ISSN 0029 - 3865

CBPF-NF-043/89

GAUGE TRANSFORMATIONS FOR SIX DIMENSIONAL SUPERFIELDS

by

C.G. BOLLINI[1]* and J.J. GIAMBIAGI[1+]

[1]Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*Departamento de Física
Facultad de Ciencias Exactas y Naturales-Universidad Nacional de La Plata
La Plata, Argentina and Comisión de Investigaciones Cientificas
de La Prov. de Buenos Aires, Argentina

+Centro Latinoamericano de Física (CLAF)
Av. Wenceslau Braz 71 (fundos)
22290 - Rio de Janeiro, RJ - Brasil

August, 1989

ABSTRACT

We give for D = 6 the general gauge transformations that keep the superfield within the Wess-Zumino gauge (any component with less than two indices of each type $\alpha,\dot{\alpha}$ is absent). We built the gauge invariant components and write down all the partial lagrangians. Finally we briefly discuss a dimensional reduction to D = 4.

Key-words: Field theory; Supersymmetry; Gauge transformations.

1§ INTRODUCTION

This work is complementary and a continuation of a previous one[1] in which we discussed the gauge superfield in six dimensions.

The choice D = 6 has for us two main reasons, namely:

In the first place, in that number of dimensions, the equations of motion resulting from the natural application of supersymmetry are of fourth order[2] for the lowest component of the gauge superfield and this is the simplest example of a higher order equation resulting from supersymmetry. In second place, among the physical components of the gauge superfields we noted the presence of several interesting fields, which we called[1]: graviton, gravitino, photon, photino and also a complex vector field and a real three vector, with the additional property that we have them all unified in a single superfield.

D = 6 has not only the appeal of the possibility of extended supersymmetric theory[6] and a realistic supersymmetric GUT[4], but for us it provides the simplest example of higher order equations of motion for physical fields[2]. In this sense, we noted previously that in higher order equations the potentials, i.e., the couplings of the different orders of derivatives should be related so as to obtain equations with physical significance[5]. We think that perhaps supersymmetry is the only relativistic symmetry that can relate the couplings in such a way that these conditions are fulfilled.

With that motivating ideas in mind we are developping the theory with the hope that it can provide us with a guidance to get a treatment fit for higher order equations. In particular one hopes that by coming down to fourth dimensions with "Kaluza Klein procedures" one can obtain here fourth order differential equations which have physical content.

With regard to this last point it is worth taking into account that by using the $D \to \infty$ method in a higher order invariant equation, one obtains a second order equation as an approximation to the exact wave equation of the theory. So in this sense an invariant higher order equation has a second order equation as an approximation. (A note on this point will be published elsewhere).

§2 NOTATIONS AND DEFINITIONS

For the sake of clarity we repeat here some definitions we have used in reference [1] which are based on Elie Cartan's book[7].

Dirac matrices in d = 6 are defined by

$$\Gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \tilde{\gamma}_\mu & 0 \end{pmatrix} ; \quad \{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu} \tag{2.1}$$

where $\gamma_\mu^+ = \gamma_\mu = \tilde{\gamma}_\mu$ for $\mu = 1,\ldots,5$ are five hermitian four-dimensional Dirac matrices and $\gamma_o = -\tilde{\gamma}_o = \mathbb{1}$.

The transposition matrix is

$$\mathbb{C} = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix} , \quad C = \gamma_2\gamma_5 \quad C^2 = -1$$

$$C\gamma^\mu = \gamma^{\mu T} C \qquad C\tilde{\gamma}^\mu = \tilde{\gamma}^{\mu T} C \tag{2.2}$$

The scalar product of two Weyl spinors of different types is defined by

$$\psi_\alpha C^{\alpha}{}_{\dot\alpha}\phi^{\dot\alpha} = \psi C\phi = \text{scalar} \tag{2.3}$$

The conjugate spinor is defined by

$$\phi^c = C\phi^* \qquad \bar\phi = \phi^+ C \tag{2.4}$$

Note that, while the transposition matrix $\mathbb{C}$ is

$\mathbb{C} = \Gamma_o\Gamma_2\Gamma_5 =$ the conjugation matrix (in 6D) is

$$\mathbb{C}' = \Gamma_2\Gamma_5 \rightarrow \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \tag{2.5}$$

In order to construct a chiral field with Weyl spinors of the first type we take the Grassmann variables as spinors of the second type $\theta_{\dot\alpha}$ and $\bar\theta_{\dot\alpha}$.

