eta)$ and $f_{aa}(\eta)$ continuous and non-negative functions in the interval $0 \le \eta \le 1$.

and

c) the integral's, $\int_0^1 f_{Na}(\eta)d\eta/\eta$ and $\int_0^1 f_{aa}(\eta)d\eta/\eta$, exist.

If these conditions are satisfied, the unique and compact solution of the systems (2.7) and (2.8) is

$$y_{a_n}(x,E) = \hat{B}P_{a_n}(x,E) = \int_0^x e^{-(x-t)/\lambda a}P_{a_n}(t,E)dt \tag{2.9}$$

where the operator $\hat{B}$ is defined by:

$$\hat{B}H(x,E) = \int_0^x e^{-(x-t)/\lambda a}H(t,E)dt \tag{2.10}$$

Then,

$$y_{a_0}(x,E) = \hat{B}P_{a_0}(x,E) = \hat{B}x^{ba/E}P_a^{NA}(x,E)$$

$$y_{a_1}(x,E) = \hat{B}P_{a_1}(x,E) = \hat{B}(1 + \hat{A}\hat{B})P_a^{NA}(x,E)x^{ba/E}$$

$$y_{a_n}(x,E) = \hat{B}(1 + \hat{A}\hat{B} + \dots + (\hat{A}\hat{B})^n)x^{b_a/E}P_a^{NA}(x,E) \tag{2.11}$$

The induction of $(n + 1)^{th}$ term is

$$y_{a_{n+1}}(x,E) = \hat{B}\left(\sum_{i=0}^{n+1} (\hat{A}\hat{B})^i\right)x^{b_a/E}P_a^{NA}(x,E)$$

The n^{th} approximation, $F_{a_n}(x,E)$, can be put in the following form,

$$y_{a_n}(x,E) = \hat{B}(1 + \hat{A}\hat{B} + \dots + \hat{A}^n\hat{B}^n)x^{b_a/E}P_a^{NA}(x,E) \tag{2.12}$$

In the apendice B we establish the equality of the equations (2.11) and (2.12) and we show that the solution, $y_a(x,E) = \lim_{n\to\infty} y_{a_n}(x,E)$, exist.

The differential flux of the secondary particles, a, at depth x, and energy between E and E + dE is

$$F_a(x,E) = x^{-ba/E}\sum_{i=0}^{\infty}\hat{A}^i\hat{B}^{i+1}x^{ba/E}P_a^{NA}(x,E) \tag{2.13}$$

3 - Differential muon and neutrino vertical fluxes

We derived from the flux of secondary particles, a, (2.13) the spectra of muouns and neutrinos, d (μ or ν) in the following way

$$F_d(x,E_d) = \int_0^x dtW(t,x,E_d) \int_{E_d^-}^{E_d^+} \frac{B_a b_a}{E_a t} F_a(t,E_a) f_{ad}(E_a,E_d) dE_a \tag{3.1}$$

where

$F_d(x,E_d)$ is the differential flux of d-lepton at depth x and energies between E_d and $E_d + dE_d$.

The values of E_d^+, E_d^- and the functions $f_{ad}(E_a,E_d)$ are obtained according to relativistic kinematics considerations[3] of the two and three body decays.

B_a is the branching ratio of secondary particles, a.

and

$W(t,x,E_d)$ is the probability that a muon or a neutrino (d), with energy, E_d, produced in a depth t, will survive until the depth x.

4 - Particular case

If the primary energy spectrum of nucleons is $F_N(0,E) = N_0 E^{-(\gamma+1)}$, the solutions (1.7) and (2.13) will take the simplified expressions as follows.

4-1 - Differential nucleon flux

The multiple integrals

$$I = \int_0^1 \cdots \int_0^1 N_0 \left(\frac{E}{\eta_1 \eta_2 \cdots \eta_n}\right)^{-(\gamma+1)} f_{NN}(\eta_1) \cdots f_{NN}(\eta_n) \frac{d\eta_1 \cdots d\eta_n}{\eta_1 \cdots \eta_n}$$

which appear in the solution (1.7), results in $NoE^{-(\gamma+1)}(C_{NN})^n$, where

$$C_{NN} = \int_0^1 \eta^\gamma f_{NN}(\eta)d\eta$$

The nucleon flux, then, becomes

$$F_N(x.E) = e^{-x/\lambda_N} \sum_{n=0}^{\infty} \frac{(x/\lambda_N)^n}{n!} (C_{NN})^n N_0 E^{-(\gamma+1)}$$

which is equivalent to the usual expression

$$F_N(x,E) = N_0 E^{-(\gamma+1)} e^{-x/L_N}$$

where

$L_N = \dfrac{\lambda_N}{1 - C_{NN}}$, is the absorption mean free path of nucleons in the atmosphere.

