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ζ - FUNCTION REGULARIZATION OF CHIRAL JACOBIANS FOR SINGULAR DIRAC OPERATORS: FINITE CHIRAL ROTATIONS

by

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ABSTRACT

The Jacobian for finite chiral rotations which preserve the topology is computed using ζ-function regularization when the Dirac operator of the theory is singular. The full generating functional, including fermionic sources, is used in the definition of the Jacobian which is shown to be algebraically identical to the Jacobian which would be obtained if the Dirac operator was non-singular.

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The investigation of chiral gauge theories in their non-trivial topological sector has been a source for some new interesting results concerning their dynamics and consistency. We could quote the works of Bardakci and Crescimanno [1] and Manias, Naón and Trobo [2,3], which found modifications on correlation functions when compared with the case of trivial topology. Banerjee et al [4] investigated the consistency of a gauge theory of SU(2) Weyl fermions, taking into account a modification in the anomaly when computed in the non-trivial sector. It would also be interesting to study anomaly cancelation [5,6,7] in the presence of zero modes. The common point to some of these problems is the need of calculating the Jacobian for finite chiral rotations in order to proceed with bosonization [2,3] or to calculate Wess-Zumino terms [6,7] necessary to ensure gauge invariance.

In a previous paper [8], we computed the Jacobian for infinitesimal chiral rotations using ζ-function regularization [9]. We showed that it was the same as the one computed in the trivial sector, at least to first order in the infinitesimal chiral parameter. In obtaining this result, it was crucial to take into account the residual coupling between the fermionic sources and the zero modes of the singular Dirac operator, and that the orthogonality of its zero eigenfunctions is not preserved by the chiral rotation. As we were forced to choose one definite way to orthonormalize the set of new zero modes, it was also not obvious if our result would be invariant with respect to the way of doing this.

In this letter we prove explicitly the invariance mentioned above, giving a generalization of the procedure followed in [8]. Besides, we prove the algebraic coincidence of the Jacobians calculated in the trivial and non-trivial topological sectors for finite chiral rotations which preserve the topology. We finally remark the equivalence between this result and the one obtained by adding a small mass to the fermions in the original Dirac operator.

Let us briefly review some basic facts stated in [8] concerning the computation of the Jacobian in the presence of zero modes. We start from the generating functional

$$Z[\bar{\eta},\eta] = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp\Big(- < \bar{\psi}\, D\, \psi > + < \bar{\eta}\, \psi > + < \bar{\psi}\, \eta >\Big) \;, \qquad (1)$$

where < > denotes integration over a d-dimensional Euclidean space and the fermions transform under a given representation of SU(N). If we change the integration variables as

$$\psi(x) \longrightarrow e^{\alpha(x)\gamma_5}\, \psi(x) \equiv \Omega_5(x)\psi(x) \quad , \qquad (2a)$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}(x) e^{\alpha(x)\gamma_5} \equiv \bar{\psi}(x)\Omega_5(x) \quad , \qquad (2b)$$

where $\alpha(x)= \alpha^a\, T_a$, then Z must remain unchanged ,

$$Z[\bar{\eta}, \eta] = Z[\bar{\eta}, \eta; \alpha] =$$

$$= J(\alpha) \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \exp\left(- < \bar{\psi} \, D(\alpha)\psi > + < \bar{\eta}(\alpha)\psi > + < \bar{\psi} \, \eta(\alpha)>\right) . \tag{3}$$

In (3) we used the definitions

$$D(\alpha) \equiv e^{\alpha(x)\gamma_5} \, D \, e^{\alpha(x)\gamma_5} , \tag{4a}$$

and

$$\bar{\eta}(x;\alpha) \equiv \bar{\eta}(x) e^{\alpha(x)\gamma_5} , \quad \eta(x;\alpha) \equiv e^{\alpha(x)\gamma_5} \eta(x) . \tag{4b}$$