§3 GAUGE TRANSFORMATIONS

As we pointed out in ref. [1], the real superfield has the general form.

$$V = \sum_{s,t=0}^{4} \bar{\theta}_{\dot{\alpha}_1} \ldots \bar{\theta}^{*}_{\dot{\alpha}_s} \; A^{\dot{\alpha}_1 \ldots \dot{\alpha}_s}_{\alpha_1 \ldots \alpha_t} \; \theta^{\alpha_1} \ldots \theta^{\alpha_t} \tag{3.1}$$

$$\left(A^{\dot{\alpha}_1 \ldots \dot{\alpha}_s}_{\alpha_1 \ldots \alpha_t} \right)^{*} = A^{\dot{\alpha}_1 \ldots \dot{\alpha}_t}_{\alpha_1 \ldots \alpha_s}$$

The abelian gauge transformation is given by

$$V' = V + i(\bar{\psi} - \psi) \tag{3.2}$$

where

$$\bar{D}_{\dot{\alpha}_1} \bar{D}_{\dot{\alpha}_2} \psi = 0 \quad \text{and} \quad D_{\alpha_1} D_{\alpha_2} \bar{\psi} = 0 \tag{3.3}$$

$D_{\dot{\alpha}}, \bar{D}_{\alpha}$ are the usual covariant derivatives.

By means of the transformation 3.2 we can go to the Wess-Zumino gauge where the components of V with less than two indices of each kind are zero. In that gauge.

$$V = \sum_{s,t=2}^{4} \bar{\theta}_{\dot{\alpha}_1} \ldots \bar{\theta}_{\dot{\alpha}_s} \; A^{\dot{\alpha}_1 \ldots \dot{\alpha}_s}_{\alpha_1 \ldots \theta_t} \; \theta^{\alpha_1} \ldots \theta^{\alpha_t} \tag{3.4}$$

We can still remain in this gauge, as can be verified, by a transformation induced by the following general double chiral superfield

$$\psi = e^{i\theta\partial\bar{\theta}} \left[\lambda + \theta^{\alpha}\lambda_{\alpha} + \bar{\lambda}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}} + \theta^{\alpha}\bar{\theta}_{\dot{\alpha}}(\lambda^{\dot{\alpha}}_{\alpha} - i\partial^{\alpha}_{\dot{\alpha}}\lambda) + 2i\theta^{\alpha_1}\theta^{\alpha_2}\bar{\theta}_{\dot{\alpha}_1}\partial^{\dot{\alpha}_1}_{\dot{\alpha}_1}\lambda_{\alpha_2} \right] \tag{3.5}$$

where $\lambda;\ \lambda_\alpha, \lambda_\alpha^{\dot\alpha}$ are arbitrary up to the restrictions

$$\lambda = \lambda^* \qquad \lambda_\alpha^{\dot\alpha} = \left(\lambda_\alpha^{\dot\alpha}\right)^* \tag{3.6}$$

The corresponding gauge transformation is:

$$\begin{aligned} i(\bar\psi-\psi) = {} & \theta^{\alpha_1}\theta^{\alpha_2}\bar\theta_{\dot\alpha_1}\bar\theta_{\dot\alpha_2}\partial_{\alpha_1}^{\dot\alpha_1}\lambda_{\alpha_2}^{\dot\alpha_2} + i\theta^{\alpha_1}\theta^{\alpha_2}\bar\theta_{\dot\alpha_1}\bar\theta_{\dot\alpha_2}\bar\theta_{\dot\alpha_3}\partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\bar\lambda^{\dot\alpha_3} + \\ & + i\theta^{\alpha_1}\theta^{\alpha_2}\theta^{\alpha_3}\bar\theta_{\dot\alpha_1}\bar\theta_{\dot\alpha_2}\partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\lambda_{\alpha_3} + \theta^{\alpha_1}\theta^{\alpha_2}\theta^{\alpha_3}\bar\theta_{\dot\alpha_1}\bar\theta_{\dot\alpha_2}\bar\theta_{\dot\alpha_3}\partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\partial_{\alpha_3}^{\dot\alpha_3}\lambda \\ & + \frac{1}{3}\theta^{\alpha_1}\theta^{\alpha_2}\theta^{\alpha_3}\bar\theta_{\dot\alpha_1}\ldots\bar\theta_{\dot\alpha_4}\partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\partial_{\alpha_3}^{\dot\alpha_3}\bar\lambda^{\dot\alpha_4} + \frac{1}{3}\theta^{\alpha_1}\ldots\theta^{\alpha_4}\bar\theta_{\dot\alpha_1}\bar\theta_{\dot\alpha_2}\bar\theta_{\dot\alpha_3}\partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\partial_{\alpha_3}^{\dot\alpha_3}\lambda_{\alpha_4} \\ & + \frac{1}{3!}\theta^{\alpha_1}\ldots\theta^{\alpha_4}\bar\theta_{\dot\alpha_1}\ldots\bar\theta_{\dot\alpha_4}\partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\partial_{\alpha_3}^{\dot\alpha_3}\lambda_{\alpha_4}^{\dot\alpha_4} \end{aligned} \tag{3.7}$$