4-2 - Differential flux of secondary particles

The production rate of secondaries, a, from the nucleon-air nuclei interaction is,

$$P_a^{NA}(x,E) = \frac{F_N(x,E)}{\lambda_N} C_{Na} \tag{4.2.1}$$

where

$$C_{Na} = \int_0^1 \eta^\gamma f_{Na}(\eta)d\eta \tag{4.2.2}$$

and the applying the operator $\hat{B}$, (n+1) times in the expression $x^{ba/E}\, P_a^{NA}(x,E)$, we obtain

$$\hat{B}^{n+1} x^{ba/E} P_a^{NA}(x,E) = \int_0^x dt_{n+1} \ldots \int_0^{t_2} dt_1 t_1^{ba/E} \,.\, e^{-(x-t_1)/\lambda_a} x^{ba/E} P_a^{NA}(t,E)$$

If we will make the following substitution, $t_1 = x - \tau$, and use the properties of the iteratives integrals[4], we obtain

$$\hat{B}^{n+1} x^{ba/E} P_a^{NA}(x,E) = e^{-x/L_N} \int_0^x e^{\left(\frac{1}{L_N} - \frac{1}{\lambda_a}\right)\tau} \frac{\tau^n}{n!} \,.\, (x-\tau)^{b_a/E} NoE^{-(\gamma+1)} \frac{C_{Na}}{\lambda_N} \, d\tau \tag{4.2.3}$$

Applying the operator $\hat{A}$ n times in the last equation, we obtain

$$\hat{A}^n\left(\hat{B}^{n+1} x^{ba/E} P_a^{NA}(x,E)\right) = NoE^{-(\gamma+1)} C_{Na} e^{-x/L_N} \,.$$

$$.\int_0^x e^{\left(\frac{1}{L_N} - \frac{1}{\lambda_a}\right)\tau} \frac{\tau^n}{n!} \, (x-\tau)^{ba/E} \lambda_N (C_{aa}/\lambda_a)^n \tag{4.2.4}$$

where

$$C_{aa} = \int_0^1 \eta^\gamma f_{aa}(\eta) d\eta$$

Finally the differential flux of secondary particles, a, is

$$F_a(X,E) = \frac{N_0 E^{-(\gamma+1)} C_{Na}}{\lambda_N} e^{-x/L_a} \int_0^x \left(\frac{t_1}{x}\right)^{b_a/E} e^{-\left(\frac{1}{L_N} - \frac{1}{L_a}\right)t_1} dt_1 \qquad (4.25)$$

where

L_a is the absorption mean free path of the secondaries, a, in the atmosphere.

If $E >> b_a$, the expression $\left(\frac{t_1}{x}\right)^{ba/E}$ is approximately 1 and the solution (4.2.6) take the well known form

$$F_a(x.E) = \frac{N_0 E^{-(\gamma+1)} C_{Na}}{\lambda_a} \frac{e^{-x/L_a} - e^{-x/L_N}}{\frac{1}{L_N} - \frac{1}{L_a}} \qquad (4.2.7)$$

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APENDICE A

Convergence of the succession $y_{Nn}(x,E)$

If G(E) is a continuous, non-negative and bounded function $(G(E) \leq M)$ in the interval $I = [E_{min,}\infty]$, $E_{min} > 0$, $E \in [E_{min},\infty]$ where M is some positive constant, and if the integral $\int_0^1 f_{NN}(\eta)d\eta/\eta = A$ exist with the functions $f_{NN}(\eta)$ positives, then the series $S = \sum_{n=0}^{\infty} u_n(x,E)$, whose n^{th} partial sum S_n is $y_{N_n}(x,E)$, is a series of positive terms. Sn is bounded in any set (T) such that $0 \leq x \leq X$; $E_{min} \leq E \leq E_{max}$. In fact, we have

$$S_n = y_{N_n}(x,E) \leq M \sum_{\nu=0}^{n} \frac{(x/\lambda)^{\nu}}{\nu!} A^{\nu} < Me^{Ax/\lambda}.$$

The uniform convergence of the exponential in the set (T) assures the uniform convegence of the series S to a function $y_N(x,E)$ in (T).

