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THE ANALYTIC REGULARIZATION ZETA FUNCTION METHOD AND THE CUT-OFF METHOD IN CASIMIR EFFECT.

by

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ABSTRACT

The zero point energy associated to a hermitian massless scalar field in the presence of perfectly reflecting plates in a three dimensional flat space-time is discussed . A new technique to unify two different methods - the zeta function and a variant of the cut-off method - used to obtain the so called Casimir energy is presented,and the proof of the analytic equivalence between both methods is given.

Key-words: Quantum field theory; Renormalization.

I-Introduction

In a previous paper[1] we have introduced a new technique of comparison between two usual methods for obtaining the Casimir energy , viz : the cut-off method[2] and the zeta function method[3][4][5].Using this approach we proved the analytic equivalence between these two methods in a two-dimensional spacetime.The purpose of this work is to extend this previous result to a higher dimension.

The problem of the renormalization of ill defined quantities leading to a physically significant result is a fundamental and ubiquitous question of quantum field theory.Although many regularization methods have been employed, a proof that all these different methods lead to the same result is still lacking[6][7].

The classical example of a ill defined quantity is that of the zero point energy of a quantum field in a flat space time. The Wick normal ordering procedure may elude this divergence.However, Casimir showed that this procedure is not adequate to the study of fields in the presence of surfaces when the fields satisfies boundary conditions.Using the idea that , although formally divergent , the zero point energy can suffer a finite change if the physical configuration is modified,he derived a finite result for the energy of the vacuum state of an electromagnetic field in the presence of conducting parallel plates.This method can be summarized in the following steps:a complete set of mode solutions and the respective eingenfrequencies of the classical wave equation satisfying appropriate boundary conditions is found;the divergent zero point energy of the quantized field is regularized by means of a cut-off function and then is renormalized using auxiliary configurations which are added and subtracted.

Subsequently another methods , like the Green function method[8][9][10],the dimensional regularization method[11][12] and the zeta function method [3][4][5] were employed to obtain a finite result for the vacuum energy.Even in the well studied case of the Casimir energy,however a proof of the equivalence between some of these different techniques was not available.In this article an analytic proof of the equivalence between the zeta function and the cut-off method for obtaining the Casimir energy of a scalar field confined in rectangular cavities satisfying Dirichlet boundary conditions is presented for the case $D=3$ (three dimensional space time).The generalization to $D > 3$ and fields of higher spin is straightforward.

This paper is organized as follows:

In section II the zeta function regularization method is briefly presented.

In section III the exponential cut-off method is carefully studied.

In section IV the zeta function method is interpreted as an "algebraic" cut-off method.

In section V the unification between these two methods is achieved using the mixed cut-off procedure.The equivalence between these methods is obtained as a consequence of the analyticity of a certain complex function of two variables.

Conclusions are given in section VI.

In this paper we use $\hbar = c = 1$.

II-The Casimir energy obtained using the zeta function method

For a massless scalar field confined in a two-dimensional rectangular box satisfying Dirichlet boundary conditions the eingenfrequencies are given by:

$$\omega_{n\,m}=\left[\left(\frac{n\,\pi}{L_1}\right)^2+\left(\frac{m\,\pi}{L_2}\right)^2\right]^{1/2} \qquad (2.1)$$

$n,m=1,2,3,\ldots$

where L_1,L_2 are the lengths of the sides of the box.

The zero point energy is

$$E(L_1,L_2)=\frac{1}{2}\sum_{n,m=1}^{\infty}\omega_{n\,m} \qquad (2.2)$$

where ω_{nm} is given by eq (2.1).This expression is divergent and can be written as

$$E(L_1,L_2,s)=\frac{1}{2}\sum_{n,m=1}^{\infty}\omega_{n\,m}^{\;-2s} \qquad (2.3)$$

for $s = -1/2$.

The expression (2.3) is analytic for $Re(s) > 1$.The zeta function method consist in evaluating the analytic extension of this function at the point $s = -1/2$, thereby obtaining a finite result.Algebraic manipulations of eq.(2.3),using eq.(2.1) gives:

$$E(L_1,L_2,s)=\frac{1}{8}\,A\left(\left(\frac{\pi}{L_1}\right)^2,\left(\frac{\pi}{L_2}\right)^2;2s\right)+$$

$$-\frac{1}{4}\left(\left(\frac{L_1}{\pi}\right)^{2s}+\left(\frac{L_2}{\pi}\right)^{2s}\right)\zeta(2s) \quad , \qquad (2.4)$$