Using 3.2 and 3.4 it is easy to see that the components of V transform in the following way:

$$A_{\alpha_1\alpha_2}^{\dot\alpha_1\dot\alpha_2\,\prime} = A_{\alpha_1\alpha_2}^{\dot\alpha_1\dot\alpha_2} + \partial_{\alpha_1}^{\dot\alpha_1}\lambda_{\alpha_2}^{\dot\alpha_2} \tag{3.8}$$

$$A_{\alpha_1\alpha_2}^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\,\prime} = A_{\alpha_1\alpha_2}^{\dot\alpha_1\dot\alpha_2\dot\alpha_3} + i\partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\bar\lambda^{\dot\alpha_3};\ A_{\alpha_1\alpha_2\alpha_3}^{\dot\alpha_1\dot\alpha_2\,\prime} = A_{\alpha_1\alpha_2\alpha_3}^{\dot\alpha_1\dot\alpha_2} + i\partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\lambda_{\alpha_3} \tag{3.9}$$

$$A_{\alpha_1\alpha_2\alpha_3}^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\,\prime} = A_{\alpha_1\alpha_2\alpha_3}^{\dot\alpha_1\dot\alpha_2\dot\alpha_3} + \partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\partial_{\alpha_3}^{\dot\alpha_3}\lambda \tag{3.10}$$

$$A_{\alpha_1\alpha_2\alpha_3}^{\dot\alpha_1\ldots\dot\alpha_4\,\prime} = A_{\alpha_1\alpha_2\alpha_3}^{\dot\alpha_1\ldots\dot\alpha_4} + \frac{1}{3}\partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\partial_{\alpha_3}^{\dot\alpha_3}\bar\lambda^{\dot\alpha_4};\ A_{\alpha_1\ldots\alpha_4}^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\,\prime} = A_{\alpha_1\ldots\alpha_4}^{\dot\alpha_1\dot\alpha_2\dot\alpha_3} + \frac{1}{3}\partial_{\alpha_1}^{\dot\alpha_1}\partial_{\alpha_2}^{\dot\alpha_2}\partial_{\alpha_3}^{\dot\alpha_3}\lambda_{\alpha_4} \tag{3.11}$$

$$A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4 '}_{\alpha_1\alpha_2\alpha_3\alpha_4} = A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} + \frac{1}{3!}\, \partial^{\dot\alpha_1}_{\alpha_1}\partial^{\dot\alpha_2}_{\alpha_2}\partial^{\dot\alpha_3}_{\alpha_3}\lambda^{\dot\alpha_4}_{\alpha_4} \tag{3.12}$$

In the right hand members of these equalities it must be understood that the terms containing the gauge parameters are to be antisymmetrized for both types of indices α and $\dot\alpha$.

It is easy to see that $A^{\dot\alpha_1\ldots\dot\alpha_4}_{\alpha_1\alpha_2\alpha_3}$, $A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3}_{\alpha_1\alpha_2\alpha_3\alpha_4}$ and $A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}$ can be replaced (or redefined) by linear combinations which are gauge invariants

$$B^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}_{\alpha_1\alpha_2\alpha_3} = A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}_{\alpha_1\alpha_2\alpha_3} + \frac{2}{3}i\partial^{\dot\alpha_1}_{\alpha_1}A^{\dot\alpha_2\dot\alpha_3\dot\alpha_4}_{\alpha_2\alpha_3}$$

$$B^{\dot\alpha_1\dot\alpha_2\dot\alpha_3}_{\alpha_1\alpha_2\alpha_3\alpha_4} = A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3}_{\alpha_1\alpha_2\alpha_3\alpha_4} + \frac{2}{3}i\partial^{\dot\alpha_1}_{\alpha_1}A^{\dot\alpha_2\dot\alpha_3}_{\alpha_2\alpha_3\alpha_4} \tag{3.13}$$

$$D^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} = D\epsilon^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4} = A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} + \partial^{\dot\alpha_1}_{\alpha_1}\partial^{\dot\alpha_2}_{\alpha_2}A^{\dot\alpha_3\dot\alpha_4}_{\alpha_3\alpha_4} \tag{3.14}$$

where again the terms with derivatives must be antisymmetrized in both types of indices.