If D would be invertible, we could construct

$$S(x,y) = \sum_n \frac{\phi_n(x) \, \phi_n^+(y)}{\lambda_n} \tag{5}$$

where

$$D\phi_n = \lambda_n \phi_n \tag{6a}$$

and

$$\sum_n \phi_n(x) \, \phi_n^+(y) = \delta(x-y)1 , \quad \int \phi_n^+(x) \, \phi_m(x) = \delta_{n,m} \tag{6b}$$

such that

$$D_x \, S(x,y) = S(x,y) \, D_y = \delta(x-y)1 . \tag{7}$$

Then, performing in (1) the shift

$$\psi(x) \longrightarrow \psi(x) + \int S(x,y) \, \eta(y) \, dy , \tag{8a}$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}(x) + \int \bar{\eta}(y) \, S(y,x) \, dy , \tag{8b}$$

we would obtain

$$Z[\bar{\eta},\eta] = e^{<\bar{\eta}S\eta>} \,\mathrm{Det}\, D\,, \tag{9}$$

where $<\bar{\eta}S\eta> = \int dxdy\; \bar{\eta}(y)S(y,x)\eta(x)$.

To calculate the r.h.s of (3) in a similar way, we shift variables in a similar fashion [8], using

$$S(x,y;\alpha)=e^{-\alpha(x)\gamma_5}S(x,y)e^{-\alpha(y)\gamma_5}, \tag{10}$$

instead of S(x,y) to get

$$Z[\bar{\eta},\eta,\alpha] = J(\alpha)\; e^{<\bar{\eta}S\eta>} \,\mathrm{Det}\, D(\alpha)\;. \tag{11}$$

The Jacobian would be given by

$$J(\alpha) = \frac{\mathrm{Det}\, D}{\mathrm{Det}\, D(\alpha)}\;. \tag{12}$$

However, if we are in the non-trivial topological sector we have to be more careful [8]. If the operator D has N zero eigenvalues, any meaningful definition of Det D must vanish, which turns (12) ill-defined. If we go further back we remember that in this case the inverse of S(x,y) does not exist because some of its terms are divided by zero (the ones corresponding to the zero eingenvalues). Nevertheless, we can still define

$$S(x,y) = \sum_{n\neq 0} \frac{\phi_n(x)\;\phi_n^+(y)}{\lambda_n}\;, \tag{13}$$

which satisfies

$$D_x\, S(x,y) = 1\, \delta(x-y) - \sum_{i=1}^{N} \phi_{oi}(x)\, \phi^{+}_{oi}(y)$$

$$= 1\, \delta(x-y) - P_o(x,y)$$

$$= S(x,y)\, D_y \quad , \tag{14}$$

where $P_o(x,y)$ is the projection operator on the sub-space generated by the zero eigenvectors of D.

Using this S(x,y) to perform the translation (8) in (1) we obtain [10]

$$Z\,[\bar{\eta},\eta] = e^{<\bar{\eta}S\eta>}\ \mathrm{Det}'D \prod_{i=1}^{N} \langle\, \phi^{+}_{oi}\ \eta\, \rangle \langle\, \bar{\eta}\ \phi_{oi} \rangle \quad , \tag{15}$$

where Det'D stands for a given regularization of the product of non-null eigenvalues of D [9]. To compute the generating functional after the change of variables (2), we translate the fermions using

$$S(x,y;\alpha) = \sum_{n\neq 0} \frac{\phi_n(x;\alpha)\ \phi^{+}_n(y;\alpha)}{\lambda_n(\alpha)} \quad , \tag{16}$$

where $\phi_n(x;\alpha)$ satisfy

$$D(\alpha)\ \phi_n(x;\alpha) = \lambda_n(\alpha)\ \phi_n(x;\alpha) \tag{17}$$

and we are assuming that the basis $\{\phi_n(x;\alpha)\}$ is also orthonormal.

The analogue of (14) is now

$$D_x(\alpha)\ S(x,y;\alpha) = S(x,y,;\alpha)\ D_y(\alpha) =$$

$$= 1\ \delta(x-y) - \sum_{i=1}^{N} \phi_{0i}(x;\alpha)\ \phi_{0i}^{+}(y;\alpha)$$

$$\equiv 1\ \delta(x-y) - P_0(x,y;\alpha)\ , \tag{18}$$

and $P_0(x,y;\alpha)$ is the projection operator on the subspace spanned by the zero eigenvectors of the rotated Dirac operator.