where $A(a,b;2s)$ is the Epstein zeta function and $\zeta(2s)$ is the Riemann zeta function. So $E(L_1,L_2;2s)$ is analytic in $s \in \mathbb{C}\backslash\{1/2,1\}$ and the evaluation of $E(L_1,L_2;-1/2)$ gives the Casimir energy $U(L_1,L_2)$,

$$U(L_1,L_2) = \frac{\pi}{48}\left(\frac{1}{L_1}+\frac{1}{L_2}\right) + - \frac{L_1 L_2}{32\,\pi} \sum_{p,q=-\infty}^{\infty}{}' \left(p^2 L_1^{\,2} + q^2 L_2^{\,2}\right)^{-3/2} . \qquad (2.5)$$

The prime sign in the summation means that the term p=q=0 is to be excluded.

To obtain the Casimir energy given by eq(2.5) through this method there is apparently no need of a normalization scheme and a finite result comes out automatically. This aspect will be discussed later.

An important remark is that the contribution of the field outside the box for the Casimir energy apparently was not taken in account at any moment. This is not a trivial matter because the two configurations illustrated by Fig.1.a and Fig.1.b are physically quite different. In the box configuration there is field in the whole space ,while in bubble configuration there is field only inside the cavity.

Fig 1.a-The box configuration

fig 1.b-The bubble configuration

III-The exponential cut-off method

The divergent expression given by eq(2.2) can be regularized using a exponential cut-off function such as

$$e^{-\lambda\,\omega_{nm}} \quad . \tag{3.1}$$

The regularized energy is then

$$E(L_1,L_2,\lambda)= \frac{1}{2}\sum_{n,m=1}^{\infty}\omega_{nm}\, e^{-\lambda\,\omega_{nm}} \tag{3.2}$$

$$Re(\lambda) > 0 \quad .$$

The function given by eq(3.2) is analytic for $Re(\lambda) > 0$ but divergent at $\lambda=0$. A renormalization procedure is thus required to enable one to take the limit $\lambda \to 0^+$ without divergences.This method was employed by Casimir, Fierz[13],Boyer[14] and others using auxiliary configurations in order to obtain a finite result for the case of parallel plates.

Let us define a function h(a,b,u) which will be of use throughout along this paper:

$$h(a,b,u)= \sum_{n,m=1}^{\infty} e^{-u\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]^{1/2}} \quad . \tag{3.3}$$

$$a,b > 0$$

Then the regularized energy given by eq.(3.2) can be expressed as:

$$E(L_1,L_2,\lambda) = -\frac{1}{2}\frac{\partial}{\partial\lambda}h(L_1,L_2,\lambda) \quad . \tag{3.4}$$

Adding and subtracting terms in eq.(3.3)in order to get a double summation in $n,m \in \mathbb{Z}$,performing this summation using the Poisson summation formula,integrating this result in polar coordinates and using the fact that the two simple summations which are introduced are geometric series, we get the following expression for the function h:

$$h(a,b,u) = \frac{a\,b}{2\,\pi}\,u^{-2} + \frac{a\,b}{2\,\pi}\,u \sum_{p,q=-\infty}^{\infty}{}' \left(u^2 + 4\,p^2a^2 + 4\,q^2b^2 \right)^{-3/2} +$$

$$- \frac{1}{2}\,\frac{1}{e^{\pi u/a} - 1} - \frac{1}{2}\,\frac{1}{e^{\pi u/b} - 1} - \frac{1}{4} \quad . (3.5)$$

Defining $r_0 = \min\{a,b\}$ we see from eq.(3.5) that $h(a,b,u)$ is analytic for $0<|u|<2r_0$,the point $u=0$ is a second order pole and the negative powers portion of the Laurent series expansions of h around $u=0$ is given by:

$$h_{polar}(a,b,u) = \frac{a\,b}{2\,\pi}\,u^{-2} - \frac{a+b}{2\,\pi}\,u^{-1} .$$

Substituting eq.(3.5) in eq.(3.4) the regularized energy becomes:

$$E(L_1,L_2,\lambda) = \frac{L_1L_2}{2\,\pi}\,\lambda^{-3} - \frac{1}{4\,\pi}\,B_0(L_1+L_2)\lambda^{-2} + \frac{\pi}{8}\,B_2\left(\frac{1}{L_1} + \frac{1}{L_2}\right) +$$

$$+ \frac{L_1L_2}{4\,\pi} \sum_{p,q=-\infty}^{\infty}{}' \left(\lambda^2 + 4\,p^2a^2 + 4\,q^2b^2 \right)^{-3/2} +$$

$$+\lambda^2 g_1(\lambda), \qquad (3.6)$$

where B_0 and B_2 are Bernoulli numbers and $g_1(\lambda)$ is analytic in

$|\lambda|<2\min\{L_1,L_2\}$.