According to Cartan ref. [7] we can express the multispinor field components into its tensor components

$$A^{\alpha}_{\dot\alpha} = A_\mu\tilde\gamma^{\mu\alpha}_{\dot\alpha} + A_{\nu_1\nu_2\nu_3}\left(\tilde\gamma^{\nu_1}\gamma^{\nu_2}\tilde\gamma^{\nu_3}\right)^{\alpha}_{\dot\alpha} \tag{3.14}$$

$A_{\nu_1\nu_2\nu_3}$ is a completely antisymmetric self-dual tensor.

$$A_{\alpha_1\alpha_2}^{\dot{\alpha}_1\dot{\alpha}_2} = (\gamma^{\mu}C)_{\alpha_1\alpha_2}(C\gamma^{\nu})^{\dot{\alpha}_1\dot{\alpha}_2} A_{\mu\nu} \tag{3.15}$$

$$A_{\alpha_1\alpha_2}^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3} = \varepsilon^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4} \bar{A}_{\dot{\alpha}_4}^{\mu}(\gamma_{\mu}C)_{\alpha_1\alpha_2} \tag{3.16}$$

$$A_{\alpha_1\alpha_2}^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4} = \varepsilon^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4} B_{\mu}(\gamma^{\mu}C)_{\alpha_1\alpha_2} \tag{3.17}$$

$$A_{\alpha_1\alpha_2\alpha_3}^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4} = \varepsilon^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4} \varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4} A^{\alpha_4} \tag{3.18}$$

Also, in particular, for $\lambda_{\alpha}^{\dot{\alpha}}$, we have

$$\lambda_{\alpha}^{\dot{\alpha}} = \lambda_{\mu}\gamma^{\mu}{}_{\alpha}{}^{\dot{\alpha}} + \lambda_{\nu_1\nu_2\nu_3}\left(\gamma^{\nu_1}\tilde{\gamma}^{\nu_2}\gamma^{\nu_3}\right)_{\alpha}^{\dot{\alpha}} \tag{3.19}$$

Using these formula, together with 3.8 to 3.12 we find:

$$A'_{(\mu\nu)} = A_{(\mu\nu)} + \partial_{\mu}\lambda_{\nu} + \partial_{\nu}\lambda_{\mu} - \eta_{\mu\nu}\partial^{\alpha}\lambda_{\alpha} \tag{3.20}$$

$$A'_{[\mu\nu]} = A_{[\mu\nu]} + \partial^{\rho}\lambda_{\rho\mu\nu} \tag{3.21}$$

$$\partial^{\mu}A_{[\mu,\nu]} = \text{gauge invariant}$$

where $(\mu\nu)$ means symmetric part and $[\mu\nu]$, antisymmetric one.

$$A'^{\alpha}_{\mu} = A^{\alpha}_{\mu} + (C\partial\!\!\!/\gamma_{\mu}\tilde{\partial})^{\alpha\beta}\lambda_{\beta} \tag{3.22}$$

$$A'_\mu = A_\mu + \partial_\mu \Box \lambda \tag{3.23}$$

$$A_{\nu_1\nu_2\nu_3} = \text{gauge invariant}$$

$$B^\alpha = \text{gauge inv.} \tag{3.24}$$

$$D = \text{gauge inv.}$$

$$B_\mu = \text{gauge inv.}$$

from 3.22

$$(\gamma^\mu C)_{\alpha\beta} A'^{\beta}_{\mu} = (\gamma^\mu C)_{\alpha\beta} A^\beta_\mu + (\gamma^\mu C)_{\alpha\beta} (C\bar{\partial}\gamma_\mu \bar{\partial})^{\beta\rho} \lambda_\rho$$

$$= (\gamma^\mu C)_{\alpha\beta} A^\beta_\mu + 4\Box\lambda_\alpha \tag{3.25}$$

So, one can adjust λ_α so as to have a zero "gamma trace" gauge.

$$(\gamma^\mu C)_{\alpha\beta} A'^{\beta}_{\mu} \equiv 0 \tag{3.26}$$

§4 LAGRANGIAN

The redefinitions we have introduced in 3.13, 3.14 to obtain gauge invariant tensors induce modifications in the partial lagrangians for the corresponding field components. In particular this is so for $A_{\alpha_1\alpha_2}^{\dot{\alpha}_1\dot{\alpha}_2}$ and $A_{\alpha_1\alpha_2}^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3}$. Instead there are no changes for the lagrangians corresponding to the rest of the fields.