APENDICE B

Convergence of the succession $y_{a_n}(x,E)$

We will show the convergence of the succession $y_{a_n} = \hat{A}^n\hat{B}^{n+1}P_{a_0}(x,E)$ by the following steps:

B-1) Convergence of the series $\hat{B}^n P_{a_0}(x,E)$

If G(E) is a continuous, non-negative and limited function $(G(E) \le M)$ in the interval $0 < E_{min} \le E < \infty$, and if the integral $\int_0^1 f_{Na}(\eta)d\eta/\eta = C_1$ exists, with the functions $f_{Na}(\eta)$ continuous and non-negative in the interval $0 \le \eta \le 1$, then the succession

$$\hat{B}^n P_{a_0}(x,E) = \int_0^x dt_n \ldots \int_0^{t_2} e^{-(x-t_1)/\lambda_a} P_{a_0}(t,E)dt_1 \tag{B.1.1}$$

where

$$P_{a_0}(x,E) = x^{b_a/E} P_a^{NA}(x,E)$$

can be put in the form

$$\hat{B}^n P_{a_0}(x,E) = \frac{1}{\lambda_N} \sum_{\nu=0}^{\infty} b_\nu(E) A_{n,\nu}(x) \tag{B.1.2}$$

where

$$b_\nu(E) = \int_0^1 d\eta_n f_{Na}(\eta) G\left(\frac{E}{\eta\eta_1 \ldots \eta_\nu}\right) \le C_1 M \tag{B.1.3}$$

and

$$A_{n,\nu}(x) = \int_0^x \frac{(x-\tau)^{n-1}}{(n-1)!} e^{-\tau/\lambda_a} a_\nu(x-\tau)d\tau \qquad (B.1.4)$$

with

$$a_\nu(x-\tau) = \frac{e^{-(x-\tau)/\lambda_N} A^\nu}{\nu!} \left(\frac{x-\tau}{\lambda_N}\right)^\nu \qquad (B.1.5)$$

The expression (B.1.2) is absolute and uniformely convergent in $T:(0 \le x \le X);\ 0 < E_{min} \le E < \infty)$, as

$$\frac{1}{\lambda_N}\left|a_\nu(E)a_{n,\nu}(x)\right| \longrightarrow \frac{MC_1}{\lambda_N} \frac{X^n}{n!} R_\nu(x) \ .\ e^{-x(1/\lambda_a+1/\lambda_N)} \qquad (B.1.6)$$

where $R_\nu(x)$ is the series $R_\nu(x) = \sum_{\nu=0}^{\infty} \left(\frac{Ax}{\lambda_N}\right)^\nu \frac{1}{\nu!}$ which converges to e^{Ax/λ_N}, and we assume the convergence of the exponentials that appears in (B.1.5)

B-2) Convergence of the series $\hat{A}^n\hat{B}^n P_{a_0}(x,E)$

$$\hat{A}^n\hat{B}^n P_{a_0}(x,E) = \frac{1}{\lambda_N} \frac{1}{\lambda_a^n} \int_{\varepsilon_1}^1 \dots \int_{\varepsilon_n}^1 \frac{f_{aa}(\eta_1)\dots f_{aa}(\eta_n)}{\eta_1\dots\eta_n} \ .$$

$$.\ d\eta_1\dots d\eta_n \sum_{\nu=0}^{\infty} b_\nu\left(\frac{E}{\eta_1\dots\eta_n}\right) a_{n,\nu}(x) \qquad (B.2.1)$$

where, $0 \le \varepsilon_i \le 1$, $i = 1\dots n$. As b_ν is positive and continuous in

