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ON THE PHYSICAL CONTENT OF THE SU(2) SKYRME MODEL

by

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Abstract

The solutions for the Skyrme model in the hedgehog $SU(2)$ representation are analized as a function of the parameters of the model. Sum rules obtained from the Euler-Lagrange differential equations are efficient tools for this analysis, and use is made also of the original Derrick's argument regarding stability in a constructive way. It is found that the evolution of the solutions seem appropriately described by a specific control parameter ϕ, which has distinctive values ϕ_1, ϕ_2, ϕ_3, ... for solutions corresponding to solitons with different integer baryon number 1, 2, 3,$\cdots$ respectively. It is emphasized that a dimensional parameter appears for the regular solutions, which can be taken as the slope of the chiral angle. The Skyrme parameter is shown to have a distinctive rôle, inasmuch as it denotes the instability of the classical soliton solution with integer baryon number. Moreover, the quantization through collective coordinates leads to an expression for the energy that has a well defined stable minimum in terms of the dimensionless Skyrme parameter for the each integer baryon number. The main features for the lower baryon numbers are presented for this solution. The baryon masses are obtained in terms of specific numbers depending on ϕ, the angular momentum and of the pion decay constant, f_π. There is a B=2. I=J=1 baryon with almost twice the value of the mass of the B=1, I=J=1/2 case, but the state B=3, I=J=1/2 is lighter than the former. All baryons turn to be narrower with the increase of B, even the nucleon candidate is essentially more little than the physical nucleon. A scheme of consistent low energy approximation is suggested.

Key-words: Baryons, Chiral Solitons, Non-Linear Sigma Model, Skyrme Model.

1 Introduction

In this work we present what we believe to be a complete study of the solutions to the differential equation of the Skyrme model using the $SU(2)$ hedgehog representation for the unitary chiral field in the sector corresponding to lower baryon numbers. From this work results a physical picture of baryonic states at the $SU(2)$ level of the model which, though not quite accurate for description of realistic states, we suggest might provide the basic starting point for a consistent physical approximation scheme that may bring to a satisfactory description for low energy hadron interactions including baryons and mesons, after new ingredients are appropriately added in several steps to the rudiments with which we deal here.

Our study was performed mainly using the numerical solutions of the non-linear differential equation for the chiral angle obtained with various integration packages and also by codes we devised. They showed satisfactory quantitative agreement, even at rather far asymptotic values. Besides, we applied the analytical solutions by power series valid for both ends of the real half line (i.e, the radial coordinate [1]) and also explored the irregular solution at its lowest orders. These tools allowed to identify properly the parameters that caracterize the solutions of the theory, particularly the physical (solitonic) ones.

We supported the consistency of the whole work by obtaining sum rules from the differential equation which provide useful information about the solutions, specially regarding asymptotic behaviour. In the same spirit, we have applied in a constructive way the original Derrick argument [2] that allowed to establish the conditions for the stability of the solutions to non-linear differential equations in varied number of dimensions of space. In fact, we found it more powerful than we originally thought, since it seems to indicate when a stable solution exists.

It is commonplace in the literature to see references to the Skyrme model as one containing "chiral solitons from the non-linear sigma model stabilized by the Skyrme term". It seems to be understood that the Skyrme term is a device to stabilize a soliton which is already present at the level of the non-linear sigma model lagrangean. Since this lagrangean is the effective lagrangean apt for the current algebra description of the low energy strong interactions, the addition of the Skyrme term would be something as a wanted complement to a sound driving dynamical contribution. Pak and Tze [3] analyzed this in a wider context of current algebra. Our study shows, however, that in the complete model baryonic solitons exist only for a very special relation among its parameters (the pion decay constant, f_π; the dimensionless Skyrme paremeter, e, and a new "size" parameter introduced below for the analytical solutions). When this relations are fulfilled, the contribution of the (complementary ?) Skyrme term to the classical mass of the soliton is equal to the one from the non-linear sigma model. Besides, we confirm the result obtained by Iwasaki and Ohyama [4] that there are not baryonic solutions for the classical pure non-linear sigma model in the hedgehog representation for the chiral unitary field.

Our results made conspicuous the rôle of the Skyrme parameter for the complete model which is completely undetermined at the classical level. As a consequence we were led to reexamine the problem of stability of the solitons at the classical level, and found that in principle the soliton can have any mass value (while preserving its functional stability). Associated to the Skyrme parameter is a dimensional "size" parameter undetermined classically. The "size" parameter can be taken as the slope at the origin of the chiral angle [1].

These features of the model had not been perceived previously [5] - [7], and they are essential for the physical characterization of the baryonic states.

We applied the usual machinery for quantization of the model with collective coordinates. It is found that the rotational contribution to the energy provides stability by allowing for a minimum. This minimal value determines a consistency condition for the value of the Skyrme parameter, and consequently, of the "size" parameter associated with it. The baryonic states at the minimum for different angular momentum (and isospin) labels have no free parameter other than the originally free pion decay constant, f_π.

These baryonic states, because of the constraints among the parameters, have well defined behaviours under changes of values for the latter, depending essentially on the dimension of the quantity of interest. These in distinction with previous work [5] - [7], which kept a residual freedom for the parameters that turned difficult to interpret physical results or its evolution under changes in the values of the parameters.

The quantum baryonic minimum energy states allow, then, to make clear the genuine content of the theory, and makes this approach attractive to evolve in the sense of an approximation scheme for the baryon sector of low energy hadron physics.

All baryons at the minimum quantum energy have 1/4 of its mass coming from the rotation contribution. This tight relationship among the different pieces of the theory has a follow up conclusion: for a given baryon number, all states with different values for their angular momentum (and isospin) keep a precise rate between their masses, determined by the simple ratio of their Casinir angular momentum operator eigenvalues.

All baryons are rather concentrated objects, with typical values of their radii being 0.2, 0.06 and 0.012 fm approximately for B=1, 2, 3 (and f_π= 0.129 Gev), as resulting from their "size" parameters. This, of course, is not in agreement with the observed fact that lower mass nuclei are extended objects with size increasing with the baryon number. Moreover, it appears that the states B=2, I=J=1 is heavier than the one with B=3, I=J=1/2, contrary to observed physics.

We have not been able to establish a link between these lower baryon number states with others referred to in previous work, as claimed by Jackson, Jackson and Pasquier [8] in the $SU(2)$ case or by Balachandran et al. [9] once extension to $SU(3)$ is carried out.

For the B=1 state, the overall set values for the physical quantities of interest show that a value like 0.129 Gev for f_π provide a resonable agreement for the masses of the nucleon (I=J=1/2) and $\Delta(1232)$ (I=J=3/2), but the global agreement is rather poor. The values for the gyromagnetic ratios of the nucleon states have the correct signs, but wrong magnitude and ratio. The choice of that value for the pion decay constant is made to allow comparison with previous works [5] - [7], but its meaning is quite different: for previous work it comes from a fitting procedure, in our case is a pure input parameter, a nice feature coming from the theory, since this parameter regulates the physical values for all physical processes in the current algebra approach to low energy hadron interactions.

The article is setup as follows. In section II we describe the solutions for the non-linear Euler-Lagrange equation for the chiral angle of the Skyrme model in the hedgehog representation for the unitary $SU(2)$ chiral field. We begin by introducing the analytic power series solution at the origin, and show that a dimensional "size" parameter F_1, the slope at origin of the chiral angle, has no determined value from the classical equations. We also show that the structure of the analytic solution is the coefficients for the regular solution for the non-linear sigma model multiplied by a rational function of the dimensionless quantity $\phi = F_1/ef_\pi$. It turns out that ϕ is a "control" parameter for the solutions, descri bing its evolution and labelling the sector with lower baryon numbers. The model has soliton solutions at specific values of ϕ, increasing approximately with the baryon number of the soliton. They are regular solutions at both ends of the real half line, the asymptotic value must differ from the value at the origin by an integer multiple of π. For values not corresponding to this set of specific characteristic values of ϕ, the solutions regular at one end of the half line are irregular at the other, i.e, not described by a power series, but rather by an oscillatig form with decreasing amplitude around an intermediate, half integer value of π. We perform the same study for the asymptotic behaviour of the chiral angle, and verify a behaviour analogous to that at the origin, but the structure of the analytic solution is rather different. The asymptotic solution for the non-linear sigma model appears as the leading contribution, to which others depending explicitly on powers of the Skyrme parameters are added, in a systematic fashion.

Again, the first non null derivative is undetermined, and we give its relationship to F_1 in terms of the behaviour of the chiral angle.

In section III we derive sum rules from the non-linear differential equation analyzed before, by suitable analytic manipulations. We show how the sum rules are quite effective to indicate, from the asymptotic behaviour of the solutions, when a specific value of ϕ related to a given baryon number appears. They complement what can be visually observed from the behaviour of the solutions. Moreover, they provide consistency relations for the parameters of the theory.

In section IV we recall the original Derrick's argument [2] about the stability of the solutions of a non-linear lagrangean theory in any number of dimensions to substantiate the results previously found. We derive a constructive way to use this argument, and the results of its application coincide with the ones found before concerning the rôle of the dimensionless control parameter ϕ. We find that Derrick's argument seems to be more useful, in the sense that it determines when a stable solution exists, more than providing necessary conditions for stability.

We check that the solutions are stable in the functional sense, of its second functional derivative of the action being positive.

When we put together all results: existence of dimensional "size" parameters undetermined at the classical level, of specific values for a control parameters to have solitonic solutions, we are able to show that keeping these specific values fixed the classical mass of the solitons is variable at will with the Skyrme parameter.