Let us recall the construction of our Lagrangian

$$L = \varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4} W_{\alpha_1\alpha_2} W_{\alpha_3\alpha_4}\Big|_{\theta^4} + \text{h.c.} \qquad (4.1)$$

(See form (24) of ref. [1]).
Where the chiral superfield strength is given by[1].

$$W_{\alpha_1\alpha_2} = \overset{4}{\bar{D}} D_{\alpha_1} D_{\alpha_2} V$$

Let us start with the "diagonal" component terms. The lagrangian is built from the following part of $W_{\alpha_1\alpha_2}$:

$$\Big\{ \partial_{\alpha_1}^{\dot\alpha_1} \partial_{\alpha_2}^{\dot\alpha_2} A_{\beta_1\beta_2}^{\dot\alpha_3\dot\alpha_4} + 2\partial_{\beta_1}^{\dot\alpha_1} \partial_{[\alpha_1}^{\dot\alpha_2} A_{\alpha_2]\beta_2}^{\dot\alpha_3\dot\alpha_4} + \partial_{\beta_1}^{\dot\alpha_1} \partial_{\beta_2}^{\dot\alpha_2} A_{\alpha_1\alpha_2}^{\dot\alpha_3\dot\alpha_4} - $$
$$ - 3i\partial_{[\alpha_1}^{\dot\alpha_1} A_{\alpha_2]\beta_1\beta_2}^{\dot\alpha_2\dot\alpha_3\dot\alpha_4} + 6i\partial_{\beta_1}^{\dot\alpha_1} A_{\alpha_1\alpha_2\beta_2}^{\dot\alpha_2\dot\alpha_3\dot\alpha_4} + 12A_{\alpha_1\alpha_2\beta_1\beta_2}^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4} \Big\}$$

but according to 3.14 we must add and subtract to the last term, the quantity.

$$12\, \partial_{\alpha_1}^{\dot\alpha_1} \partial_{\alpha_2}^{\dot\alpha_2} A_{\alpha_3\alpha_4}^{\dot\alpha_3\dot\alpha_4} \qquad (4.2)$$

properly antisymmetrized, then $D_{\alpha_1\ldots\alpha_4}^{\dot\alpha_1\ldots\dot\alpha_4}$ appears in the lagrangian explicitly, and at the same time, the lagrangian of $A_{\alpha_1\alpha_2}^{\dot\alpha_1\dot\alpha_2}$ is now

$$\mathcal{L}_{2,2} = \left(\partial_{\alpha_1}^{\dot\alpha_1} \partial_{\alpha_2}^{\dot\alpha_2} A_{\beta_1\beta_2}^{\dot\alpha_3\dot\alpha_4} + 2\,\partial_{\beta_1}^{\dot\alpha_1} \partial_{\alpha_2}^{\dot\alpha_2} A_{\alpha_1\beta_2}^{\dot\alpha_3\dot\alpha_4} + \partial_{\beta_1}^{\dot\alpha_1} \partial_{\beta_2}^{\dot\alpha_2} A_{\alpha_1\alpha_2}^{\dot\alpha_3\dot\alpha_4} \right) . \varepsilon^{\alpha_1\ldots\alpha} \varepsilon_{\dot\alpha_1\ldots\dot\alpha_4} \varepsilon^{\beta_1\ldots\beta_4}$$

$$\left(\partial^{\dot\beta_1}_{\alpha_3}\partial^{\dot\beta_2}_{\alpha_4}A^{\dot\beta_3\dot\beta_4}_{\beta_3\beta_4} + 2\,\partial^{\dot\beta_1}_{\beta_3}\partial^{\dot\beta_2}_{\alpha_4}A^{\dot\beta_3\dot\beta_4}_{\alpha_3\beta_4} + \partial^{\dot\beta_1}_{\beta_3}\partial^{\dot\beta_2}_{\beta_4}A^{\dot\beta_1\dot\beta_4}_{\alpha_3\alpha_4}\right)\cdot\ \varepsilon_{\dot\beta_1\ldots\dot\beta_4} \tag{4.3}$$