The two divergent terms in eq.(3.6) are proportional to the "volume" and to the "perimeter" of the cavity;thus following the Casimir approach we need to add and subtract auxiliary configurations in order that:

a)The final result is a difference between "isovolumetric" and "isoperimetric" configuration sets.

b)The auxiliary configurations should not give contributions to the finite renormalized energy.

This second prescription is achieved if the distance between the opposite sides of the auxiliary cavities becomes infinite, so that the field inside this auxiliary boxes tends to the free , unconstrained field.

A naive procedure to obtain the Casimir energy in the rectangular cavity case would be to define:

$$U(L_1,L_2)= \lim_{\substack{\lambda\to 0\\ R,R'\to\infty}} E(L_1,L_2,\lambda) + E(R-L_1,L_2,\lambda) + E(L_1,R'-L_2,\lambda)+ + E(R-L_1,R'-L_2,\lambda) - 4\, E(R/2,R'/2,\lambda). \quad (3.7)$$

This can be visualized by Fig(2.a) and Fig.(2.b).

Fig. 2.a and 2.b-Set of configurations employed to otain $U(L_1,L_2)$ of eq.(3.7)

The problem of this renormalization is that we are adding up "plates" to the initial configuration.The field inside the cavities of sides $(R-L_1,L_2)$ and $(L_1,R'-L_2)$ will never tend to the free unconstrained field as $R,R'\to\infty$ like in the two plates Casimir approach, and prescription (b) is thus not satisfied.This (eq.(3.7)) renormalized energy must be corrected by removing this "plate" effect.This can be done by means of the following renormalization:

$$Y(L_1,L_2,R,R',\lambda) = E(L_1,L_2,\lambda) + E(R-L_1,L_2,\lambda) + E(L_1,R'-L_2,\lambda) + E(R-L_1,R'-L_2,\lambda) - 4\,E(R/2,R'/2,\lambda) + \\ -\Big[E(L_1,R'-L_2,\lambda)+E(R-L_1,R'-L_2,\lambda) -2E(R/2,R'-L_2,\lambda)\Big] + \\ -\Big[E(R-L_1,L_2,\lambda)+E(R-L_1,R'-L_2,\lambda) -2E(R-L_1,R'/2,\lambda)\Big]. \tag{3.8}$$

Then:

$$U(L_1,L_2) = \lim_{\substack{\lambda\to 0 \\ R,R'\to\infty}} Y(L_1,L_2,R,R',\lambda). \tag{3.9}$$

The evaluation of $U(L_1,L_2)$ by eq.(3.9) gives the same result as that obtained using eq.(2.5). It is easy to see that if we use the following auxiliary configurations (illustrated in Fig(3.a) and (3.b)),which are "isovolumetric" and "isoperimetric",the finite contribution of the auxiliary cavities to the Casimir energy vanish as $R.R'\to\infty$.This strange configuration gives the same result as that obtained from eq.(3.9) as expected.

Fig 3.a and 3.b- Set of configurations that gives the same $U(L_1,L_2)$ as that obtained by eqs.(2.5) and (3.9)

IV-The zeta function method as an "algebraic" cut-off method

A regularized energy can be obtained from eq.(2.2) using an "algebraic" cut-off:

$$\omega_{n\,m}^{\;-\sigma} \qquad (4.1)$$

and the regularized energy becomes

$$E(L_1,L_2,\sigma)= \frac{1}{2}\sum_{n,m=1}^{\infty} \omega_{n\,m}\,\omega_{n\,m}^{\;-\sigma} \quad , \qquad (4.2)$$

$$\mathrm{Re}(\sigma) > 3$$

which is convergent and analytic for $\mathrm{Re}(\sigma)>3$.Algebraic manipulations of eq.(4.2) using eq.(2.1) gives:

$$E(L_1,L_2,\sigma)= \frac{1}{8}\, A\left(\left(\frac{\pi}{L_1}\right)^2,\left(\frac{\pi}{L_2}\right)^2;\sigma-1\right)+$$

$$- \frac{1}{4}\left(\left(\frac{L_1}{\pi}\right)^{\sigma-1} + \left(\frac{L_2}{\pi}\right)^{\sigma-1}\right)\zeta(\sigma-1) \quad . \qquad (4.3)$$

If $\sigma > 3$,this cut-off works finely and we get a finite energy.As in any cut-off method, we want to take the limit $\sigma\to 0$ starting from $\sigma>3$.Eq.(4.3) defines an analytic function in $\sigma \in \mathbf{C}\setminus\{2,3\}$.(See Fig.(4))

Fig 4-The summation of eq.(4.2) is convergent in the shadowed region,while the analytic extension of this function is a meromorphic function with its poles indicated.