We must remark that, if $\phi_{0i}(x)$ is a zero eigenvector of D, $\psi_{0i} = e^{-\alpha(x)\gamma_5}\ \phi_{0i}$ is a zero eigenvector of $D(\alpha)$. However, this set of zero modes is not orthonormal,

$$\langle\ \psi_{0i}^{+}\ \psi_{0j}\ \rangle = \langle\ \phi_{0i}^{+}\ e^{-2\alpha\gamma_5}\ \phi_{0j}\ \rangle \neq \delta_{ij}\ . \tag{19}$$

We use an orthonormal set given by linear combinations of elements of the previous one,

$$\phi_{0i}(x;\alpha) = \sum_{j=1}^{N} B_{ij}[\alpha]\ e^{-\alpha(x)\gamma_5}\ \phi_{0j}(x) \tag{20}$$

After the translation on $Z\ [\bar{\eta},\ \eta;\ \alpha]$, we get

$$Z\ [\bar{\eta},\ \eta;\ \alpha] = J(\alpha)\ e^{\langle\bar{\eta}(\alpha)S(\alpha)\eta(\alpha)\rangle}\ \mathrm{Det}'D(\alpha) \prod_{i=1}^{N} \langle\phi_{0i}^{+}(\alpha)\ \eta(\alpha)\rangle\ .$$

$$\langle\ \bar{\eta}(\alpha)\phi_{0i}(\alpha)\rangle\ . \tag{21}$$

It must be noticed that the method employed to obtain (21) is slightly different from the one used in [8]. There, we used a

$S(x, y; \alpha)$ given by

$$S(x, y; \alpha) = e^{-\alpha(x)\gamma_5} \sum_{n\neq 0} \frac{\phi_n(x)\ \phi_n^+(y)}{\lambda_n}\ e^{-\alpha(y)\gamma_5}\ , \tag{22}$$

and $\phi_n(x)$ are eigenfunctions of D.

The difference is in the exponential term. In [8] it is

$$\exp\left(\langle\ \bar{\eta}\ S\ \eta\ \rangle + \sum_j \frac{\bar{\xi}_j(\alpha)\ \xi_j(\alpha)}{\lambda_j(\alpha)}\right)\ , \tag{23}$$

with ξ_j, $\bar{\xi}_j$ given by

$$\xi_j(\alpha) = \sum_{k=1}^{N}\ \langle\ \phi_{0k}^+\ \eta\ \rangle \langle\ \phi_j^+(\alpha)\ e^{\alpha\gamma_5}\ \phi_{0k}\ \rangle\ , \tag{24a}$$

$$\bar{\xi}_j(\alpha) = \sum_{k=1}^{N}\ \langle\ \bar{\eta}\ \phi_{0k}\ \rangle \langle\ \phi_{0k}^+\ e^{\alpha\gamma_5}\ \phi_j(\alpha)\rangle\ . \tag{24b}$$

For an infinitesimal chiral rotation, the second term in the exponential is of order α^2, and it can be neglected if we keep only the terms up to first order in α. However, it must be further analysed if we want to consider a finite α rotation.

Let us call

$$S(x,\ y;\ \alpha) = \sum_{n\neq 0} \frac{\phi_n(x;\ \alpha)\ \phi_n^+(y;\ \alpha)}{\lambda_n(\alpha)} \equiv \langle\, x|S(\alpha)|y\,\rangle\ , \tag{25}$$

and

$$S'(x,\ y;\ \alpha) = e^{-\alpha(x)\gamma_5} \sum_{n\neq 0} \frac{\phi_n(x)\ \phi_n^+(y)}{\lambda_n}\ e^{-\alpha(y)\gamma_5} \equiv \langle\, x|S'(\alpha)|y\,\rangle . \tag{26}$$