using now 3.15 we obtain

$$\mathcal{L}_{2,2} \cong 8\partial^\mu\partial^\nu A_{(\mu\nu)}\partial^\rho\partial^\sigma A_{(\rho\sigma)} + 6\Box A_{(\mu\nu)}\Box A^{(\mu\nu)} - \Box A_\mu{}^\mu\Box A_\rho{}^\rho$$

$$- 12\,\Box A_{(\mu\nu)}\partial^\mu\partial_\sigma A^{\nu\sigma} + 4\Box A^\mu_\mu\partial^\rho\partial^\sigma A_{(\rho\sigma)} + 12\partial_\rho\partial^\mu A_{[\mu\nu]}\partial^\rho\partial_\sigma A^{[\sigma\nu]} \tag{4.4}$$

The last term is the only contribution of the antisymmetric part of $A_{\mu\nu}$, which appears through its gauge invariant divergence $\partial^\mu A_{[\mu\nu]}$. For the symmetric part we can then choose "De Donder gauge" $\partial_\mu A^\nu_\nu = 2\partial^\nu A_{\mu\nu}$ for which the lagrangian takes the simplest form:

$$\mathcal{L}'_{22} \doteq \Box A_{(\mu\nu)}\Box A^{(\mu\nu)} \;;\ \mathcal{L}''_{2,2} \doteq \partial_\rho\partial^\mu A_{[\mu\nu]}\partial^\rho\partial_\sigma A^{[\sigma\nu]} \tag{4.5}$$

It is easy to see that one can still remain in this gauge if we make transformations generated by λ_α such that $\Box\lambda_\alpha = 0$ $\partial^\alpha\lambda_\alpha = 0$ A similar procedure can be followed for the component $A^{\dot\alpha_1\dot\alpha_2}_{\alpha_1\alpha_2\alpha_3}$ for which we find (taking into account 3.13)

$$\mathcal{L}_{2,3} \doteq \left(\partial^{\dot\alpha_1}_{\alpha_1}\partial^{\dot\alpha_2}_{\alpha_2}A^{\dot\alpha_3\dot\alpha_4}_{\beta_1\beta_2\beta_3} + 2\,\partial^{\dot\alpha_1}_{\beta_1}\partial^{\dot\alpha_2}_{\beta_2}A^{\dot\alpha_3\dot\alpha_4}_{\alpha_1\alpha_2\beta_3}\right)\varepsilon_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}\varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}$$

$$\left(\partial^{\dot\beta_4}_{\beta_4}\bar A^{\dot\beta_1\dot\beta_2\dot\beta_3}_{\alpha_4\alpha_3} + \partial^{\dot\beta_4}_{\alpha_3}A^{\dot\beta_1\dot\beta_2\dot\beta_3}_{\alpha_4\beta_4}\right)\ \varepsilon_{\dot\beta_1\dot\beta_2\dot\beta_3\dot\beta_4}\varepsilon^{\beta_1\beta_2\beta_3\beta_4} \tag{4.6}$$

And with the use of 3.16 and the "gamma gauge" 3.26

$$\mathcal{L}_{2,3} = 4i\partial^{\mu}A^{\alpha\dot\alpha}_{\mu}\partial_{\alpha\dot\alpha}\partial_{\nu}\bar{A}^{\nu}_{\dot\alpha} + i\ A^{\alpha}_{\mu}\partial^{\dot\alpha}_{\alpha}\bar{A}^{\mu}_{\dot\alpha} \tag{4.7}$$

The lagrangian for $A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}_{\alpha_1\alpha_2}$ is easily written down

$$\mathcal{L}_{2,4} = A^{\dot\beta_1\dot\beta_2\dot\beta_3\dot\beta_4}_{\alpha_3\alpha_4}\left\{\partial^{\dot\alpha_1}_{\alpha_1}\partial^{\dot\alpha_2}_{\alpha_2}A^{\dot\alpha_3\dot\alpha_4}_{\beta_1\beta_2\beta_3\beta_4} + 8\,\partial^{\dot\alpha_1}_{\beta_1}\partial^{\dot\alpha_2}_{\alpha_1}A^{\dot\alpha_3\dot\alpha_4}_{\alpha_2\beta_2\beta_3\beta_4} + \right.$$
$$\left. + 6\,\partial^{\dot\alpha_1}_{\beta_1}\partial^{\dot\alpha_2}_{\beta_2}A^{\dot\alpha_3\dot\alpha_4}_{\alpha_1\alpha_2\beta_3\beta_4}\right\}\varepsilon^{\alpha_1\ldots\alpha_4}\varepsilon_{\dot\alpha_1\ldots\dot\alpha_4}\varepsilon^{\beta_1\ldots\beta_1}\varepsilon_{\dot\beta_1\ldots\dot\beta_4} \tag{4.8}$$