In section V we incorporate quantization into the model in the usual manner with resort to collective coordinates. This gives the quantum hamiltonian for the baryonic soliton as the one of a rotating top. From the classical indeterminacy of the Skyrme parameter (and "size") and the classical mass, we prove that the quantum energy for the baryonic soliton has a minimum for a specific value of the Skyrme parameter, which takes the form of a consistency condiction. We extract and calculate several consequences of this, that were mentioned before, and discuss the mass spectrum for the lower baryonic number minimal quantum energy states.

Section VI is devoted to the presentation and comments on the results for the physical quantities of the nucleon state. We emphasize the way the behaviour of the values for these physical quantities is predictable, according to its dimensional character, from the variation of the sole free parameter of the theory, the pion decay constant.

Finally, in section VII we make summary of the work and argue in favour of the view of taking this rudimentary dynamical theory as the germ for the framing a consistent, systematic, chiral dynamic approach to low energy hadron interaction. In our view, f_π may be taken as an overall consistency parameter for the whole scheme, varying its value as new ingredients are included (pion mass and currents, $SU(3)$, etc) until it reaches its physically known value ($f_\pi = 0.186$ Gev).

2 The Solution for the Hedgehog in the Skyrme Model

The lagrangean density for the Skyrme model reads:

$$\mathcal{L} = \frac{1}{16} f_\pi^2 \mathrm{Tr}[(\partial_k U)(\partial_k U^\dagger)] + \frac{1}{32e^2}\mathrm{Tr}[U^\dagger(\partial_k U), U^\dagger(\partial_l U)]^2 \qquad (1)$$

where repeated indices are summed, f_π is the pion decay constant (physically with 0.186 Gev value) and e is the dimensionless Skyrme parameter. Introducing the well known hedgehog ansatz in $SU(2)$:

$$U = \exp\left[i\boldsymbol{n}\cdot\boldsymbol{\tau}F(r)\right] \qquad (2)$$

with $\boldsymbol{n} = \boldsymbol{r}/r$; $r = |\boldsymbol{r}|$, $\boldsymbol{\tau}$ the vector formed with the three Pauli matrices and $F(r)$ the profile function chiral angle, we have for the Euler-Lagrange equation from Eq.(1):

$$\begin{aligned}\left[\frac{1}{4}r^2 + \frac{2}{e^2f_\pi^2}\sin^2[F(r)]\right]\frac{d^2F(r)}{dr^2} + \frac{1}{e^2f_\pi^2}\sin[2F(r)]\left[\frac{dF(r)}{dr}\right]^2 \\ +\frac{1}{2}r\frac{dF(r)}{dr} - \frac{1}{4}\sin[2F(r)] - \frac{1}{e^2f_\pi^2r^2}\sin^2[F(r)]\sin[2F(r)] = 0\,. \end{aligned} \qquad (3)$$

This differential equation is singular at both ends of the real half- line. We shall first consider its regular solution at each end, and then the leading terms for the irregular solutions.

Lets us begin with the origin. A power series solution:

$$F(r) = F_0 + F_1 r + \frac{1}{2!}F_2 r^2 + \frac{1}{3!}F_3 r^3 + \cdots \qquad (4)$$

exists, after substituting in Eq.(3), provided

$$F_0 = n_0\pi, \qquad n_0 \text{ an integer}\,. \qquad (5)$$

The next coefficient, F_1, turns to be undertermined. All even powers have null coefficients, and the odd powers coefficients are written as a power of the first one times a rational function of the dimensionless parameter [1],

$$\phi = \frac{F_1}{e f_\pi}. \tag{6}$$

The results have been published elsewhere [1], they are given here for completeness:

$$F_3 = -\frac{4}{5} F_1^3 \frac{1+2\phi^2}{1+8\phi^2} \tag{7}$$

$$F_5 = \frac{24}{7} F_1^5 \frac{1+\frac{32}{5}\phi^2+\frac{88}{5}\phi^4+\frac{448}{5}\phi^6}{1+24\phi^2+192\phi^4+512\phi^6} \tag{8}$$

$$F_7 = -\frac{544}{15}\frac{N_7(\phi^2)}{D_7(\phi^2)} \tag{9}$$

$$N_7(\phi^2) = 1+\frac{496}{85}\phi^2+\frac{1432}{85}\phi^4+\frac{37008}{85}\phi^6+\frac{203520}{85}\phi^8+\frac{181248}{85}\phi^{10}$$

$$D_7(\phi^2) = 1+40\phi^2+640\phi^4+5120\phi^6+20400\phi^8+32768\phi^{10}$$

$$\vdots$$

and, in geral:

$$F_{2n+1} = C_{2n+1} F_1^{2n+1} \frac{N_{2n+1}(\phi^2)}{D_{2n+1}(\phi^2)} \tag{10}$$

The constants C_{2n+1} are the coefficients for the analytic solution of the pure non-linear sigma model. Notice that F_1 is a kind of "size" parameter, since it gives for the analytic solution the region where the chiral angle varies the more.

The irregular solutions at the origin are found inspired by the work of Iwasaki and Ohyama [4] (from here onwards referred as Iwasaki and Ohyama) for the non-linear sigma model lagrangean density. In fact, in our case, the strongest singularities, related the terms containing the Skyrme parameter, are quite like those of the non-linear sigma model at infinity. We propose:

$$F(r) = m_0\frac{\pi}{2} + \alpha\, r^{\frac{1}{2}}\ \cos(\frac{\sqrt{3}}{2}\ \ln\gamma r) + \cdots \tag{11}$$

with m_0 an odd integer.

Notice that Iwasaki and Ohyama put the dimensional parameters α, γ equal to the undertermined coefficient of the regular solution. This we not do here, though it is an interesting possibility.

Moreover, the ansatz of Eq.(11) only works for the leading singular terms, $O(r^{-3/2})$. The dependence on $r^{-1/2}$ is not cancelled by any improvement we have attempted, and to our knowledge, this is a shortcoming for the ansatz in the non-linear sigma model, also.

We have not been able to follow in detail the behaviour of the irregular solution for quite low values of r, since we can only study it from the analytic solution at infinity.

To handle the equation at infinity, we perform the usual change of variable:

$$r = \frac{1}{\rho}\,. \tag{12}$$

Eq.(3) reads, now:

$$\begin{aligned}&\left[\frac{1}{4}\rho^2 + \frac{2}{e^2 f_\pi^2}\rho^4 \sin^2 K(\rho)\right]\frac{d^2K(\rho)}{d\rho^2} + \frac{4}{e^2 f_\pi^2}\rho^3 \sin^2\left[K(\rho)\right]\frac{dK(\rho)}{d\rho}\\ &+\frac{1}{e^2 f_\pi^2}\rho^4 \sin\left[2K(\rho)\right]\left[\frac{dK(\rho)}{d\rho}\right]^2 - \frac{1}{4}\sin\left[2K(\rho)\right]\\ &-\frac{1}{e^2 f_\pi^2}\rho^2 \sin^2\left[K(\rho)\right]\sin\left[2K(\rho)\right] = 0\end{aligned} \tag{13}$$

where we have taken:

$$K(\rho) = F(r)\,,\quad r \longrightarrow \infty\,. \tag{14}$$

As before, we use a series expansion:

$$K(\rho) = K_0 + K_1\rho + \frac{1}{2!}K_2\rho^2 + \frac{1}{3!}K_3\rho^3 + \cdots \,. \tag{15}$$

Again, to have a regular solution and eliminate the singularities, we need to have:

$$K_0 = n_\infty \pi, \qquad n_\infty \text{ an integer} \,. \tag{16}$$

In this region, odd powers of ρ have vanishing coefficients, and the even ones ($K_4 = 0$) have coefficients written as powers of K_2 (which is completely undertermined from the differential equation) and combinations with powers of ef_π. The first coefficients are:

$$K_6 = -\frac{30}{7}K_2^3 \tag{17}$$

$$K_8 = \frac{1}{e^2 f_\pi^2}(-6720K_2^3) \tag{18}$$

$$K_{10} = \frac{6480}{11}K_2^5 \tag{19}$$

$$K_{12} = \frac{1}{e^2 f_\pi^2}\frac{969\,408\,00}{13}K_2^5 \tag{20}$$

$$K_{14} = -449\,280K_2^7 + \frac{1}{e^4 f_\pi^4}18\,888\,629\,760K_2^5 \tag{21}$$

$$\vdots$$

From this we find that the structure of the series solution is:

$$K(\rho) = \frac{1}{2!}K^0(K_2\rho^2) - \frac{6720}{8!}\frac{\rho^8}{e^8 f_\pi^8}K^1(K_2\rho^2) + \frac{18\,888\,629\,760}{14!}\frac{\rho^{14}}{e^{14} f_\pi^{14}}K^2(K_2\rho^2) - \cdots \tag{22}$$

$$= \frac{1}{2}K^0(K_2\rho^2) - \frac{1}{6}\left(\frac{\rho}{ef_\pi}\right)^8 K^1(K_2\rho^2) + \frac{13}{60}\left(\frac{\rho}{ef_\pi}\right)^{14} K^2(K_2\rho^2) - K_{20}\left(\frac{\rho}{ef_\pi}\right)^{20} K^3(K_2\rho^2) + \cdots \,. \tag{23}$$

The first function $K^0(K_2\rho^2)$ corresponds precisely to the series solution for the pure non-linear sigma model that naturally controls the asymptotic solution for the chiral angle. In the case of unit baryon number, its first coefficient, K_2, is directly related to the axial weak coupling constant, g_A [5] - [7].

