ISSN 0029 - 3865

CBPF-NF-042/91

FERMION DETERMINANT FOR THE GENERALIZAED SCHWINGER MODEL

by

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CBPF-NF-042/91

Abstract

The fermion determinant for the generalized Schwinger model is explicitly computed, using point-splitting regularization, and, from it, the Wess–Zumino functional for that theory is obtained. The relation between our results and those that have already appeared in the literature is also drawn.

Key-words: Two-dimensional models; Fermion determinant.

Our main purpose in this note is to give an alternative derivation of the fermion determinant for the generalized Schwinger model (GSM). This fermion determinant has been computed by a number of authors [1,2,3,4] and it has proven to be very useful either in testing consistency of two-dimensional gauge theories [3] or in proving its equivalence with a generalized sine-Gordon model (in the massive fermion case) [4]. However, there is a lack (to our knowledge) in the literature of the same computation using Schwinger's point-splitting technique [5] that should be compared with previous results. We performed this calculation and show our findings below and it checks with the particular cases of the standard (SSM), axial (ASM) and chiral (CSM) Schwinger models, presented in Jackiw's lectures [6]. Further, we present the Wess–Zumino functional for the generalized model, which also agrees with the known expressions for the restricted models above mentioned. The relation between our results and those of other groups [3,4] is also given.

The GSM is defined by the Lagrangian density (in two dimensional Euclidean space) [7]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \overline{\psi}\gamma_\mu \left(i\partial_\mu + e_R A_\mu P_+ + e_L A_\mu P_-\right)\psi. \tag{1}$$

The covariant Dirac operator appearing in (1) can be rewritten as

$$D = \gamma^\mu (i\partial_\mu^y + B_\mu(y; g_+, g_-))\delta^2(x-y), \tag{2}$$

where

$$B_\mu(x; g_+, g_-) = \left(g_+\delta_{\alpha\mu} + ig_-\epsilon_{\alpha\mu}\right) A_\alpha \tag{3}$$

and

$$g_\pm = \frac{1}{2}\left(e_R \pm e_L\right). \tag{4}$$

From the identity $\det D = \exp \mathrm{Tr} \ln D$, the determinant is seen to satisfy

$$\frac{\partial}{\partial g_+} \ln \det D = \int d^2x \, \mathrm{tr} \left[G(x,x;B) \gamma_\mu \right] A_\mu(x) \tag{5}$$

$$\frac{\partial}{\partial g_-} \ln \det D = i \int d^2x \, \mathrm{tr} \left[G(x,x;B) \gamma_\mu \right] \epsilon_{\alpha\mu} A_\alpha(x), \tag{6}$$

thus reducing the problem of computing $\ln \det D$ to the one of solving the system (5), (6) of coupled partial differential equations in g_+, g_- [8]. Here, $G(x,y;B)$ denotes the inverse of the operator D, defined by

$$\gamma^\mu (i\partial^x_\mu + B_\mu(x)) G(x,y;B) = \delta^2(x-y). \tag{7}$$

The Ansatz [5,9]

$$G(x,y;B) = e^{i(\phi(x)-\phi(y))} G_F(x-y), \tag{8}$$

where G_F is the inverse of the free Dirac operator, implies, upon substitution in eq. (7) and introduction of the free laplacian Green's function, D_F,

$$\phi(x) = \int d^2x' D_F(x-x') \left[\partial^{x'}_\mu B_\mu(x') + i\gamma_5 \tilde{\partial}^{x'}_\mu B_\mu(x') \right]. \tag{9}$$

Eq. (8) may then be reexpressed as

$$\begin{aligned} & G(x,y;B) \\ = \; & \exp\left[-i \int d^2x' B_\mu(x') \left(\partial^{x'}_\mu + i\tilde{\partial}^{x'}_\mu \right) \left(D_F(x-x') - D_F(y-x') \right) \right] \\ & \times P_+ G_F(x-y) \\ & + \exp\left[-i \int d^2x' B_\mu(x') \left(\partial^{x'}_\mu - i\tilde{\partial}^{x'}_\mu \right) \left(D_F(x-x') - D_F(y-x') \right) \right] \\ & \times P_- G_F(x-y). \end{aligned} \tag{10}$$