So, the following relations are satisfied:

$$D(\alpha)\ S(\alpha) = S(\alpha)\ D(\alpha) = 1 - P_O(\alpha)\ , \tag{27a}$$

$$D(\alpha)\ S'(\alpha) = 1 - P_R(\alpha)\ , \tag{27b}$$

$$S'(\alpha)\ D(\alpha) = 1 - P_L(\alpha)\ , \tag{27c}$$

where

$$\langle\, x|P_O(\alpha)|\ y\,\rangle = \sum_{i=1}^{N} \phi_{0i}(x;\ \alpha)\ \phi_{0i}^+\ (y;\ \alpha)\quad , \tag{28a}$$

$$\langle\, x|P_R(\alpha)|\ y\,\rangle = e^{\alpha(x)\gamma_5} \sum_{i=1}^{N} \phi_{0i}(x)\ \phi_{0i}^+(y)\ e^{-\alpha(y)\gamma_5}\ , \tag{28b}$$

$$\langle\, x|P_L(\alpha)|\ y\,\rangle = e^{-\alpha(x)\gamma_5} \sum_{i=1}^{N} \phi_{0i}(x)\ \phi_{0i}^+(y)\ e^{\alpha(y)\gamma_5}\ . \tag{28c}$$

Thus, $S(\alpha)$ obeys the consistency equations

$$S(\alpha) = S'(\alpha) - S'(\alpha)P_O(\alpha) + P_L(\alpha)\ S(\alpha)\quad , \tag{29a}$$

and

$$S(\alpha) = S'(\alpha) - P_O(\alpha)\ S'(\alpha) + S(\alpha)P_R(\alpha)\quad , \tag{29b}$$

which follow from the application on the right and on the left of

$S'(\alpha)$ to (27a). Then,

$$\exp < \bar{\eta}(\alpha)\ S(\alpha)\ \eta(\alpha) > = \exp < \bar{\eta}(\alpha)\ S'(\alpha)\ \eta(\alpha)> .$$

$$. \exp - < \bar{\eta}(\alpha)\ S'(\alpha)\ P_O(\alpha)\ \eta(\alpha)>. \exp< \bar{\eta}(\alpha)\ P_L(\alpha)\ S(\alpha)\ \eta(\alpha)> =$$

$$= \exp< \bar{\eta}\ S\ \eta > . \exp \left\{ - \sum_{i=1}^{N} < \bar{\eta}(\alpha)\ S'(\alpha)\ \phi_{0i}(\alpha)> < \phi_{0i}^{+}(\alpha)\ \eta(\alpha)> \right\} .$$

$$. \exp\left\{ \sum_{i=1}^{N} B_{ij}^{-1}[\alpha] < \bar{\eta}(\alpha)\ \phi_{0j}(\alpha)> < \phi_{0i}^{+}\ e^{\alpha\gamma_5}\ S(\alpha)\ \eta(\alpha)> \right\}$$

$$= \exp < \bar{\eta}\ S\ \eta > . \exp\left(\sum_{j=1}^{N} M_j < \phi_{0j}^{+}(\alpha)\ \eta(\alpha)> + < \bar{\eta}(\alpha)\ \phi_{0j}(\alpha)> N_j \right). \tag{30}$$

Now we remember that $< \phi_{0j}^{+}(\alpha)\eta(\alpha)>$ and $< \bar{\eta}(\alpha)\phi_{0j}(\alpha)>$ are Grassmann numbers and that the exponential (30) in the expression of $Z[\bar{\eta}, \eta; \alpha]$ is multiplied by the product

$$\prod_{j=1}^{N} < \phi_{0j}^{+}(\alpha)\ \eta(\alpha)> < \bar{\eta}(\alpha)\phi_{0j}(\alpha)> \quad . \tag{31}$$

Therefore, the only contribution from (30) to (21) is $\exp <\bar{\eta}\ S\ \eta>$, and $Z[\bar{\eta}, \eta; \alpha]$ becomes

$$Z[\bar{\eta}, \eta; \alpha] = J(\alpha)\ e^{<\bar{\eta}\ S\ \eta>}\ \mathrm{Det}'D(\alpha) \prod_{i=1}^{N} < \phi_{0i}^{+}(\alpha)\ \eta(\alpha)> < \bar{\eta}(\alpha)\phi_{0i}(\alpha)>. \tag{32}$$

In reference [8], if we identify

$$\sum_j \frac{\bar{\xi}_j(\alpha)\,\xi_j(\alpha)}{\lambda_j(\alpha)} = \langle\, \bar{\eta}(\alpha)\; P_L(\alpha)\; S(\alpha)\; P_R(\alpha)\; \eta(\alpha)\rangle\;, \qquad (33)$$

we see that a similar reasoning conducts us to the same result.