The appearance of this new undertermined parameter for the regular solution around infinity was not known in the literature until we uncovered it [1].

The irregular solution in this case is precisely the one proposed by Iwasaki and Ohyama since the leading asymptotic behaviour is given by the non-linear sigma model contribution. That is, we have:

$$K(\rho) = m_\infty \frac{\pi}{2} + \alpha' \rho^{1/2} \cos\left(\frac{\sqrt{7}}{2} \ln \gamma'\rho\right) + \cdots \tag{24}$$

Notice again that there are persistent singularities $O\left(\rho^{-\frac{1}{2}}\right)$ which are not eliminated through quite natural extensions of Eq.(24).

Let us comment for the moment on the main features of the results above.

First, the appearance of dimensional undetermined parameters, F_1 and K_2, for the regular solutions at origin and infinity, respectively. This property is already present for the regular solutions in the Lagrangean for the pure non-linear sigma model [10]. We think that they result from the fact that conditions Eq.(5) and Eq.(14) prevent the fulfillment of the Lipshitz conditions. In several examples in the mathematical literature, when the conditions are not satisfied, undetermined parameters come into play [11, 12]. However, we have not been able to find a positive statement, of the kind "whenever the Lipshitz condition is not satisfied, an undetermined parameter results".

The parameter F_1 is present in the work by Jackson, Jackson and Pasquier [8] through a related quantity, τ. They qualify it as irrelevant, but, as we shall show, F_1 plays an important rôle for the stability of the quantum chiral soliton. The same authors use a parameter A proportional to our ϕ^2, Eq.(6).

That this "size" parameter is important can be seen from another angle. The numerical part of the work by Adkins, Nappi and Witten [5] (from here onwards referred as Adkins, Nappi and Witten) can be practically taken as an attempt to fix the value of F_1 and e, and this they succeed to do through fitting by introducing the mass of the nucleon and delta resonance as starting points and at the price of adjusting also the pion decay constant, f_π.

For the solutions of the Skyrme model with hedgehog, as we shall see, the "natural" dimensionless parameter ϕ introduced in Eq.(6) labels the solutions. This quantity has been named A also [13].

Notice that the analytic solutions at both ends have to be an integer multiple of π; when a solution is analytic at both ends, its baryon number should be integer because of its regularity. It is not imposed by the topological baryon current but rather the reverse.

What about the irregular solutions at the origin and infinity? Are they also controlled by some undertermined parameter? This is quite plausible, since for most values of the parameters, as we shall describe below, to a regular solution in one end there corresponds an irregular one at the other, as numerical integration of the differential equation demonstrates. We have not carried out in detail our analysis of the link between regular and irregular solutions starting from the numerical solutions, but there seems to be a continuous variation with the parameters e and F_1, when working with the physical radius r, or, ϕ, when working with the dimensionless radius $\tilde{r}$ (see below). (Eq.(27)). We present in Fig. 1 the variation of the amplitude and phase at infinity for the irregular solution as a function of ϕ.

Related to these parameters are the "natural" variables of the problem. From the form of the power series near the origin (Eq.(4)) and "near" infinity ($\rho \sim 0$, Eq.(15)) it looks like having as candidates:

$$s = F_1 r \tag{25}$$

$$\eta = K_2 \rho^2 . \tag{26}$$

There is a commonly used natural variable, introduced originally by Adkins, Nappi and Witten:

$$\tilde{r} = e f_\pi r . \tag{27}$$

It is easy to see the relationship to Eq.(26):

$$s = F_1 r = \phi \tilde{r} \tag{28}$$

Notice that ϕ is the slope of the profile function in terms of $\tilde{r}$. (See Eq.(4)). At infinity, other equivalent dimensionless variable is familiar, but, of course, a quite natural choice would be:

$$\tilde{\rho} = \frac{\rho}{e f_\pi} \tag{29}$$

Associated to this variable, a dimensionless parameter analogous to ϕ is $\kappa \equiv K_2 (e f_\pi)^2$.

We have performed the integration of the differential equations Eq.(3) and Eq.(13) by several computer routine packages and also by assuming the knowledge of the regular solutions Eqs.(8) - (10) and Eqs.(18) - (22) and integrating by steps. Both methods agree to better than 1% even quite far from the initial point (typically, $\tilde{r} = 1000$).

We have further used the numerical integration to test the sum rules which can be obtained from the differential equation with moderate assumptions on the first derivative, and also to test the application of Derrick's theorem, as is shown in later sections.

We have looked for the solutions varying F_1 and e (or K_2 and e) or ϕ (or κ). The solutions change continuously with these parameters, and the irregular solutions linked to the regular ones appear to oscillate with a decreasing amplitude with the correct power.

The results can be described as follows, in terms of the dimensionless radial variable $\tilde{r}$. For values of ϕ between 0 and $1.00376\cdots$ the solutions regular at origin with value $F(0) = -\pi$ grow and oscillate around $-\pi/2$ as $\tilde{r}$ goes to infinity.

A regular solution at infinity, with $K(0) = 0$ decreases and begins to oscilate with decreasing amplitude tending to $K \simeq -\pi/2$ as $\tilde{\rho}$ goes to infinity. These are represented in Fig. 1 , not to scale.

As ϕ approaches the value $1.00376\cdots$ the oscillation at large values for $\tilde{r}$ is slower and its amplitude grows but it still decreases with $\tilde{r}$ for the solution regular at $\tilde{r} = 0$. For the special value $\phi = 1.00376\cdots$ the solution is smooth at both ends of $\tilde{r}$ and it is zero at infinity. This is the value used for Adkins, Nappi and Witten and many other authors to work out the dynamical properties of the nucleon [5, 7].

For values of the ϕ in the range $1.00376\cdots < \phi < 1.9650\cdots$ the regular solution at the origin beginning at $-\pi$ increases until it begins to oscillate around $F(\tilde{r}) = \pi/2$. Analogously, the solution regular at infinity where its value is zero decreases and oscillates around $K(\rho) = -3\pi/2$. Only at the value $\phi = 1.9650\cdots$ the solution is regular at both ends and jumps 2π between them. The situation looks as the one for the solution near the origin. Fig. 1 illustrates the behaviour for both solution.

Physically, the baryon number of the hedgehog soliton is given by

$$B = n_0 - n_\infty \tag{30}$$

and, as was apparently already known [8], there are solutions with well defined values of B $(1,\ 2,\ 3,\cdots)$ only for well determined values of ϕ ($\phi_1 = 1.00376\cdots$, $\phi_2 = 1.9650\cdots$, $\phi_3 = 2.8882\cdots$). These values of ϕ are usually determined through the use of integration routines calculating by "shooting" procedures [5] - [7]. We can obtain precise values using sum rules or Derrick's theorem, as will be shown below.

One can also work with the physical variable, r, to confirm the result that ϕ is the control parameter for the solutions (or κ). Incidentally, there is a corresponding value of κ for each value of ϕ related to a given value of the baryon number ($\kappa_1 = 17.277\cdots$, $\kappa_2 = 51.673\cdots$, $\kappa_3 = 103.8239\cdots$, numbers which seem to suggest $\frac{\kappa_{B+1}}{\kappa_B} = \frac{B+2}{B}$).

Physically, only solutions with the special values of ϕ are interesting. However, what is the criterium to select one value for the couple (F_1, e)? This is precisely what Adkins, Nappi and Witten did , as referred previously. Giving the value of ϕ, selecting two values out of f_π, e and F_1 is required. The two numbers chosen to define the situation seem to have been f_π and e, and they were fixed from the masses of the nucleon and the $\Delta(1232)$ (with a convenient, unphysical value of 0.129 Gev for f_π).

It can be argued which couple of values is more meaningful. In what follows, our choice is to fix the physical values for f_π and let e and F_1 to be chosen from a given criterium. As we shall show, there is a natural criterium coming from the stability of the mass of the soliton after quantization.

We can establish a formal link between the values of the control parameters ϕ and κ. We chose $\sim\ \pi$ as the starting point for $F(\tilde{r})$ at the origin and asymptotic value for $K(\tilde{\rho})$. For a regular solution with baryon number n_B at sufficiently small values of $\tilde{r}$ and $\tilde{\rho}$, knowing the numerical values of $F(\tilde{r})$ and $K(\tilde{\rho})$, we have:

$$F_B(\tilde{r}\ \sim\ 0) = -\pi + \phi_B\tilde{r} + O(\tilde{r}^3) \tag{31}$$

$$F_B(\tilde{r}\ \sim\ \infty) = K_B(\tilde{\rho}\ \sim\ 0) = (n_B - 1)\pi - \frac{1}{2}\kappa_B\tilde{\rho}^2 + O(\tilde{\rho^6})\,. \tag{32}$$

We can safely write, subtracting both expressions:

$$\kappa_B = \frac{2}{\tilde{\rho}^2}\left\{-\phi_B\tilde{r}\ +\ n_B\pi\ +\ [F_B(\tilde{r}\ \sim\ 0) - F_B(\tilde{r}\ \sim\ \infty)]\right\}\,. \tag{33}$$

Knowing $\tilde{r}$, $\tilde{\rho}$, $F_B(\tilde{r})$, n_B and ϕ_B, κ_B is determined. The values so obtained for κ_B confirm the results obtained numerically from the limit

$$\kappa_B = \lim_{\tilde{r}\to\infty} \tilde{r}^2\left[(n_B - 1)\pi - F(\tilde{r})\right] \tag{34}$$

to several places.

The values for κ_B do not show the regularity with n_B that seems to be valid for ϕ_B. Probably this reflects the different structures of the analytical solutions (Eqs.(8) - (10) , Eqs.(18) - (22) and Eq.(23)).