It is interesting to note that eq.(4.3), when evaluated at $\sigma=0$, gives the Casimir energy derived in section II. This later result (of section II) is based on the use of the analytic continuation of a complex function. Although it seems to be quite obvious ,we want to point out that analytic continuations:

a)Are to be performed in open connected domains.

b)Make use of paths (entirely contained in the domain) in order to extend the function which is initially defined only in a subset of the domain.

Since we are dealing with the zeta method as a cut-off method, we will move σ (our regularization parameter) only along the real axis and certain procedures related with the physics of the problem will be employed.

A careful study of eq.(4.3) leads us to the expression:

$$E(L_1,L_2,\sigma)= G_1(L_1,L_2,\sigma) + \frac{1}{4\,\Gamma\left(\frac{\sigma-1}{2}\right)}\left(\frac{L_1\,L_2}{\pi(\sigma-3)} - \frac{L_1+L_2}{\pi^{3/2}(\sigma-2)}\right), \qquad (4.4)$$

where $G_1(L_1,L_2,\sigma)$ is analytic in the whole σ-complex plane. As we move along the real axis from $\sigma>3$ toward $\sigma=0$, we find, first a divergence proportional to the "volume" of the cavity and, after that, a divergence proportional to the "perimeter" of the cavity. Again, it is clear that it is necessary to use auxiliary configurations with "isovolumetric" and "isoperimetric" subtractions in order to eliminate the divergences along the path. If we take the auxiliary configurations as in eq.(3.8) or as in Fig(3.a) and Fig(3.b) the result will be the same as that of Sec.II, since the auxiliary configurations will not disturb the value of the analytic continuation of eq.(4.3) at the point $\sigma=0$ in the limit $R,R'\to\infty$.

Since the exponential cut-off is a strong factor of convergence, the regularized energy $E(L_1,L_2,\lambda)$ becomes singular only when $\lambda\to 0^+$. Being a weaker factor of convergence, the algebraic cut-off scatters the singularities of the regularized energy $E(L_1,L_2,\sigma)$ along the path towards the origin, leaving the origin itself free from singularities when we take the analytic continuation.

V-The mixed cut-off as a tool for unification.

The procedure of the preceding Sections III and IV can be unified by the use of a mixed cut-off function:

$$\omega_{nm}^{-\sigma}\, e^{-\lambda\, \omega_{nm}} \quad . \tag{5.1}$$

In this case the regularized energy is:

$$E(L_1,L_2,\lambda,\sigma)= \frac{1}{2} \sum_{n,m=1}^{\infty} \omega_{nm}\, \omega_{nm}^{-\sigma}\, e^{-\lambda\, \omega_{nm}} \tag{5.2}$$

$$\mathrm{Re}(\lambda) > 0 \;, \sigma\in\mathbb{C} \quad \text{or}$$

$$\mathrm{Re}(\lambda)=0 \;, \mathrm{Re}(\sigma)>3 \quad .$$

The regularized energy given by eq.(5.2) as a function of λ and σ is analytic in $\mathrm{Re}(\lambda)>0, \sigma\in\mathbb{C}$ and is continuous in $\mathrm{Re}(\lambda)\geq 0, \mathrm{Re}(\sigma)>3$. In $\lambda=0$, as a function of σ, it can be analytically extended to $\sigma\in\mathbb{C}\backslash\{2,3\}$.

Fig 5-The summation of eq.(5.2) converges to an analytic function(in λ,σ) in the shadowed region. It converges also in $\mathrm{Re}(\lambda)=0, \mathrm{Re}(\sigma)>3$. The points $\sigma=2,3$ are poles of the analytic extension of $E(L_1,L_2,\lambda=0,\sigma)$.

It is interesting to stress that:

(i) $\lim_{\lambda\to 0^+} E(L_1,L_2,\lambda,\sigma=0)$ does not exist,and

(ii) $\lim_{\sigma\to 0^+} E(L_1,L_2,\lambda=0,\sigma)$ does exist (if we are dealing with the analytic continuation).