Before we continue, we must consider the problem of performing arbitrary chiral rotations in a non-trivial topological sector. These arbitrary transformations are ill- defined in general at some points of the (compactified) base space, introducing difficulties in the compactification procedure [1,3]. We shall avoid these difficulties performing chiral rotations parametrized by an $\alpha(x)$ which obeys trivial boundary conditions so that the base manifold can be compactified in the same way which is done for the infinitesimal case. This prevents changes in the topology of the fiber bundle in which sections D is supposed to act and enables us to apply the methods stated in references [9] and [11]. We will have no great loss of generality, as we shall discuss in the conclusions.

We can now proceed with the computation of the finite Jacobian. We use the method stated in [11] where one performs a finite rotation parametrized by r, $0 \le r \le 1$

$$\psi \longrightarrow e^{r\alpha(x)\gamma_5}\,\psi \quad , \qquad \bar{\psi} \longrightarrow \bar{\psi}\, e^{r\alpha(x)\gamma_5} \quad , \quad \alpha(x) \text{ finite.} \qquad (34)$$

The generating functional becomes

$$Z[\bar{\eta}, \eta; r]=J(r\alpha)\ \exp< \bar{\eta}\ S\ \eta >\ \mathrm{Det}'D\ (r\alpha) \prod_{i=1}^{N} < \phi_{0i}^{+}\ \eta > < \bar{\eta}\phi_{0i}> N(r\alpha), \tag{35}$$

and we should stress that ϕ_{0i} are the zero eigenfunctions of D, and η and $\bar{\eta}$ are the unrotated external currents. The factor $N(r\alpha)$ comes from the Grassmanian nature of the sources and will be computed later.

Imposing the invariance of Z with respect to the change of variables (34), we get

$$\frac{dz}{dr} = 0 = \frac{d}{dr}\ (\ell nJ(r\alpha)) + \frac{d}{dr}\ (\ell n\ \mathrm{Det}'D(r\alpha)) + \frac{d}{dr}\ (\ell n\ N(r\alpha)), \tag{36}$$

from where we read the finite Jacobian,

$$J(\alpha) = \exp\left[- \int_0^1 dr\ \omega'(r\alpha)\right]\ (N(\alpha))^{-1} \quad , \tag{37}$$

with

$$\omega'(r\alpha) = \frac{d}{dr}\ \ell n\ \mathrm{Det}'D(r\alpha).$$

It can be shown that $\omega'(r\alpha)$ can be written as

$$\omega'(r) = \lim_{\epsilon \longrightarrow 0} \mathrm{Tr}\left[(D_\epsilon(r\alpha))^{-s-1}\ A_1(r\alpha)\right] \Bigg|_{s=0} \quad , \tag{38}$$

where

$$D_\epsilon(r\alpha) = e^{r\alpha(x)\gamma_5} (D + \epsilon\, 1)\, e^{r\alpha(x)\gamma_5} , \qquad (39.a)$$

and

$$A_1(r\alpha) = \{\alpha\gamma_5, D(r\alpha)\} = \{\alpha\gamma_5, D_\epsilon(r\alpha)\} - 2\epsilon\alpha\gamma_5 . \qquad (39.b)$$