A final comment: unphysical solutions are those which are irregular at one end, since they correspond to half integer baryon number and, worse, they have divergent values for the mass and momemt of inertia of the solution. There seems to be however solutions which are regular around half integer values of π at both ends of $\tilde{r}$ corresponding to the values $\phi_1, \phi_2, \cdots$ of ϕ. They would have integer baryon number, but would have divergent values for the classical mass and moment of inertia. They seem to appear as a consistent requirement since for non-special values of ϕ there seems to be always two solutions.

3 Sum Rules for the Solutions of the Differential Equation

The use of shooting methods to solve the differential equation Eq.(3) and Eq.(13) is a fast and reliable way to obtain the value for the parameter ϕ that labels a classical barionic solutions.

Procedures integrating by steps in the independent variable, as the ones we have used, let you know when a significant ϕ value for integer baryon number has been attained only visually after solution jumps from a half integer value of π to the next.

Iwasaki and Ohyama developed sum rules for the non-linear sigma model which allowed to undestand the main features of the solutions in the asymptotic region. The sum rules can be seen also as a consistency check for the solution.

Let us show the derivation of an analogous set of sum rules for the Skyrme model. Let us multiply the differential equation Eq.(3) by $r^{2N}\ dF(r)/dr$, where

N is an integer, and then integrate all terms between a pair of values r_1 and r_2 ($r_1 < r_2$). In several terms, integration by parts may be used, and we have, at the end:

$$\frac{1}{2} r^{N+4} \left[\frac{dF(r)}{dr}\right]^2 \Bigg|_{r_1}^{r_2} - \frac{1}{2} N \int_{r_1}^{r_2} dr\, r^{N+3} \left[\frac{dF(r)}{dr}\right]^2 - r^{N+2} \sin^2[F(r)] \Big|_{r_1}^{r_2}$$
$$+ (N+2) \int_{r_1}^{r_2} dr\, r^{N+1} \sin^2[F(r)] + \frac{1}{e^2 f_\pi^2} \left\{ 4\, r^{N+2} \sin^2[F(r)] \left[\frac{dF(r)}{dr}\right]^2 \Bigg|_{r_1}^{r_2} \right.$$
$$-4\,(N+2) \int_{r_1}^{r_2} dr\, r^{N+1} \sin^2[F(r)] \left[\frac{dF(r)}{dr}\right]^2 - 2 r^N \sin^4[F(r)] \Big|_{r_1}^{r_2}$$
$$\left. +2N \int_{r_1}^{r_2} dr\, r^{N-1} \sin^4[F(r)] \right\} = 0\,. \tag{35}$$

This complicated expression merits some comments:

i) When multiplied by ef_π to the power $N+2$, it goes immediatly into an expression in terms of the dimensionless variable $\tilde{r}$;

ii) Changing variable to $\rho = 1/r$ provides an almost identical expression with $K(\rho)$ instead of $F(r)$, which is precisely the kind of sum rule coming from Eq.(13);

iii) In the limit $e \longrightarrow \infty$, we recover the expressions used in the non-linear sigma model by Iwasaki and Ohyama.

It is interesting to look at the limits when r_1 is let to zero and r_2 grows indefinitely. This we shall not do in general, but only for the special cases with $N = -2$ and $N = -4$, where some terms cancel from the start (in $N = -2$), and some integrals are rapidly convergent at infinity (for $N = -4$). By the way, these are the cases studied for the non-linear sigma model by Iwasaki and Ohyama.

We assume that the value at the origin of the chiral angle is $-\pi$,

$$F(0) = -\pi \tag{36}$$

as required for the regularity of the Euler-Lagrange equation Eq.(3) [1, 7, 10]. We require also that, at the origin, the first derivative is finite,

$$F(r) \stackrel{r\to 0}{\longrightarrow} const\ . \tag{37}$$

Asymptotically, we also demand the solution to be finite:

$$F(r) \stackrel{r\to\infty}{\longrightarrow} const \tag{38}$$

which covers the case of regular and irregular solutions.

In order to make sense of the integrals involved, it is needed that at infinity the first derivative of the solution is zero, which is consistent with the preceding condition:

$$\frac{dF(r)}{dr} \stackrel{r\to\infty}{\longrightarrow} 0\ . \tag{39}$$

Notice that the last three conditions are fulfillled by the regular and irregular solutions found in the previous section.

We have, then, the following results for $N = -2$ and $N = -4$:

$$\sin^2\left[F(r\to\infty)\right] = \int_0^\infty dr\ r\ \left[\frac{dF(r)}{dr}\right]^2 - \frac{4}{e^2 f_\pi^2}\int_0^\infty dr\ \frac{\sin^4\left[F(r)\right]}{r^3} \tag{40}$$

$$\begin{aligned}
&\left[\frac{dF(r\to 0)}{dr}\right]^2 + \frac{8}{e^2 f_\pi^2}\left[\frac{dF(r\to 0)}{dr}\right]^4 \\
&= 4\int_0^\infty dr\ \frac{1}{r}\ \left[\frac{dF(r)}{dr}\right]^2 - 4\int_0^\infty dr\ \frac{\sin^2\left[F(r)\right]}{r^3} \\
&\quad + \frac{16}{e^2 f_\pi^2}\int_0^\infty dr\ \frac{\sin^2\left[F(r)\right]}{r^3}\left\{\left[\frac{dF(r)}{dr}\right]^2 - \frac{\sin^2\left[F(r)\right]}{r^2}\right\}\ .
\end{aligned} \tag{41}$$

The first equation informs about the asymptotic behaviour for the chiral angle, and is the analog of the Eq.(14) of Iwasaki and Ohyama for the non-linear sigma model, which it reproduces when $e \longrightarrow \infty$.

We immediately see the possibility of a result which differs from the case of the sigma model, where, unavoidably, the right hand side being positive definite the only solution was $F(r \to \infty) = -\pi/2$ (that is, no finite energy soliton with integer baryon number).

We study the sum rule by integrating carefully the numerical solution for the differential equation as ϕ grows from zero (the sigma model limit). The integration is performed by simple summing of rectangles with base the step of integration and height the value obtained for the quantities in the numerical integration of the differential equation.

The result is sketched in Fig. 2 for the left and right-hand sides as functions of ϕ. For most values of ϕ, the modulus of the sine is 1, except for a few particular values near to integers, as mentioned above. In practice, in the integration, at the vicinity of one of these values, a gentle dip appears. If care is taken in the integration, reducing the steps of the calculation and sweeping the region of ϕ with detail, it turns out that the dip goes almost to zero, at the same time that its border gets neat and narrower. In the process, more figures for ϕ give better results, in the sense that the cancellations are more complete. The numbers, however, are quite sensible. If one changes the interval of integration in the right hand side to $[0,1]$ by means of a change of variable:

$$\tilde{r} = \frac{u}{1-u} \tag{42}$$

the results do not match further than three decimal figures. The results for the chiral angle are those described previously, and represented in Fig. 2.

For what regards the second sum rule, it is not as useful as the first one, but it is important as a check of consistency between the input on the left hand side and the global behaviour of the solution in the right hand side. Its validity helps to support the idea that there is a continuous variation of the solutions, regular and irregular, with the parameters of the theory.

The numbers obtained for the unit baryon number ($\phi = 1.003764...$ [14]) can be compared to those obtained by G. S. Adkins [7], and Ananias and Ferreira [13].

As we mentioned before, the existence of this special value when combined with the indefinite character of the parameters e and F_1 allows to demonstrate that the classical mass for the Skyrme model is not stable as we show below. It is only at the quantum level that both properties combine to provide a well defined condition for a state of definite minimun energy.

For $B = 2$ there are candidates proposed by Jackson, Jackson and Pasquier [8] and Balachandran et al. [9]. The last authors find a $B = 2$ state only by appealing to an $SU(3)$ imbedding of the chiral soliton, and the former, propose that a $B = 2$ state in the $SU(2)$ framework, spherically symmetric, is either unstable (with mass three times that of the $B = 1$ state) or it is not a minimum but a maximum of the action. In our case, we have explored the stability equation for $B = 2$ and found that we are dealing with a stable solution. The stability condition is analyzed in the next section.

Higher B numbers do not seem to be of strong physical interest, but they are just a feature of the solutions of the differential equation for the Skyrme model, and may be of some use for the description of nuclei.

4 Classical Functional Stability, Derrick's Argument and the Instability of the Soliton

The Euler-Lagrange equation for the Skyrme model in the hedgehog configuration, Eq.(3), is not clearly a functionally stable solution. By this we mean that the expression for the functional second derivative of the lagrangean is not explicitly positive definite.