The regularized energy thus obtained in eq.(5.2) can be renormalized using the same procedures as in Sec. III and IV:addition and subtraction of auxiliary configurations.This procedure can be formalized in the following way.Define:

$$Y(L_1,L_2, R ,R',\lambda,\sigma)= \sum_{i=1}^{N} E(L_{1i},L_{2i},\lambda,\sigma) - \sum_{i=N+1}^{2N} E(L_{1i},L_{2i},\lambda,\sigma), \tag{5.3}$$

where the original cavity with lengths (L_1,L_2) appears as(L_{11},L_{21}).The other L_{ki},k=1,2 ,i=2,3...2N are monotonous functions of R,R′ in such a way that

$$\lim_{R,R'\to\infty} L_{ki}= \infty \quad , \quad L_{ki}\geq L_{k1} \qquad i\neq 1 \quad . \tag{5.4}$$

Since we want "isovolumetric" and "isoperimetric" subtraction,it must be imposed that

$$\sum_{i=1}^{N} L_{1i}+ L_{2i} = \sum_{i=N+1}^{2N} L_{1i}+ L_{2i} \tag{5.5}$$

and

$$\sum_{i=1}^{N} L_{1i}\, L_{2i} = \sum_{i=N+1}^{2N} L_{1i}\, L_{2i} \quad . \tag{5.6}$$

In Sec.IV it was proved that the zeta function method is equivalent to the algebraic cut-off method,which is a particular case of the use of the mixed cut-off method.This happens when we evaluate:

$$U_{alg}(L_1,L_2)= \lim_{\substack{\sigma\to 0^+ \\ R,R'\to\infty}} Y(L_1,L_2,R,R',\lambda=0,\sigma) \qquad (5.7)$$

and the Casimir energy derived by the exponential cut-off method is given by:

$$U_{exp}(L_1,L_2)= \lim_{\substack{\lambda\to {}^+0 \\ R,R'\to\infty}} Y(L_1,L_2,R,R',\lambda,\sigma=0) \qquad (5.8)$$

We claim that the "isovolumetric" and "isoperimetric" subtraction performed in eq.(5.3) renders the function $Y(L_1,L_2,R,R',\lambda,\sigma)$ analytic in $|\lambda|<\rho_0,\sigma\in\mathbf{C}$ for some $\rho_0>0$. Consequently, eq.(5.7) and eq.(5.8) gives the same result and the two methods- cut-off and zeta function - are analytically equivalent.Now we conclude the proof demonstrating the analyticity of $Y(L_1,L_2,R,R',\lambda,\sigma)$ in a domain $|\lambda|<\rho_0,\sigma\in\mathbf{C}$ for some $\rho_0>0$.

Let us call

$$\Omega(\rho_0) = \{\lambda \in \mathbf{C}; |\lambda|<\rho_0\} \times \{\sigma\in\mathbf{C}\} \quad \rho_0>0 \quad . \qquad (5.9)$$

Using an integral representation of the Γ function , the regularized energy given by eq.(5.2) can be expressed for $Re(\lambda)>0$ or $Re(\lambda)=0$ and $Re(\sigma)>3$ as:

$$E(L_1,L_2,\lambda,\sigma)=\frac{1}{2}\,\frac{1}{\Gamma(\sigma-1)} \int_0^\infty dx\; x^{\sigma-2} \sum_{n,m=1}^{\infty} e^{-(\lambda+x)\;\omega_{nm}} \qquad (5.10)$$

$Re(\lambda)>0$ or

$Re(\lambda)=0$ and $Re(\sigma)>3$.

Using eq.(2.1) ,eq.(3.3) and splitting the above integral in ρ_0 we get:

$$E(L_1,L_2,\lambda,\sigma)=\frac{1}{2}\frac{1}{\Gamma(\sigma-1)}\int_0^{\rho_0}dx\ x^{\sigma-2}\ h(L_1,L_2,\lambda+x)+g_2(L_1,L_2,\lambda,\sigma) \tag{5.11}$$

Re(λ)>0 or

Re(λ)=0 and Re(σ)>3,

where $g_2(L_1,L_2,\lambda,\sigma)$ is analytic in $\Omega(\rho_0)$.