Evaluating the trace in (38), as in [9], we come to

$$\omega'(r) = \lim_{\epsilon\to 0} \left(2 \int dx \; tr \left[K_0(D_\epsilon(r\alpha); x, x)\; \gamma_5\alpha(x)\right]\right) - 2 \sum_i tr\left[\int dx\; \phi^+_{0i}(x; r\alpha)\; \gamma_5\alpha(x)\; \phi_{0i}(x; r\alpha)\right], \qquad (40)$$

where $\phi_{0i}(x; r\alpha)$ belongs to the orthonormal set of zero modes of the operator $D(r\alpha)$. The first term in (40) gives the usual result [11] in the absence of zero modes, K_0 is the analitical extension at s=0 of the Kernel $K_{-s}(D_\epsilon(r\alpha); x,x))$, and can be computed using Seeley's coefficients [12]. Thus, if we call J_T, the algebraic expression of the Jacobian in the trivial sector, we can say that

$$J_{NT}(\alpha) = J_T(\alpha)\, \exp\left[\int_0^1 dr\; tr\left(\sum_i \int dx\; \phi^+_{0i}(x; r\alpha)\; \alpha(x)\gamma_5\phi_{0i}(x; r\alpha)\right].\right. \cdot \left[N(\alpha)\right]^{-1} \qquad (41)$$

Let us consider first the term $(N(\alpha))^{-1}$. It comes from (21),

$$\prod_{i=1}^{N} < \phi^{+}_{0i}(\alpha)\ \eta(\alpha)> < \bar{\eta}(\alpha)\ \phi_{0i}(\alpha)> =$$

$$= \prod_{i=1}^{N} \left(\sum_{j=1}^{N} B^{*}_{ij}(\alpha) < \phi^{+}_{0i}\ \eta > \right) \left(\sum_{k=1}^{N} B_{ik}(\alpha) < \bar{\eta}\ \phi_{0i}> \right) =$$

$$= |\mathrm{Det}\ \mathbb{B}|^{2} \prod_{i=1}^{N} < \phi^{+}_{0i}\ \eta > < \bar{\eta}\ \phi_{0i}> = N(\alpha) \prod_{i=1}^{N} < \phi^{+}_{0i}\ \eta > < \eta\ \phi_{0i}> \tag{42}$$

as is proved in the Appendix. If we consider the orthonormality condition of $\phi_{0i}(\alpha)$, we have

$$< \phi^{+}_{0i}(\alpha)\ \phi_{0j}(\alpha)> = \delta_{ij} = \sum_{k,l} B^{*}_{ik}(\alpha)\ < \phi^{+}_{0k}\ e^{-2\alpha\gamma_5}\ \phi_{0l} > B_{jl}\ , \tag{43}$$

we see that, in matrix notation, it can be expressed as

$$\mathbb{B}^{*}\mathbb{C}\ \mathbb{B}^{t} = 1 = \mathbb{B}\ \mathbb{C}^{t}\ \mathbb{B}^{+}\ , \tag{44}$$

with

$$C_{kl} = < \phi^{+}_{0k}\ e^{-2\alpha\gamma_5}\ \phi_{0l} > .$$

For any choice of the B_{ij}'s ,

$$|\ \mathrm{Det}\ \mathbb{B}\ |^{2} = \frac{1}{\mathrm{Det}\ \mathbb{C}} = N(\alpha)\ . \tag{45}$$

Keeping (44) in mind, it is straightforward to evaluate the second term in (41), that is

$$\exp\left[2\int_0^1 dr \quad tr \quad \left(\sum_i \int dx\ \phi_{0i}^+(x;\ r\alpha)\ \alpha(x)\gamma_5\ \phi_{0i}(x;\ r\alpha)\right)\right] =$$

$$= \exp\left[\int_0^1 dr \quad tr \left(\sum_{i,k,l} B_{ik}^* < \phi_{0k}^+ 2\alpha\gamma_5\ e^{-2r\alpha\gamma_5}\ \phi_{0l} > B_{il}\right)\right] =$$

$$= \exp\left[-\int_0^1 dr \quad tr \quad \left(B\ \frac{d}{dr}\ (C^t)\ B^+\right)\right]. \tag{46}$$

From (44) we obtain,

$$C^t = (B^+\ B)^{-1} \quad , \tag{47}$$

so that

$$\exp\left[-\int_0^1 dr\ tr\ \left(B\ \frac{d}{dr}\ (C^t)\ B^+\right)\right] =$$

$$= \exp\left[-\int_0^1 dr\ \left(\frac{d}{dr}\ tr\ \ell n\ C^t\right)\right] = Det(C^t)^{-1} = \frac{1}{Det\ C} . \tag{48}$$

Combining (45) with (48) we obtain

$$J_{NT}(\alpha) = J_T(\alpha) . \tag{49}$$

where $\alpha(x)$ is now a finite parameter of the chiral local rotation, obeying trivial boundary conditions.