which, with 3.17, takes the form:

$$\mathcal{L}_{2,4} = 2\bar{B}^{\mu}\partial_{\mu}\partial^{\nu}B_{\nu} - \bar{B}^{\mu}\Box B_{\mu} \tag{4.9}$$

For $A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3}_{\alpha_1\alpha_2\alpha_3}$ we have

$$\mathcal{L}_{3,3} = \left\{\partial^{\dot\alpha_4}_{\alpha_2}A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3}_{\alpha_1\beta_1\beta_2} + \partial^{\dot\alpha_4}_{\beta_1}A^{\dot\alpha_1\dot\alpha_2\dot\alpha_3}_{\alpha_1\alpha_2\beta_2}\right\}\left\{\partial^{\dot\beta_4}_{\alpha_4}A^{\dot\beta_1\dot\beta_2\dot\beta_3}_{\alpha_3\beta_3\beta_4} + \partial^{\dot\beta_4}_{\beta_3}A^{\dot\beta_1\dot\beta_2\dot\beta_3}_{\alpha_3\alpha_4\beta_4}\right\}\cdot$$
$$\varepsilon^{\alpha_1\ldots\alpha_4}\varepsilon_{\dot\alpha_1\ldots\dot\alpha_4}\varepsilon^{\beta_1\ldots\beta_4}\varepsilon_{\dot\beta_1\ldots\dot\beta_4} \tag{4.10}$$

Using 3.14, we obtain

$$\mathcal{L}'_{3,3} = F^{\mu\nu}F_{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{4.11}$$

$$\mathcal{L}''_{3,3} \doteq G^{\mu\nu}G_{\mu\nu} \quad \text{with} \quad G_{\mu\nu} = \partial^{\rho} A_{\mu\nu\rho} \tag{4.12}$$

For $A_{\alpha_1\alpha_2\alpha_3}^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}$, taking into account 3.13:

$$\mathcal{L}_{3,4} \doteq B_{\alpha_1\alpha_2\beta_4}^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4} . i\partial_{\beta_1}^{\dot\beta_4} B_{\alpha_3\alpha_4\beta_2\beta_3}^{\dot\beta_1\dot\beta_2\dot\beta_3} \; \varepsilon^{\alpha_1\ldots\alpha_4}\varepsilon_{\dot\alpha_1\ldots\dot\alpha_4}\varepsilon^{\beta_1\ldots\beta_4}\varepsilon_{\dot\beta_1\ldots\dot\beta_4} \tag{4.13}$$

And using 3.18 we obtain

$$\mathcal{L}_{3,4} \doteq iB^{\alpha}\partial_{\alpha}{}^{\dot\alpha}\bar{B}_{\dot\alpha} \tag{4.14}$$

Finally, with the definition 3.14

$$\mathcal{L}_{4,4} \doteq D^2 \tag{4.15}$$

§5 DISCUSSION

The gauge superfield has the following tensor content: A second rank tensor, $A_{\mu\nu}$, a real vector A_μ and a real self-dual antisymmetric three-vector $A_{\nu_1\nu_2\nu_3}$, a vector spinor A_μ^α a complex vector B_μ a spinor B^α and an auxiliary scalar field D.

It is perhaps interesting to perform a naive dimensional reduction to four dimensions (fields independent of x_4, x_5) We shall do that together with a brief comment on each of them.

$A_{\mu\nu}$ The symmetric part when reduced to four dimensions (independence of x_4, x_5), gives rise to a symetric tensor A_{ij}, two vectors A_{i4}, A_{i5} and three scalars: A_{44}, A_{55}, A_{45}. All of them obeying $\Box\Box A_{\mu\nu} = 0$ The antisymmetric part appears (see comment below 4.4) only through its gauge invariante divergence which generates a four vector and two scalars satisfying the usual wave equation.