In order to cope with the regularization freedom of the theory, we substitute the traces in eqs. (5) and (6) according to

$$\begin{aligned} &\mathrm{tr}\,(G(x,y;B)\gamma_\mu) \\ &\quad\longrightarrow \mathrm{tr}\left(G(x,y;B)\gamma_\mu \exp\left[i\,(a_R e_R P_+ + a_L e_L P_-)\int_x^y A_\nu dz_\nu\right]\right), \end{aligned} \tag{11}$$

where a_R and a_L are arbitrary parameters which parametrize this ambiguity. After using the explicit expression for G_F, computing all necessary traces, expanding $D_F(x-y)$ around $|x-y|=0$ to first order and taking the symmetric limit $x\to y$, we find that the differential equations (5) and (6) become

$$-\frac{\partial\Gamma}{\partial g_+} = g_+F + g_-H \tag{12}$$

$$-\frac{\partial\Gamma}{\partial g_-} = g_-F + g_+H, \tag{13}$$

with

$$\Gamma[A] = \ln\det D, \tag{14}$$

$$F[A] = \frac{1}{2\pi}\int d^2x\; A_\mu(x)\left(-a_+\delta_{\mu\nu} + \frac{\partial_\mu\partial_\nu - \tilde\partial_\mu\tilde\partial_\nu}{\Box}\right)A_\nu(x) \tag{15}$$

$$H[A] = \frac{i}{2\pi}\int d^2x\; A_\mu(x)\left(ia_-\delta_{\mu\nu} + \frac{\partial_\mu\tilde\partial_\nu + \tilde\partial_\mu\partial_\nu}{\Box}\right)A_\nu(x), \tag{16}$$

and $a_\pm = \frac{1}{2}(a_R \pm a_L)$. Solving the system (12), (13) we obtain

$$-\Gamma[A] = \frac{1}{2}g_+^2F + g_+g_-H + \phi_-(g_-) \tag{17}$$

$$= \frac{1}{2}g_-^2F + g_+g_-H + \phi_+(g_+). \tag{18}$$

Compatibility between (17) and (18) requires the functions $\phi_\pm$ to be given by

$$\phi_\pm = \frac{1}{2}Fg_\pm^2. \tag{19}$$

This gives us the desired effective action,

$$\begin{aligned}\Gamma[A] &= \frac{1}{4\pi}\left(g_+^2+g_-^2\right)\int d^2x\, A_\mu(x)\left(a_+\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu-\tilde\partial_\mu\tilde\partial_\nu}{\Box}\right)A_\nu(x)\\ &\quad -\frac{i}{2\pi}g_+g_-\int d^2x\, A_\mu(x)\left(ia_-\delta_{\mu\nu}+\frac{\partial_\mu\tilde\partial_\nu+\tilde\partial_\mu\partial_\nu}{\Box}\right)A_\nu(x) \qquad (20)\\ &= \frac{1}{2\pi}\int d^2x\,\left\{\frac{1}{4}\left[(e_R^2+e_L^2)(a_++1)+(e_R^2-e_L^2)a_-\right]A_\mu^2\right.\\ &\quad \left.-\frac{1}{2}\left[(e_R^2+e_L^2)A_\mu\frac{\partial_\mu\partial_\nu}{\Box}A_\nu+i(e_R^2-e_L^2)A_\mu\frac{\partial_\mu\tilde\partial_\nu}{\Box}A_\nu\right]\right\}. \qquad (21)\end{aligned}$$

This expression for $\Gamma[A]$ allows us to compute immediately the Wess–Zumino functional, defined by

$$\alpha_{\rm WZ}[A;\theta]=\Gamma[A^\theta]-\Gamma[A]; \qquad A_\mu^\theta=A_\mu+\partial_\mu\theta. \qquad (22)$$

We find

$$\begin{aligned}&\alpha_{\rm WZ}[A;\theta]\\ &= \frac{1}{8\pi}\left[(e_R^2+e_L^2)(a_+-1)+(e_R^2-e_L^2)a_-\right]\int d^2x(\partial_\mu\theta\partial_\mu\theta-2\theta\partial_\mu A_\mu)\\ &\quad -\frac{i}{4\pi}(e_R^2-e_L^2)\int d^2x\theta\tilde\partial_\mu A_\mu. \qquad (23)\end{aligned}$$

It is easy to see that expression (21) reduces to the known effective actions [6] for the SSM ($e_R=e_L$), ASM ($e_R=-e_L$) and CSM ($e_{\substack{R\\L}}=0$), and that (23) gives the correct Wess–Zumino functionals for the same cases [10].