Of course, the equivalence is only algebraic, because the fields involved in J_{NT} obey different boundary conditions than that of the trivial sector. The index theorem [13]

$$\int dx\ tr\ \left[K_o(D;\ x,x)\gamma_5\right] = n_+ - n_- \tag{50}$$

gives us immediately the Jacobian associated to global (α = constant) chiral rotations, from (49)

$$J_{global} = \exp\ (-2\alpha(n_+ - n_-)). \tag{51}$$

Our result for finite chiral rotations agrees with that obtained by the addition of a small mass to the Dirac operator [8]. In this case, the sources still couple to the zero modes, but only in the limit of vanishing mass. Before taking this limit, the operators D and D_α are non singular and the Jacobian is

$$J(\alpha) = \lim_{m\to 0} \frac{\mathrm{Det}(D + m\ 1)}{\mathrm{Det}(\Omega_5(D + m\ 1)\Omega_5}\quad , \tag{52}$$

which is clearly different from [9],

$$J'(\alpha) = \lim_{m\to 0} \frac{\mathrm{Det}\ (D + m1)}{\mathrm{Det}(\Omega_5 D\ \Omega_5 + m1)}\ . \tag{53}$$

Since there is no singularity in (52) as m is non-zero, it gives exactly the same result that we have obtained.

As we have previously stated, our result is general enough to handle with many interesting cases as, for example, bosonization and anomaly cancelation. One can follow the method used in [1], [2] and [3] where is always possible to decompose

$$A_\mu^{(N)} = A_\mu^{(c)} + a_\mu, \tag{54}$$

where $A_\mu^{(N)}$ is an arbitrary point in the space of gauge connections with topological charge N, $A_\mu^{(c)}$ is a fixed point in this space and a_μ obeys trivial boundary conditions. One is left, at the end, with the simpler problem of calculating determinants in the presence of fixed external fields $A_\mu^{(c)}$.

Finally, we would like to stress the great importance of working carefully with the sources when we are in non-trivial topological sectors. This observation enables our work to make contact with other works [14,15] were the sources are equally important in the definition of the theory.

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APPENDIX

We will now prove that $N(\alpha) = |\mathrm{Det}\ B(\alpha)|^2$. Let us first define

$$\eta_i = \langle \phi^+_{oi}\ \eta \rangle \tag{A1}$$

and

$$\bar\eta_i = \langle \bar\eta\ \phi_{oi} \rangle\ . \tag{A2}$$

Now we will evaluate $\prod_{i=1}^{N} \langle \phi^+_{0i}(\alpha)\ \eta(\alpha) \rangle \langle \bar\eta(\alpha)\ \phi_{0i}(\alpha) \rangle$ for N=2:

$$F_2 \equiv \prod_{i=1}^{N} \langle \phi^+_{0i}(\alpha)\ \eta(\alpha) \rangle \langle \bar\eta(\alpha)\ \phi_{0i}(\alpha) \rangle = \prod_{i=1}^{2} \left(\sum_{j=1}^{2} B^*_{ij}\ \eta_j \right) \cdot$$

$$\left(\sum_{j=1}^{2} B_{ij} \bar\eta_j \right) = - \prod_{i=1}^{2} \left(\sum_{j=1}^{2} B^*_{ij}\ \eta_j \right) \prod_{i=1}^{2} \left(\sum_{j=1}^{2} B_{ij}\ \bar\eta_j \right) . \tag{A3}$$