Defining

$$H(L_1,L_2,R,R',u)=\sum_{i=1}^{N}h(L_{1i},L_{2i},u)-\sum_{i=N+1}^{2N}h(L_{1i},L_{2i},u)\ , \tag{5.12}$$

and

$$G_2(L_1,L_2,R,R',\lambda,\sigma)=\sum_{i=1}^{N}g_2(L_{1i},L_{2i},\lambda,\sigma)-\sum_{i=N+1}^{2N}g_2(L_{1i},L_{2i},\lambda,\sigma)\ , \tag{5.13}$$

then from eqs.(5.3) ,(5.11),(5,12) and (5.13) we have

$$Y(L_1,L_2,R,R',\lambda,\sigma)=\frac{1}{2}\frac{1}{\Gamma(\sigma-1)}\int_0^{\rho_0}dx\ x^{\sigma-2}\ H(L_1,L_2,\lambda+x)\quad + \\ +G_2(L_1,L_2,R,R',\lambda,\sigma)\quad . \tag{5.14}$$

Since $G_2(L_1,L_2,R,R',\lambda,\sigma)$ is a sum of analytic functions in

$(\lambda,\sigma)\epsilon\Omega(\rho_0)$,this function is analytic in the same domain.

$H(L_1,L_2,R,R',u)$ is a sum of 2N functions ,each one analytic in $0<|u|<2\mathrm{Min}\{L_{1i},L_{2i}\}$,with a second order pole at u=0.Then if we take

$$\rho_0 = \mathrm{Min}\ \{L_1,L_2\} \tag{5.15}$$

thus from eq.(5.4) it follows that $H(L_1,L_2,R,R',u)$ is analytic at $0<|u|<2\rho_0$ and has ,at worst, a second order pole at u=0.

Using eq.(5.12),the polar portion of each $h(L_{1i},L_{2i},u)$ derived in Sec.III and the restrictions imposed upon L_{ki} by eqs.(5.5) and (5.6), we find that that the coefficients of the negative portion of the Laurent series of $H(L_1,L_2,R,R',u)$ (around u=0) vanish. Then $H(L_1,L_2,R,R',u)$ is analytic at $|u|<2\rho_0$ .Thus using eq.(5.9) and the properties of the Γ function we see that $Y(L_1,L_2,R,R',\lambda,\sigma)$ has an analytic extension in $(\lambda,\sigma)\epsilon\Omega(\rho_0)$ as we claimed.

VI-Conclusions

In this paper we developed a consistent method to unify two hitherto unrelated regularization methods employed to obtain the Casimir energy , the zeta function method and the exponential cut-off method.

Rectangular cavities with Dirichlet boundary conditions in a three dimensional space-time (D=3) were studied.We proved the analytic equivalence between the zeta function method and a variant of the exponential cut-off method for this configurations. The generalization for higher dimensional space-times is straightforward.

It is important to remark that it was showed that the zeta function method perform virtual subtractions of auxiliary configurations which do not display geometry of the space outside the box upon which Dirichlet boundary conditions were imposed.In Casimir's[2] original work and in Fierz[13] and Boyer's[14] papers,once the region outside the plates is the union of two simple connected domains (in fact,two semi-spaces) this kind of problem does not exist and therefore the contribution of the exterior modes are canceled out in the renormalization procedure . We have called the cut-off method employed in the article a variant of the cut-off method because none of the auxiliary configurations employed reproduce in any sense the geometry of the space outside the original cavity. In the case of spherical shells[14][15][16] the cut-off method has been employed with the use of concentrical auxiliary cavities; then one of the auxiliary cavities reproduce the geometry of the space outside the original shell. For D-2 dimensional parallel plates in a D dimensional spacetime such problems do not appear and so it is straightforward to prove the analytic equivalence between the zeta an the exponential cut-off method for these configurations by means of the mixed cut-off method.

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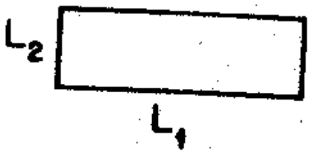