$E \geq E_{min} > 0$, then $b_{\nu}\left(\frac{E}{\eta_1 \ldots \eta_n}\right)$ is absolute and uniformely convergent in T. Putting

$$b_{n,\nu,\varepsilon_1\ldots\varepsilon_n} = \int_{\varepsilon}^{1} \prod_{i=1}^{n} \frac{f_{aa}(\eta_i)d\eta_i}{\eta_i} \, . \, b_{\nu}\left(\frac{E}{\eta_i}\right) \tag{B.2.2}$$

the expression (B.2.1) becomes:

$$\hat{A}^n\hat{B}^nP_{a_0}(x,E) = \frac{1}{\lambda_N \lambda_a^n} \sum_{\nu=0}^{\infty} A_{n,\nu}(X) b_{n,\nu,\varepsilon_1\ldots\varepsilon_n} \tag{B.2.3}$$

Assuming the existence of the integrals $\int_0^1 f_{aa}(\eta)\frac{d\eta}{\eta} = C_2$, the expression (B.2.2) converge absolute and uniformely. Then $\hat{A}^n\hat{B}^nP_{a_0}(x,E)$ converges to $\frac{MC_1}{\lambda_N} \frac{X^n}{n!} R_{\nu}(X) \frac{C_2^n}{\lambda_a^n} \, . \, e^{-x\left(\frac{1}{\lambda_N} + \frac{1}{\lambda_a}\right)}$ where we assume again the convergence of the exponentials.

B-3) Equalities $\hat{A}^n\hat{B}^nP_{a_0}(x,E) = (AB)^nP_{a_0}(x,E) = \hat{B}^n\hat{A}^nP_{a_0}(x,E)$ we have

$$\hat{A}^n\left(\hat{B}^nP_{a_0}(x,E)\right) = \frac{1}{\lambda_a^n} \prod_{i=1}^{n} \int_{\varepsilon_1}^{1} \frac{f_{aa}(\eta_i)d\eta_i}{\eta_i} \, . \hat{B}^nP_{a_0}(x,E/\eta_1\ldots\eta_n) \tag{B.3.1}$$

The expression (B.3.1) can be put in the form

$$\hat{A}^n\left(\hat{B}^n P_{a_0}(x,E)\right) = \frac{\hat{B}^n}{\lambda_a^n} \prod_{i=1}^{n} \int_{\varepsilon_i}^{1} f_a(\eta_i) \frac{d\eta_i}{\eta_i} P_{a_0}(x,E/\eta_i) \qquad \text{(B.3.1)}$$

The equality of the expressions (B.3.1) and (B.3.2) is satisfied for ε_i arbitrary and non-negative numbers. So, these expressions are non-negatives and they converge to the same limit when the numbers ε_i tend to zero, and if we suppose the existence of this limit. As we showed in apendice (B.2), the limit (B.3.1) exists and it is equal to $\hat{A}^n\hat{B}^n P_{a_0}(x,E)$. So, the limit of (B.3.2) exists and it is equal to $\hat{B}^n\hat{A}^n P_{a_0}(x,E)$.

The equality of $\hat{A}^n\hat{B}^n P_{a_0}(x,E)$ with $(AB)^n P_{a_0}(x,E)$ is easily verified because the order of integration in (B.3.1) is irrelevant.

B-4) Convergence of the succession $\hat{B}\hat{A}^n\hat{B}^n P_{a_0}(x,E)$

This series is absolute and uniformely convergent, as

$$\hat{B}\hat{A}^n\hat{B}^n P_{a_0}(x,E) = \frac{1}{\lambda_N \lambda_a^n} \sum_{\nu=0}^{n} A_{n+1,\nu}(x) b_{n,\nu}(E) \qquad \text{(B.4.1)}$$

The right side of the expression (B.4.1) is smaller than $\dfrac{MC_1 C_2^n X^{n+1} e^{\beta_N X}}{\lambda_N \lambda_a^n (n+1)!}$

The uniform convergence of the exponential in T assures the absolute and uniform convergence of the succession $y_{a_n}(x,E) = \hat{B}\hat{A}^n\hat{B}^n P_{a_0}(x,E)$.

Then the sum $\sum_{n=0}^{\infty} y_{a_n}(x,E)$ is uniformely convergent to a function $y_a(x,E)$ in T.

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