We can calculate separately each product to obtain

$$F_2 = - (B^*_{11}B^*_{11} - B^*_{12}B^*_{21}) \eta_1\eta_2 (B_{11}B_{22} - B_{12}B_{21}) \bar\eta_1\bar\eta_2 =$$

$$= |\mathrm{Det}\ B|^2 \eta_1\bar\eta_1\eta_2\bar\eta_2\ . \tag{A4}$$

We take as hypothesis that the result is valid for N-1 and prove its validity for N. So,

$$F_N = \prod_{i=1}^{N} < \phi^{+}_{0i}(\alpha)\ \eta(\alpha)\ > < \bar{\eta}(\alpha)\ \phi_{0i}(\alpha) > = (-1)^P \prod_{i=1}^{N} \left(\sum_{j=1}^{N} B_{ij}^{*} \eta_j \right) \cdot$$

$$\prod_{i=1}^{N} \left(\sum_{j=1}^{N} B_{ij}\ \bar{\eta}_j \right) , \qquad \text{(A5)}$$

with P being an even or odd number, according to the number of permutations of the η's necessary to bring F_N to the form (A5). We consider now one of the products

$$\prod_{i=1}^{N} \left(\sum_{j=1}^{N} B_{ij}\ \bar{\eta}_j \right) = \left(\sum_{j=1}^{N} B_{1j}\ \bar{\eta}_j \right) \prod_{i=2}^{N} \left(\sum_{j=1}^{N} B_{ij}\ \bar{\eta}_j \right) =$$

$$= B_{11}\bar{\eta}_1 \prod_{i=2}^{N} \left(\sum_{j=2}^{N} B_{ij}\ \bar{\eta}_j \right) + B_{12}\ \bar{\eta}_2 \prod_{i=2}^{N} \left(\sum_{\substack{j=1 \\ j\neq 2}}^{N} B_{ij}\ \bar{\eta}_j \right) + \ldots +$$

$$+ B_{1N}\ \bar{\eta}_N \prod_{i=2}^{N} \left(\sum_{j=1}^{N-1} B_{ij}\ \bar{\eta}_j \right). \qquad \text{(A6)}$$

Applying the hypothesis made for N-1 to each of the terms of the sum (A6), we get

$$\prod_{i=1}^{N} \left(\sum_{j=1}^{N} B_{ij} \bar{\eta}_j \right) = B_{11}\bar{\eta}_1 \ \mathrm{Det} \begin{vmatrix} B_{22} & B_{23} \ldots\ldots & B_{2N} \\ \vdots & & \\ B_{N2} & B_{N3} \ldots\ldots & B_{NN} \end{vmatrix} \bar{\eta}_2 \ldots\ldots \bar{\eta}_N \ +$$

$$+ B_{12} \ \bar{\eta}_2 \ \mathrm{Det} \begin{vmatrix} B_{21} & B_{23} \ldots\ldots & B_{2N} \\ \vdots & & \\ B_{N1} & B_{N3} \ldots\ldots & B_{NN} \end{vmatrix} \bar{\eta}_1 \bar{\eta}_3 \ldots \bar{\eta}_N \ + \ \ldots +$$

$$+ B_{1N}\bar{\eta}_N \ \mathrm{Det} \begin{vmatrix} B_{21} & B_{22} \ldots\ldots & B_{2,N-1} \\ \vdots & & \\ B_{N1} & B_{N2} \ldots\ldots & B_{N,N-1} \end{vmatrix} \bar{\eta}_1 \bar{\eta}_2 \ldots \bar{\eta}_{N-1} = \mathrm{Det}\ \mathbb{B}\ \bar{\eta}_1 \bar{\eta}_2 \ldots \bar{\eta}_N . \tag{A7}$$

Because the same algebra is valid for the other product in (A5),

$$F_N = (-1)^P \ |\mathrm{Det}\ \mathbb{B}|^2 \ \eta_1 \eta_2 \ldots \eta_N \bar{\eta}_1 \bar{\eta}_2 \ldots \bar{\eta}_N = \tag{A8}$$

$$= |\ \mathrm{Det}\ \mathbb{B}|^2 \ \eta_1 \bar{\eta}_1 \ldots \eta_N \bar{\eta}_N \ . \tag{A9}$$

The permutation of the η's and $\bar{\eta}$'s from (A8) to (A9) is exactly the opposite to that performed in (A5), giving us then the same factor $(-1)^P$.

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