To show this, we write for a general chiral angle a Taylor expansion around the solution of the Euler-Lagrange equation:

$$F(r) = F_0 + \varepsilon\, u_1(r) + \frac{1}{2}\varepsilon^2 u_2(r) \tag{43}$$

Here, ε is an infinitesimal parameter, $F_0(r)$ is a solution for Eq(3), and $u_1(r)$ and $u_2(r)$ are suitable smooth localized functions, vanishing at the limits of integration for the lagrangean. Inserting $F(r)$ in Eq(1), we have for the action functional:

$$L\{F\} = \int \mathcal{L}\, d^3x = L_0\{F_0\} + \varepsilon\, L_1\{F_0\} + \frac{1}{2}\varepsilon^2 L_2\{F_0\}\ . \tag{44}$$

The extremum condition

$$L_1\{F_0\} = 0 \tag{45}$$

produces the Euler-Lagrange equation we are discussing (Eqs.(3) and (13)). The quadratic term coming from Eq(43) into Eq(44) is:

$$L_2\{F_0,\, u_1,\, u_2\} = \frac{1}{2}\pi\frac{f_\pi}{e}\int_0^\infty d\tilde{r}\left\{\left[\frac{1}{4}\tilde{r}^2 + 2\sin^2[F_0(\tilde{r},\phi)]\right]\left[\frac{du_1(\tilde{r})}{d\tilde{r}}\right]^2\right.$$
$$+\left[\frac{1}{2}\cos[2F_0(\tilde{r},\,\phi)] + \frac{1}{\tilde{r}^2}\sin^2[2F_0(\tilde{r},\,\phi)] + \frac{2}{\tilde{r}^2}\sin^2[F_0(\tilde{r},\,\phi)]\cos[2F_0(\tilde{r},\,\phi)]\right.$$
$$\left.\left.-\,2\sin[2F_0(\tilde{r},\,\phi)]\frac{d^2F_0(\tilde{r},\,\phi)}{d\tilde{r}^2} - 2\cos[2F_0(\tilde{r},\,\phi)]\left(\frac{dF_0(\tilde{r},\,\phi)}{d\tilde{r}}\right)^2\right]u_1^2(\tilde{r})\right\}\ . \tag{46}$$

The second term in square brackets is not positive definite. The reader may ask about the function $u_2(r)$. It happens that its contribution to L_2 is exactly of the type that renders L_1 equal to zero in Eq(45).

Notice that in the limit $e \to \infty$ we have an undetermined result, since $F_0 \overset{r\to\infty}{\to} \pi/2$. Notice also that for $\phi \neq \phi_k$, the integrals always diverge, making L_2 meaningless, because of the term in $(du/d\tilde{r})^2$.

We have analyzed for B=1, 2, 3 the behaviour of L_2 for functions u_1 which were non negligible and finite in a finite portion of the real axis. The first term in Eq(44), proportional to the square of the derivative of u_1, overwhelms always the second. For the asymptotic region, the first term $(\tilde{r}^2)$ is by large the most important. Asymptotically, the second bracket receives contribution mainly from the first term $\cos[2F_0(\tilde{r},\phi)]$. Nowhere the integrand is negative, even when the second bracket is at values of $F_0(\tilde{r}) \simeq \pi/2$.

Another aspect of the stability problem for solitons was raised almost immediately to the work by Skyrme. It was challenged by Derrick [2] in a simple argument which we reproduce.

Consider a system with coordinates θ in (spatial) dimension N, such that its energy reads:

$$E_\lambda = \int \left[(\nabla\theta_\lambda)^2 + f(\theta_\lambda) \right] d^N x \tag{47}$$

where $\theta_\lambda(\boldsymbol{x}) = \theta(\lambda \boldsymbol{x})$. Notice that θ is a solution for the dynamical problem of the system. If one performs a scale change in the independent variable

$$\boldsymbol{x}' = \lambda \boldsymbol{x} \tag{48}$$

we have, for E_λ:

$$\begin{aligned} E_\lambda &= \int \left[\lambda^2 (\nabla\theta)^2 + f(\theta)\right] \lambda^{-N} d^N x' && (49) \\ &= \lambda^{2-N} \int (\nabla\theta)^2 d^N x' + \lambda^{-N} \int f(\theta) d^N x' \\ &= \lambda^{2-N} I_1 + \lambda^{-N} I_2 && (50) \end{aligned}$$

In the neighbourhood of the solution ($\lambda = 1$), the conditions for stability of the solution θ are:

$$\left.\frac{dE_\lambda}{d\lambda}\right|_{\lambda=1} = (2-N)\, I_1 - N\, I_2 = 0 \qquad (51)$$

$$\left.\frac{d^2E_\lambda}{d\lambda^2}\right|_{\lambda=1} = (2-N)(1-N)\, I_1 + N(N+1)\, I_2 > 0\,. \qquad (52)$$

In our case, we have for the classical mass that N=1,

$$M_0 = E = \frac{1}{2}\pi\, f_\pi^2 \int_0^\infty dr\, r^2 \left\{ \left[\frac{dF(r)}{dr}\right]^2 + \frac{2}{r^2}\sin^2[F(r)] + \frac{4}{e^2 f_\pi^2}\sin^2[F(r)] \left(2\left[\frac{dF(r)}{dr}\right]^2 + \frac{\sin^2[F(r)]}{r^2} \right) \right\}. \qquad (53)$$

After performing the change for the independent variable

$$E_\lambda = f_\pi^2 \left\{ \frac{I_1}{\lambda} + \lambda\, I_2 \right\}\,. \qquad (54)$$

At the minimum:

$$\lambda^2 = \frac{I_1}{I_2} \qquad (55)$$

$$\frac{d^2E_\lambda}{d\lambda^2} = 2I_2 \left(\frac{I_1}{I_2}\right)^{\frac{1}{2}} > 0\,. \qquad (56)$$

For the Skyrme soliton $\lambda = 1$ and from Eq.(55) we have

$$I_2 = I_1\,. \qquad (57)$$

For a stable soliton, the contribution of the Skyrme term to the classical mass is equal to that of the non-linear sigma model. At the same time, Eq(56) is satisfied only for $I_2 > 0$.

We have verified numerically that the condition Eq(57) is only satisfied by the solutions of the Skyrme model with integer baryon number.

Moreover, notice that for the pure non-linear sigma model the question of stability is crucial. Without the Skyrme term, there is no soliton solution amenable to treatment as a nucleon state.

Our results seems to indicate that the argument by Derrick is more powerful than originally proposed. It is usually taken as an extra condition on an already existing solution; from our work it looks that it indicates when a solution exists, and, besides, it is stable.

We are now ready to establish an important result for the baryonic solitons of the model at the classical level. Namely, we have asserted that the solutions for these cases are stable from the point of view of the action functional, and also from the application of Derrick's argument. Both studies were performed numerically, no special property was used of the solutions.

We now take into account that the regular solutions depend in an indetermined way on the parameter F_1, or, equivalently, on the value of the Skyrme parameter. Let us write the contributions to the classical mass coming from the non-linear sigma model term and from the Skyrme term using the dimensionless variable $\tilde{r}$. They result:

$$M(sigma\ model) = \frac{1}{2}\pi\frac{f_\pi}{e}\int_0^\infty dr\ \tilde{r}^2\left\{\left[\frac{dF(\tilde{r},\ \phi)}{d\tilde{r}}\right]^2 + \frac{2}{\tilde{r}^2}\sin^2\left[F(\tilde{r},\ \phi)\right]\right\} \quad (58)$$

$$= \frac{1}{2}\pi\frac{f_\pi}{e}a_1(\phi) \quad (59)$$

$$M(Skyrme\ term) = \frac{1}{2}\pi\frac{f_\pi}{e}\int_0^\infty dr\ \left\{4\sin^2\left[F(\tilde{r},\ \phi)\right]\left[2\left(\frac{dF(\tilde{r},\ \phi)}{d\tilde{r}}\right)^2 + \frac{2}{\tilde{r}^2}\sin^2\left[F(\tilde{r},\ \phi)\right]\right]\right\} \quad (60)$$

$$= \frac{1}{2}\pi\frac{f_\pi}{e}a_2(\phi)\ . \quad (61)$$

The pure numbers coming from the integrals, $a_1(\phi)$ and $a_2(\phi)$, are equal for a baryonic soliton solution, from Derrick's argument, Eq(57). So, at the end, we have

$$M = M(classical) = \pi\frac{f_\pi}{e}a_1(\phi)\ . \quad (62)$$

As long as we keep ϕ fixed at a special integer baryon number value, we can vary e at will and have any value for M. This is precisely the kind of argument used long time ago to demonstrate the instability of an eventual soliton from the non-linear sigma model (which has none). By the way, at the limit $e \to \infty$ the non-linear sigma model cannot have a soliton and $M \to 0$.

This result translates the corresponding indeterminacy of the dimensional parameter F_1 at the classical level, and we enunciate it clearly again: in the $SU(2)$ Skyrme model with the hedgehog representation for the chiral angle, the value for the classical mass of a soliton with integer baryon number is completely undetermined.

5 Quantization of the Skyrme Model: The Stable Baryon

The problem of the correct introduction of quantization for solitons deserves considerable attention [15, 16]. In the realm of the study of baryons as solitons, the current procedure is rather heuristic and limited in scope. It only pretends to provide an approximate description for the lower energy configurationa, and it would be necessary, and even urgent, to devise a more formal justification for it [17].