FIG.1.a

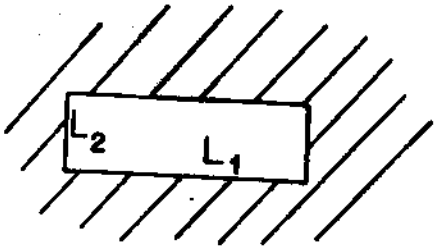

FIG.1.b

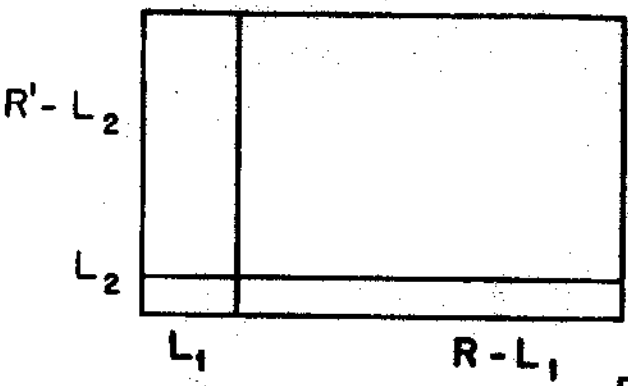

FIG.2.a

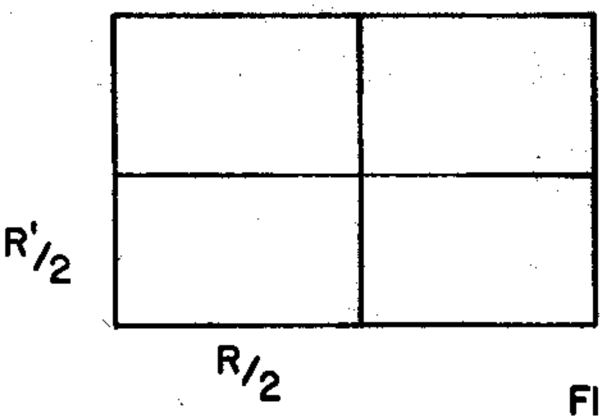

FIG.2.b

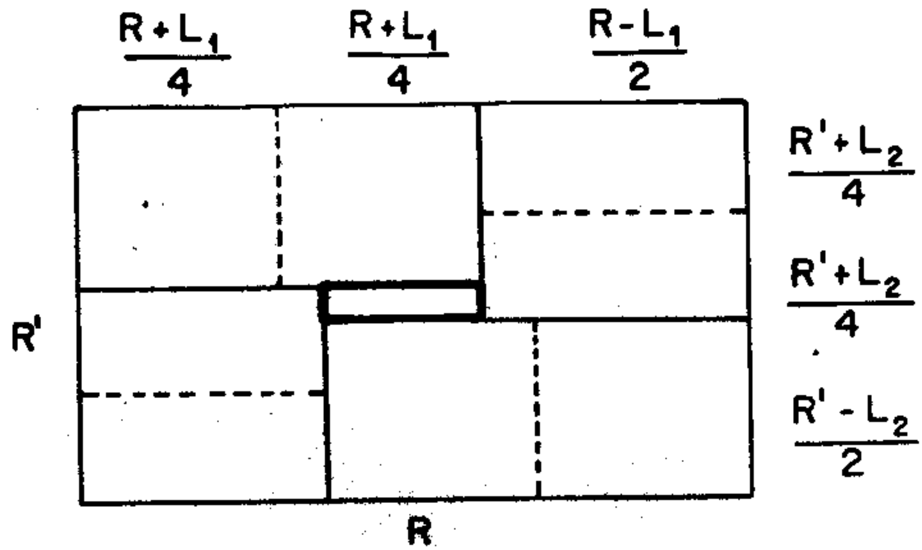