We would like to compare our findings with two other results, namely, those from Alonso, Cortés and Rivas (ACR) [3] and Naón and Wotzasek (NW) [4]. The effective action for ACR is

$$\begin{aligned}\overline{W}[A] &= \frac{1}{2\pi}\int d^2x\left\{\frac{1}{2}(\tilde e_R\tilde e_L+\bar e_R^2+\bar e_L^2)A_\mu^2-\frac{1}{2}\left[(\bar e_R^2+\bar e_L^2)A_\mu\frac{\partial_\mu\partial_\nu}{\Box}A_\nu\right.\right.\\ &\quad \left.\left.+i\,(\bar e_R^2-\bar e_L^2)A_\mu\frac{\partial_\mu\tilde\partial_\nu}{\Box}A_\nu\right]\right\}, \qquad (24)\end{aligned}$$

where

$$\overline{e}_{\substack{L\\R}} = \left[\widetilde{e}^2_{\substack{L\\R}} + (\widetilde{e}_{\substack{L\\R}} - e_{\substack{L\\R}})^2\right]^{1/2}, \tag{25}$$

with $\widetilde{e}_L$ and $\widetilde{e}_R$ being arbitrary couplings associated to the freedom of choice of the regularizing Dirac operator of the theory. Compatibility between (21) and (24) requires

$$\begin{aligned} \overline{e}^2_R + \overline{e}^2_L &= e^2_R + e^2_L, \\ \overline{e}^2_R - \overline{e}^2_L &= e^2_R - e^2_L \qquad \text{and} \\ \frac{1}{2}(\widetilde{e}_R\widetilde{e}_L + \overline{e}^2_R + \overline{e}^2_L) &= \frac{1}{4}\left[(e^2_R + e^2_L)(a_+ + 1) + (e^2_R - e^2_L)a_-\right]. \end{aligned} \tag{26}$$

The first two imply $\overline{e}_R = e_R$, $\overline{e}_L = e_L$, which, in virtue of (25) gives four solutions for $\widetilde{e}_L$ and $\widetilde{e}_R$:

$$\begin{array}{llll} i) & \widetilde{e}_L = 0 & \text{or} & \widetilde{e}_L = e_L; \\ ii) & \widetilde{e}_R = 0 & \text{or} & \widetilde{e}_R = e_R. \end{array}$$

The possibility of both $\widetilde{e}_L$ and $\widetilde{e}_R$ being different from zero is ruled out by the non-existence of a crossed term in the right-hand side of (26). All the remaining options imply $a_+ = 1$, $a_- = 0$ or, in other words, $a_R = a_L = 1$. Thus, from our point of view, the result of ACR seems to belong to a different class of regularizations of the fermion determinant, which only makes contact with our class in this special case. It would be interesting to study the consistency requirements of the theory in our class as well, along the lines of Ref. [3].

In order to compare with NW, we first decompose A_μ as

$$A_\mu = \partial_\mu\eta + \epsilon_{\mu\nu}\partial_\nu\phi. \tag{27}$$

We thus obtain

$$\Gamma[\eta,\phi]=\int d^2x\left\{\kappa_\eta\eta\Box\eta+\kappa_\phi\phi\Box\phi-i\frac{g_+g_-}{\pi}\eta\Box\phi\right\} \tag{28}$$

with

$$\kappa_\eta=\frac{g_+^2+g_-^2}{4\pi}(a_+-1)+\frac{g_+g_-}{2\pi}a_- \tag{29}$$

and

$$\kappa_\phi=\frac{g_+^2+g_-^2}{4\pi}(a_++1)+\frac{g_+g_-}{2\pi}a_-. \tag{30}$$

From Ref. [4], the NW effective action is

$$W[\eta,\phi]=\int d^2x\left\{\left(\kappa+\frac{g_+^2}{2\pi}\right)\phi\Box\phi+\left(\kappa-\frac{g_-^2}{2\pi}\right)\eta\Box\eta-i\frac{g_+g_-}{\pi}\eta\Box\phi\right\}, \tag{31}$$

where κ is an arbitrary parameter that accounts for the freedom of regularization. Comparing (28) and (31) we obtain

$$\kappa=\frac{g_+^2+g_-^2}{4\pi}a_++\frac{g_+g_-}{2\pi}a_--\frac{g_+^2-g_-^2}{4\pi}, \tag{32}$$

which can be checked to give the correct expression for the fermion determinant in all the particular cases mentioned above.

Acknowledgment C.A. Linhares is partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

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[7] Our conventions are $\gamma_0 = \sigma_1$, $\gamma_1 = -\sigma_2$; $\gamma_5 = i\gamma_0\gamma_1 = \sigma_3$; $\epsilon_{01} = +1$; $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$; $\tilde{\partial}_\mu = \epsilon_{\mu\nu}\partial_\nu$.

[8] We thank Prof. H. O. Girotti for explaining this method to us in the particular case of the CSM.

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