The starting point is to recognize that all configurations obtained from the hedgehog through an isospin rotation are of the same finite energy. It can be expected that, for the lower energy states, flavour rotation generate modes that approximate them. The rotations are introduced as collective, time dependent coordinates, and are quantized applying standard procedures. In more concret terms, given a hedgehog type solution to the Euler-Lagrange equation Eq.(3) for the Skyrme model, with a ϕ parameter corresponding to a well defined baryon number, we approximate a time dependent solution by:

$$\begin{aligned} U(\boldsymbol{r},\, t) &= A(t)U(\boldsymbol{r})A^{\dagger}(t) \\ &= \cos\left[F(r)\right] + i\tau_k D_{kl}(t)\, n_l \sin\left[F(r)\right] \end{aligned} \tag{63}$$

with $A(t) \in SU(2)$ and D_{kl} a rotation matrix in the fundamental representation. The lagrangean Eq.(1) with this field $U(\boldsymbol{r},\ t)$ instead of the original static one, reads:

$$L\left\{U(\boldsymbol{r},\, t)\right\} = -M_0 + \theta\ \mathrm{Tr}[\partial_0 A\ \partial_0 A^{\dagger}] \tag{64}$$

where ∂_0 indicates time partial derivative,

$$\theta = \frac{2}{3}\pi \, \frac{1}{e^3 f_\pi} \int_0^\infty d\tilde{r} \, \tilde{r}^2 \sin^2 [F(\tilde{r}, \, \phi)] \left\{ 1 + 4 \left(\left[\frac{dF(\tilde{r} \, \phi)}{d\tilde{r}} \right]^2 + \frac{\sin^2 [F(\tilde{r}, \, \phi)]}{\tilde{r}^2} \right) \right\} \quad (65)$$

and M_0 is the classical mass, Eq.(53). The quantization of this lagrangean is well known and is given in several articles ([5] - [7]). Let us recall two main results. First. the states of the quantum hamiltonian for the $SU(2)$ hedgehog should be labelled by the same eigenvalue of the angular momentum, J^2, and of isospin, I^2. The second is that the hamiltonian turns to be the one of a rigid rotator, with energies given by:

$$E = M + \frac{J^2}{2\theta} \, . \quad (66)$$

Notice that both M and θ are functionals of the chiral angle $F(\tilde{r}, \, \phi)$ from the classical description. For quantization of the solitons as a fermion the eigenvalue of the angular momentum should be half-integer.

The main point here is that, as shown from the explicit expressions for M and θ, the dependence of E above on e (or F_1) allows for a minimum. To wit:

$$\begin{aligned} M &= M_1 + M_2 \\ &= \frac{1}{2}\pi \, \frac{f_\pi}{e} \, [a_1(\phi) + a_2(\phi)] \end{aligned} \quad (67)$$

and θ is given above. Writing

$$\theta = \frac{2}{3}\pi \, \frac{1}{e^3 f_\pi} b(\phi) \quad (68)$$

the expression for the energy now reads:

$$E = \frac{1}{2}\pi \frac{f_\pi}{e} \, [a_1(\phi) + a_2(\phi)] + \frac{1}{2} J^2 \frac{e^3 f_\pi}{\left[\frac{2}{3}\pi \, b(\phi)\right]} \, . \quad (69)$$

We recall that for a baryon solution ϕ is fixed. For the nucleon and the Δ states, the eigenvalues of the J^2 are j=1/2 and 3/2 respectively.

It is now simple to find the minimum as a function of F_1:

$$\frac{dE}{dF_1} = 0 \tag{70}$$

and we get, taking into account that $a_2(\phi) = a_1(\phi)$ at the special ϕ value,

$$e^4 = \frac{4}{9}\frac{\pi^2}{j(j+1)}a_1(\phi)b(\phi) \tag{71}$$

or, equivalenttly, from the definition of ϕ:

$$F_1^4 = \frac{4}{9}\frac{\pi^2}{j(j+1)}\phi^4 a_1(\phi)b(\phi)f_\pi^4\,. \tag{72}$$

Eq.(71) above for e, the Skyrme parameter, may be considered as a self consistency condition for the model. It is an expression written in terms of quantities all internal to the description by the hedgehog. That the classical description and its quantum transcription are equally taken into account is witnessed by Eq.(71), where $a_1(\phi)$ comes from the purely classical treatment of the model, while $b(\phi)$, which is defined classically, intervenes only at the quantum level, after quantization of the rotation degrees of freedon introduced by quantization.

Since we are working with expressions for definite baryon number, the parameter ϕ links the Skyrme parameter with the "size" parameter F_1. This is the content on Eq.(72)

When substituting Eq.(71) in Eq.(66), the final result for the energy at the quantum (stable) minimum is:

$$E = \left(\frac{8}{3}\pi\right)^{1/2}[j(j+1)]^{1/4}\left[\frac{a_1^3(\phi)}{b(\phi)}\right]^{1/4} f_\pi\,. \tag{73}$$

Notice that appart from the value of j the dependence of this result is all on the parameter ϕ. The quantity f_π sets the scale for the minimum quantum energy.

The appearance of the minimum for the quantum energy is the most prominent feature of the formalism of quantization. The only ingredient added to the usual procedure is that the value of the parameter ϕ corresponding to baryonic solutions with B=1, 2, 3, $\cdots$ is taken into account.

We interpret the state at minimum energy as indicating the stable configuration in the quantum domain at this level of the dynamics. It might be, after all, an artifact ensuing from the rather primitive quantization procedure generally adopted, but in any case it is consistent and calls for the development of more accurate formalism for quantization, as stressed at the beginning of this section.

These important characteristics of the theory have not been properly accounted for by other work on the subject ([5] - [7]). Three quantities are introduced in the model: f_π, e and F_1. Together, they form the dimensionless parameter ϕ. Former work placed not at the quantum minimum, and two quantities had to be fixed (ϕ is selected from the need of having integer baryon number). Ignoring the relevance of F_1, the values of e and f_π had to be chosen by fitting, using as input the spins and masses of known baryonic states, in particular the masses of the nucleon and $\Delta(1232)$ isobar.

In our view, f_π should be taken to be properly defined at the level of the complete theory of hadron interactions at lower energies. This should include also massless pion-pion scattering, and at each level of addition of new dynamical ingredients, the value of f_π should compatibilize all numerical results. The other two meaningful quantities, e and F_1, are proper to the model.

It is now a matter of giving numerical values to the relevant quantities for the nucleon state, the $\Delta(1232)$ and even to play with baryon numbers bigger than one, to look for the results.

Without further work, however, we can get a value for the ratio of the masses of the $\Delta(1232)$ and the nucleon:

$$\frac{E(i=j=\frac{3}{2})}{E(i=j=\frac{1}{2})}=5^{1/4}=1.4953\cdots . \qquad (74)$$

For comparison, the experimental number is:

$$\frac{M(\Delta(1232))}{M(N(938))}=1.313 \qquad (75)$$

The agreement is better than 20 %, and is an exact prediction from the Skyrme model.

Another interesting ratio, at this level, is the one between the contribution of rotation and the static classical results:

$$\frac{\frac{3}{4}\frac{j(j+1)f_\pi e^3}{\pi b(\phi)}}{\pi\frac{f_\pi}{e}a_1(\phi)}=\frac{1}{3} \qquad (76)$$

This may be coming from the restrictions seen from Derrick's argument. The amazing thing is that this result does not depend on J^2 neither on the baryon number.

The ratio $F_1(i=j=3/2)/F_1(i=j=1/2) \;=\; 5^{1/4}$ also.

In Table 1 we list the results for B=1, 2, 3 giving two values to f_π: 0.129 Gev, as in [5, 7], and 0.186 Gev, as coming from experiment. We take the eigenvalues 1/2 and 3/2 for B=1, 1 for B=2, 1/2 and 3/2 for B=3. Fig. 3 depicts the main features of the results.

The results of Table 1 deserve some comments. In the first place, notice that the baryon states with B=2 and 3 are almost degenerate, quite in disagreement with the observed states. This should not bother much, since probably a description of the observed states demand to consider them as composites.

Second, the higher the baryon number the narrower the state in space again contrary to the observation. It would be interesting, however , to look for remnants of this structure in the observed form factors of deuteron, tritium and He^3.

As a last comment, it seems that the ratio of F_1 for the B=2 state and the B=1, I=J=1/2 state is approximately equal to the rate of the F_1 value for B=3, I=J=1/2 to the B=2 value. This is perhaps an intrinsic property of the model at the minimum for the quantum energy.

6 Nucleon Properties for the Quantum Stable Soliton

In the Table 2 we present the numerical results for nucleon properties obtained for the quantun stable nucleon, as compared to the current values obtained by Adkins, Nappi and Witten.

The definitions are the same as used by those authors, and the two list of values corresponds to the two options chosen for f_π, i.e., 0.129 Gev as in Adkins, Nappi and Witten and 0.186 Gev, which is the current experimental value. In any case, we just propose these values as illustration since we think that the right value at any stage of an approximation scheme should provene from a global consistency of all results obtained from the theory.

Let us make a few comments on some of the values quoted.

The axial weak coupling is known to come from the $1/r^2$ asymptotic contribution of the hedgehog's chiral angle [5]. Its value is, then, a direct measure of the dimensional parameter K_2 appearing in Eq.(14). The product of K_2 and F_1^2 is a pure number and turns out to be 17.45 for f_π=0.129 Gev and B=1. It is 199.66 for B=2, and 867 for B=3.

The isoscalar radii are smaller than the measured ones, but its ratio is the same as in the calculation by Adkins, Nappi and Witten.

It is interesting to exploit the fact that the parameters of the Skyrme soliton at the quantum minimum of energy are well determined to extract information on other physical quantities. Consider the isoscalar and isovector magnetic moments. Following Adkins, Nappi and Witten

$$(\mu_{I=0}) = \frac{1}{4\pi}e^3 < r^2_{I=0} > \frac{1}{b(\phi)} \tag{77}$$

$$(\mu_{I=1})_3 = \frac{2}{9}\pi\frac{1}{e^3}b(\phi)\,. \tag{78}$$

From inspection:

$$(\mu_{I=1})_3 = \frac{1}{18}\frac{< r^2_{I=0} >}{\mu_{I=0}}\,. \tag{79}$$

Using the relationship between magnetic moment and gyromagnetic ratio, we get

$$g_{I=1} = \frac{8}{9}M^2 < r^2_{I=0} > \frac{1}{g_{I=0}}\,. \tag{80}$$

This relation is well satisfied (within 10%) by the experimental numbers. We obtain $g_{I=1} = 10.4$ (experimentally, 9.4).