A_μ^α reduces to a four vector-spinor A_i^α and two Dirac spinors $A_4^\alpha A_5^\alpha$. The Lagrangian is:

$$\mathcal{L} = 4\partial^i A_i^\alpha \partial_\alpha^{\dot\alpha} \partial_j A_{\dot\alpha}^j + \Box A_j^\alpha \partial_\alpha^{\dot\alpha} A_\alpha^j + \Box A_4^\alpha A_{\dot\alpha}^j + \Box A_4^\alpha \partial_\alpha^{\dot\alpha} A_{\dot\alpha}^4 + \Box A_5^\alpha \partial_\alpha^{\dot\alpha} A_{\dot\alpha}^5 \quad (5.1)$$

with the corresponding third order equations of motion:

$$\Box \partial_\alpha^{\dot\alpha} A_j^\alpha - 4\partial_j \partial^i \partial_\alpha^{\dot\alpha} A_i^{\dot\alpha} = 0 \text{ ("gravitino" eq.)} \quad (5.2)$$

$$\Box \partial_{\alpha}^{\dot\alpha} A_{(4)}^\alpha = 0 \quad (5.3)$$

$A_{(5)}^\alpha$ is not really independent as the "gamma gauge" condition $(\gamma^\mu C)_{\alpha\beta} A_\mu^\beta = 0$ can be used to eliminate it.

A_μ leads to a four vector and two scalar, one of which can be eliminated with the gauge condition.

It is easy to see that the Lagrangian 4.11 reduces to the usual Maxwell lagrangian for the four vector A_i together with the wave Lagrangian for the scalar.

$A_{\mu_1\mu_2\mu_3}$ it reduces to a pseudo vector $\hat{A}^\ell$: $A_{ijk} = \varepsilon_{ijk\ell}\hat{A}^\ell$, and an antisymmetric tensor A_{ij4}. Due to self duality A_{ij5} is not independent of A_{ij4} and A_{i45} is not independent of A_{ijk}.

It is perharps amusing to see that the Lagrangian 4.12 implies:

$$\mathcal{L}_{33} = \partial^i A_{i\nu\rho} \partial_j A^{j\nu\rho} \qquad (5.4)$$

and splitting the pseudovector part.

$$\mathcal{L}'_{33'} \doteq \partial^i \varepsilon_{ijkm} \hat{A}^\mu \partial_\ell \varepsilon^{\ell jkn} \hat{A}_n +$$

$$\doteq \hat{F}^{\ell n} \hat{F}_{\ell n} \quad \text{where} \quad \hat{F}_{\ell n} = \partial_\ell \hat{A}_n - \partial_n \hat{A}_\ell \qquad (5.5)$$

This lagrangian gives Maxwell equations for the pseudo vector $\hat{A}_\ell$ who should be generated by pseudo scalar charges (of the type of magnetic monopoles), while A_ℓ corresponds to an electromagnetism generated by charges of the electric type.

The other part of the Lagrangian

$$\mathcal{L}''_{33,} \doteq \partial^i A_{ij4} \partial_\ell A^{\ell j4}$$

generates as eqs. of motion

$$\partial_i \partial^\ell A_{\ell j4} = \partial_j \partial^\ell A_{\ell i4}$$

which means that

$$\partial^{\ell} A_{\ell i 4} = \partial_i \phi \quad \text{with} \quad \Box \phi = 0 \tag{5.6}$$

B_μ It gives rise to a four vector B_i and two complex scalars obeying the eqs. of motion:

$$2\partial_i \partial^j B_j = \Box B_i$$

$$\text{and } \Box B_4 = \Box B_5 = 0$$

B^α $\partial_\alpha^{\dot\alpha} B_{\dot\alpha}$

when reduced to four dimensions it gives Dirac-massless equations for B^α.

REFERENCES

[1] C.G. Bollini & J.J. Giambiagi Phys. Rev. D 39, 1169, 1989.

[2] C.G. Bollini & J.J. Giambiagi Phys. Rev. D 32, 3316 (1985).

[3] P. Fayet Phys. Lett. 195, 395, 1987.

[4] P. Fayet P.L. 195 395, 1987

B. Delamotte & P. Fayet. 195, 563, 1987.

[5] C.G. Bollini & J.J. Giambiagi in: J.Leite Lopes Festschrift. World Scientific (1988).

[6] E. Witten : Physics Today, July 1980, p. 38.

[7] Elie Cartan: Leçons sur la theorie des spineurs (Hermann Paris, 1938).