FIG.3.a

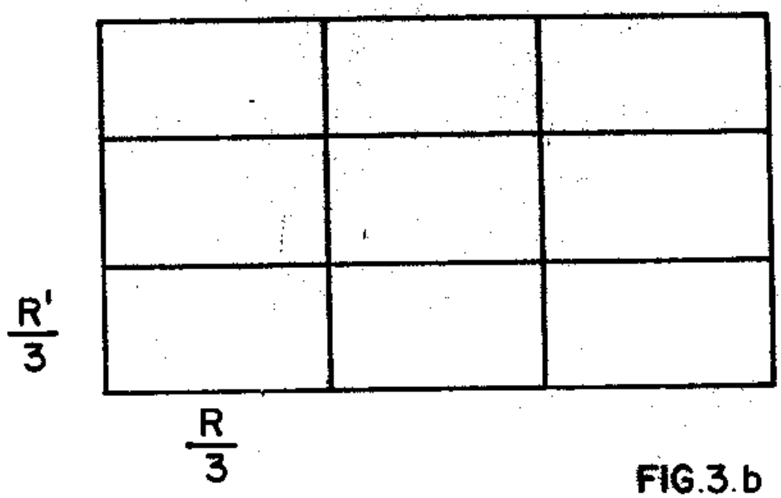

FIG.3.b

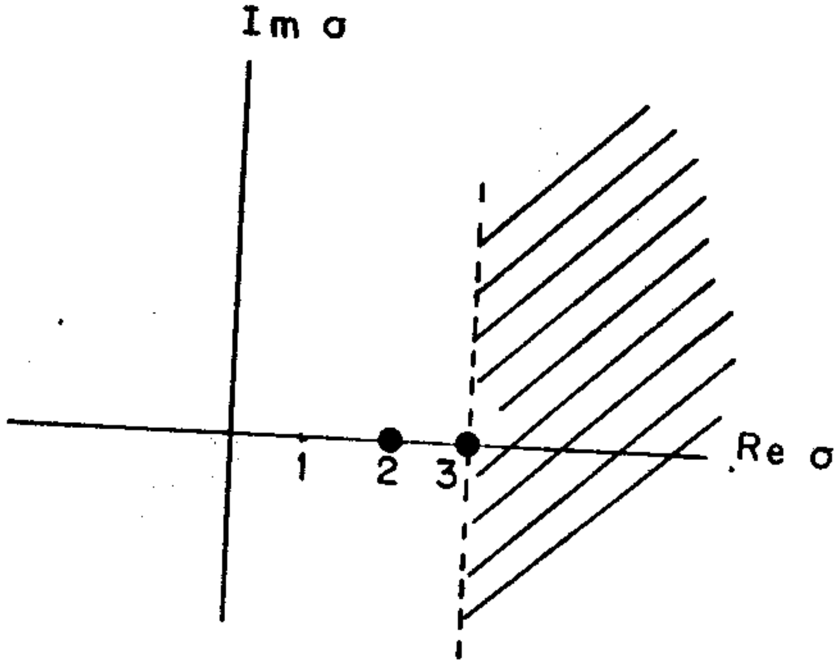

FIG. 4

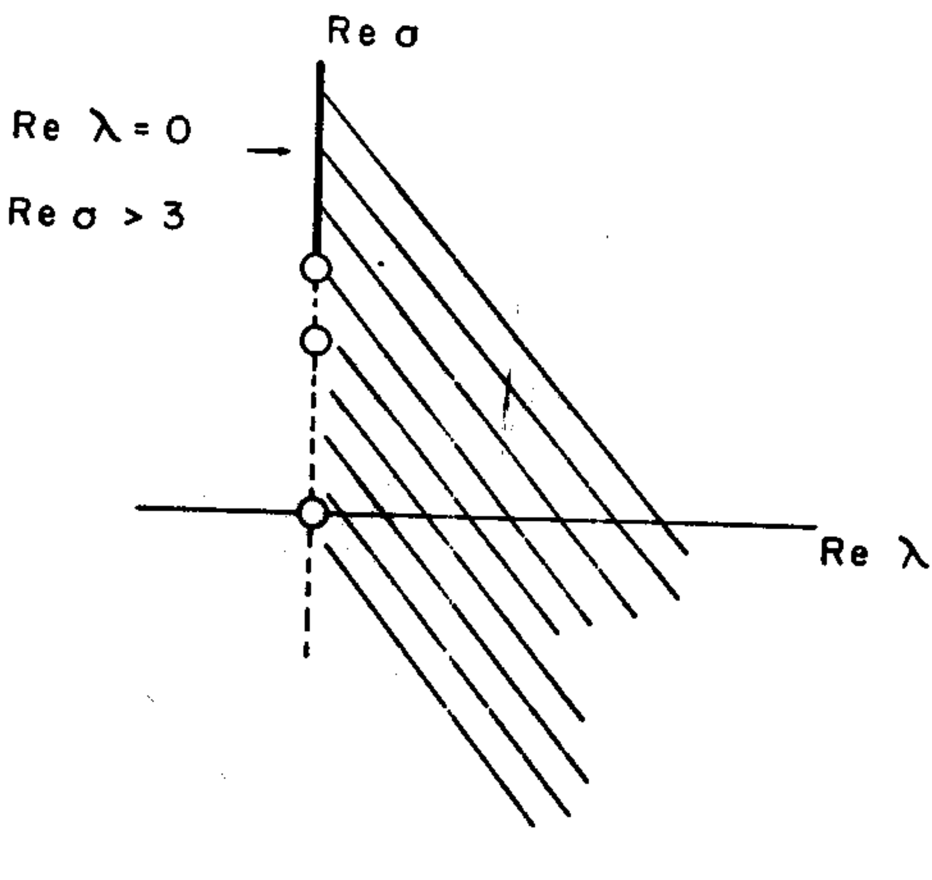

FIG. 5

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