Up to now, we have not used any information from the minimum of quantum energy. We do so by recalling that rotation contributes one fourth for the rest mass at the minimum:

$$\frac{1}{4}M(J^2) = \frac{3}{4}\frac{J^2\, F_1^3 e^3 f_\pi}{\pi b(\phi)}\ . \tag{81}$$

Substituting in the expression for the gyromagnetic ratio:

$$g_{I=1} = 4M\,(\mu_{I=1})_3 = \frac{8}{3}J^2 \tag{82}$$

$$g_{I=0} = \frac{1}{3}M^2 < r_{I=0}^2 > \frac{1}{J^2}\ . \tag{83}$$

These expressions are predictions of the model at the minimum quantum energy. Numerically, for the nucleon,

$$g_{I=1} = 2\ \ . \tag{84}$$

$$g_{I=0} = 1.35 \tag{85}$$

and we recover the values calculated from integration.

Our last comment in this section is about the value for the axial weak coupling constant. In terms of the coefficient K_2 in the expansion of the chiral angle at infinity Eq.(14), we have:

$$g_A = \frac{\pi}{3}K_2 f_\pi^2 = \frac{\pi}{3}\left(K_2 e^2 f_\pi^2\right)\frac{1}{e^2} \tag{86}$$

The solutions are controlled by the dimensionless parameter $K_2 e^2 f_\pi^2$, which is fixed for a B=1 soliton. So, as the Skyrme parameter changes, g_A goes as e^{-2}. The chiral angle is more concentrated about the origin for the "minimal" baryon of the Skyrme model as compared with the case evaluated by Adkins, Nappi and Wittten.

To summarize, it seems that in order to get a better agreement to the experimental values, a repulsive interaction would be necessary to make the nucleon broader. In particular bad shape is the isovector gyromagnetic ratio, which is almost five times smaller than the experimental value. We have reasons to believe that the inclusion of the pions provide the would be welcomed repulsive contribution, contrary to previous evidence.

Notice that f_π plays somehow a control rôle for all quantities. Masses grow with f_π, as well as the "shape" parameter F_1, making the chiral angle to grow faster, or, in other terms, to shrink. This translates, in turn, into smaller charge and magnetic radii, while the axial weak coupling constant remaing constant. These are all consequences in the "wrong" direction. A smaller value for f_π then seems to fit better with the description of the nucleon, as long as the mass is not too low. As we want to emphasize again, the fact that one is at the quantum minimum allows to estimate better what are the necessary changes to improve the model, since the relevant quantities appear with well defined behavour under changes in the model parameters. This is a consequence of the scale properties of the regular solutions for the chiral angle.

7 Discussions and Conclusions

In this work we have analized numerically and analytically the solutions for the differential equations of the Skyrme model in the hedgehog parametrization for the unitary chiral $SU(2)$ field. The aim of this research is to study the way a chiral soliton can be used to describe as a first approximation the nucleon states. We have also found a systematic for $B > 1$ states.

We have shown, by inspection of the solutions, through the study of sum rules derived from the differential equations, and, finally, from the application of Derrick's argument, that there is a dimensionless "control" parameter ϕ. It is made up of the quotient between the first derivative of the regular solution at the origin and the product of the dimensionless Skyrme parameter e and the pion decay constant. The parameter ϕ itself is the slope at the origin for the chiral angle in terms of the dimensionless variable $\tilde{r}$.

In terms of this parameter ϕ it is seen that there are solitons with finite energy and integer baryon number only for a countable set of values. For the rest of the domain; solutions which are at one extreme of the real half-line behave irregularly at the other.

The regular solutions carry a dependence on a dimensional parameter which is essentially the value of the first non-null derivative at the origin or at infinity of the chiral angle. For the baryonic solution which are regular at both ends of the real half line, we have selected the value at the origin of the first derivative as an interesting parameter.

The sum rules derived in Section III proved to be quite sensible to the asymptotic behaviour of the chiral angle. They provide also a consistency check for the value of the first derivative of the chiral angle in real space.

Throughout these calculations the differential equation is integrated numerically. There is a delicate fine-tuning involved in the obtention of consistent information about the values of the control parameter ϕ where baryonic states manifest. The agreement between different determinations is found at the level of one part in a thousand.

We have made an application of the Derrick's argument (or theorem) to the present case. It proves powerful, and the interesting point is that it not only seems to show when the eventual solution is stable, it seems to give precisely an indication about when the non-linear differential equation does really have a soliton solution.

As stated before, different results are quite consistent, and this is to be expected since to obtain them we apply the same numerically integration techniques, but there is a satisfactory agreement when different numerical procedures are used.

Moreover, the results previously obtained (and widely accepted) by Adkins, Nappi and Witten are reproduced.

The main point of these results is that the classical mass for the Skyrme model exhibits the same kind of the instability as the one present in the pure non-linear sigma model. As shown in Section IV, one can keep fixed the value of the control parameter ϕ changing F_1 and e appropriately, but the mass will vary at will with the variation of the Skyrme parameter which is a parameter that cannot be fixed at the classical level.

We have applied the same, rudimentary, quantization procedure through collective coordinates which is costumarily used by many authors. The results, as is well known, are that the soliton behaves as a rotating top at the quantum level.

The interesting point, and this is an exact consequence of the scale invariance of the soliton solution, is that the parameter e factors out neatly in the classical mass and the moment of the inertia of the solution. Then, the quantum energy depends on e in such a way that it has a minimum. This can be taken as a self consistency condition for the model.

The minimum singles out a quantum stable soliton solution, for each value of the baryon number and angular momentum. The stability is an important feature, since it prevents the state to dwindle away, as it was possible in the classical case.

This state is, in our opinion, the one that should be taken as representative of the approximation by the Skyrme lagrangean of the nucleon state.

Previous results for the nucleon parameters were not obtained for this case. We have recalculated the interesting quantities, and verified that as the parameter f_π is taken to vary, the numerical results scale in the way the scale invariance of the classical soliton solution predicts. This is quite satisfactory to have confidence on the theoretical framework.

In our opinion, in order to establish a consistent approximation scheme in the chiral theory for strong interactions, several steps suggest by themselves. The optics is that the complete, exact theory, should provide the physically observed values, whereas at intermediate levels of the approximation only a significant overall consistency should be demanded.

To begin with, Derrick's argument enforces the point that far from being a kind of correction term, the Skyrme term for the baryonic states is as important as the non-linear sigma model term put forward originally to describe low energy pion physics. Accordingly, this feature should be incorporated into the picture provided by current algebra, a task accomplished rather long ago by Pak and Tze [3].

Beyond this level, it is necessary to take into account the mass of the pion, and the fluctuations of the pion field around the nucleon state. This amounts to an improved treatment of quantization, such as has been elaborated recently by Hayashi, Saito and Uehara [18].

At each step, however, care should be taken to revise the modification for the classical soliton non-linear differential equation coming from new contributions, and check the resulting picture for the nucleon state.

A significant point in the results presented here is that the structure of the theory is well understood, and eventual variation of the significant parameters is under control.

What seems to be needed to improve the nucleon description contained at the present level is to make the chiral angle of the theory larger in space. This would mean to decrease F_1, which in turn would contribute to reduce the contribution of rotation and allow for an increase of the value of f_π which is welcomed, to approach the physical value of the nucleon mass. Non central repulsive contributions are probably welcomed to get this improvement, and we are looking at present for them.

We believe to have shown that a better understanding of the features to the differential equation central to the Skyrme model helps to device ways to improve the physics of it.

In conclusion, we think we have a complete study of the $SU(2)$ Skyrme model in the hedgehog configuration and the sector of lower baryon numbers. Part of the study is based on numerical integration of the nonlinear differential equation for the chiral angle, but the conclusions are safe since different numerical procedures agree, and the known results are reproduced. The physical content of the model has been put to light, as a result, and the ways to improve it seem clearly understood.

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Figure Captions

FIG. 1a. - Plot of the regular solution at origin for the three values of the parameter ϕ: 0.90, 1.003764... and 1.10 .

FIG. 1b. - Plot of the regular solution at infinity for three values of the parameter κ: -15.0, -17.2771... and -45.0 .

FIG. 2 - The behaviour of the sum rule for the asymptotic chiral angle, Eq.(40), as a function of ϕ .

FIG 3 - Spin and masses for the lower baryon number minimal quantum energy states with $f_\pi = 0.129$ Gev.

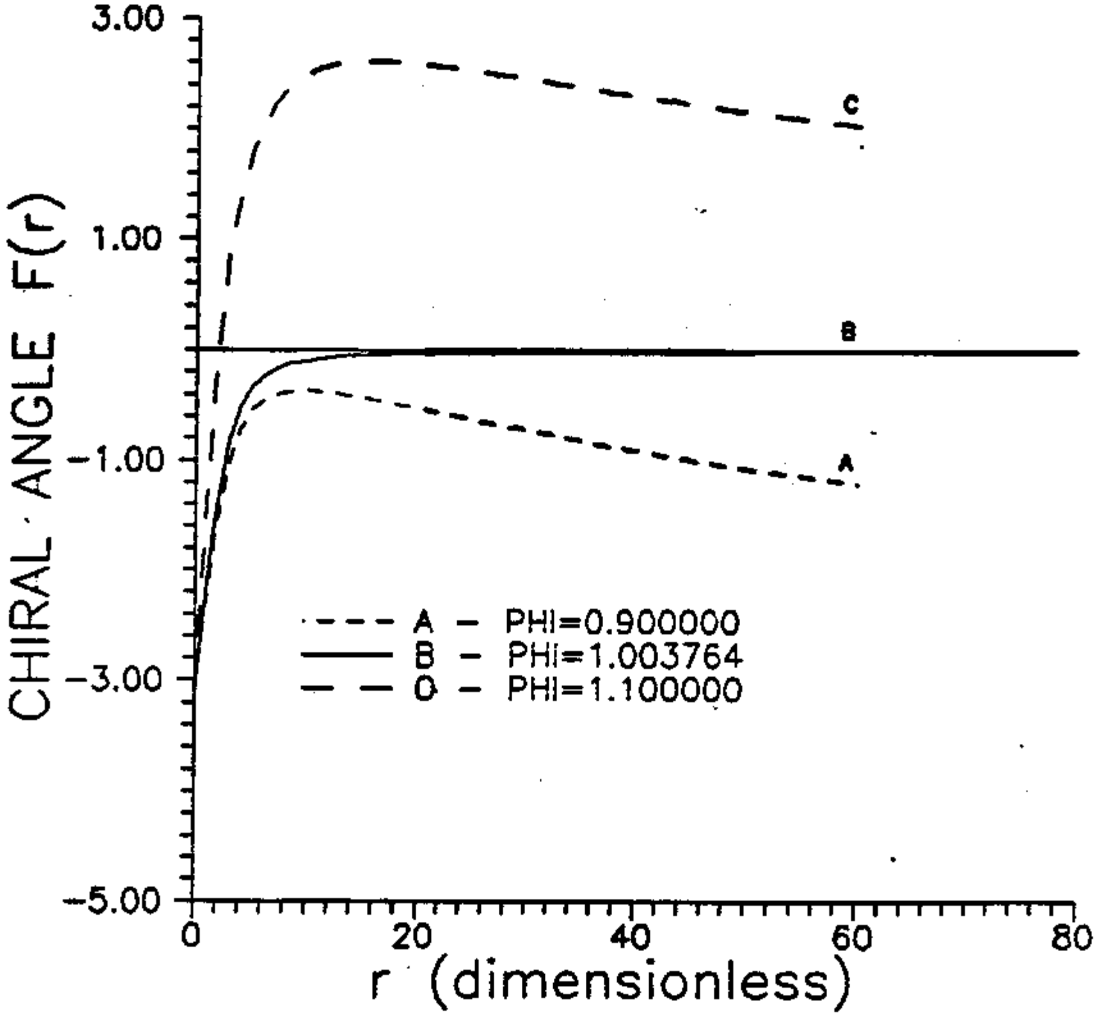
3.00
1.00
-1.00
-3.00
-5.00
0
20
40
60
80
CHIRAL ANGLE F(r)
r (dimensionless)
C
B
A
A - PHI=0.900000
B - PHI=1.003764
C - PHI=1.100000

FIG. 1a.

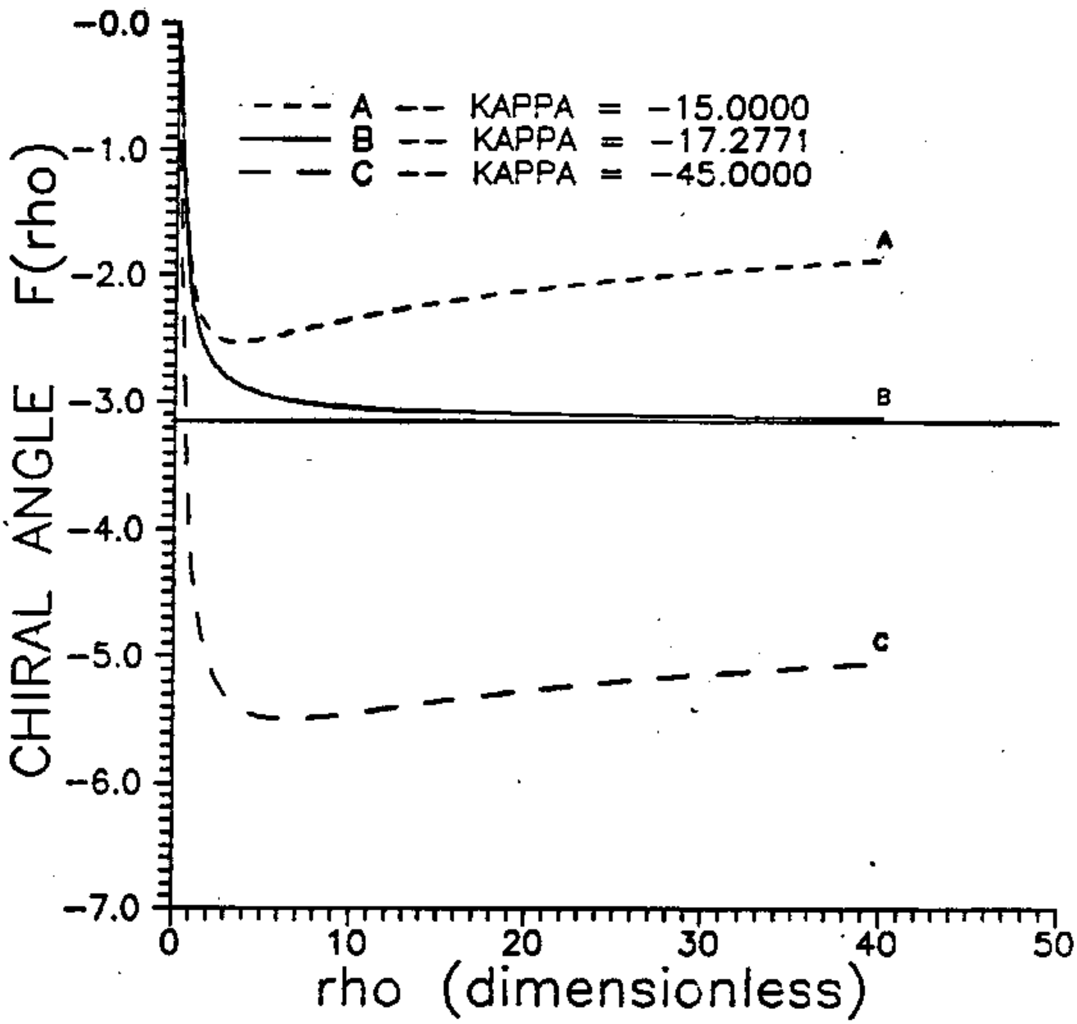
A -- KAPPA = -15.0000
B -- KAPPA = -17.2771
C -- KAPPA = -45.0000
CHIRAL ANGLE F(rho)
rho (dimensionless)
-0.0
-1.0
-2.0
-3.0
-4.0
-5.0
-6.0
-7.0
0
10
20
30
40
50
A
B
C

FIG. 1b.

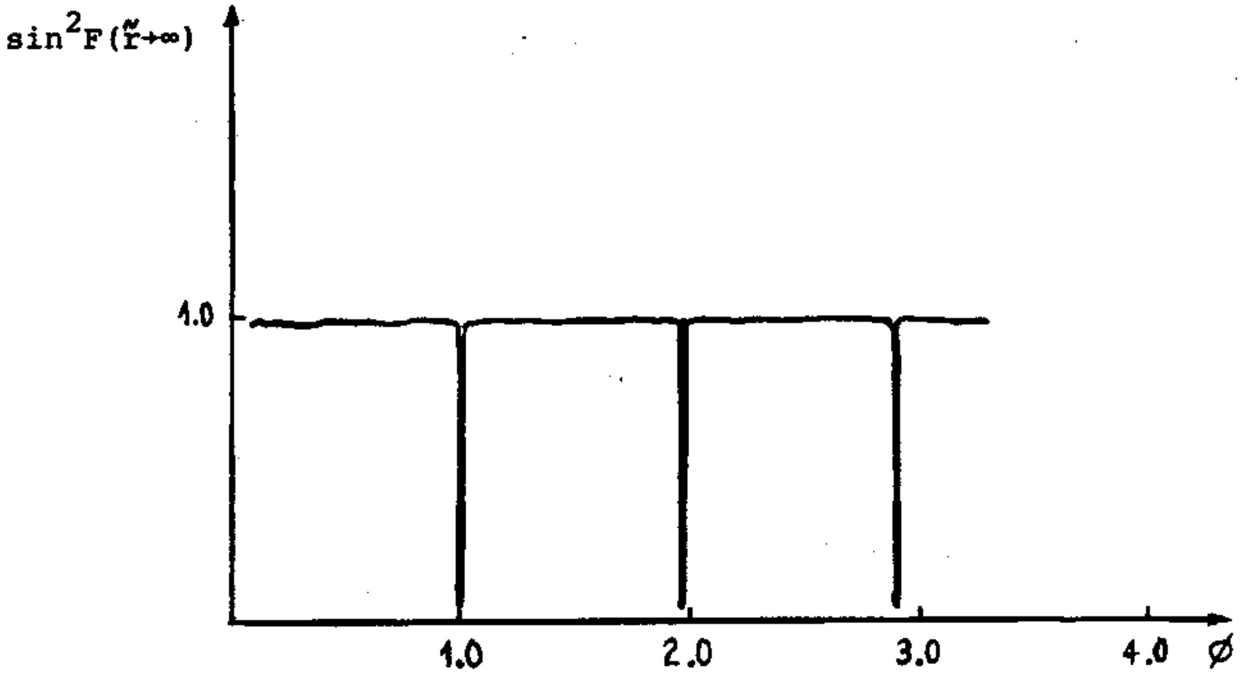

Fig.2

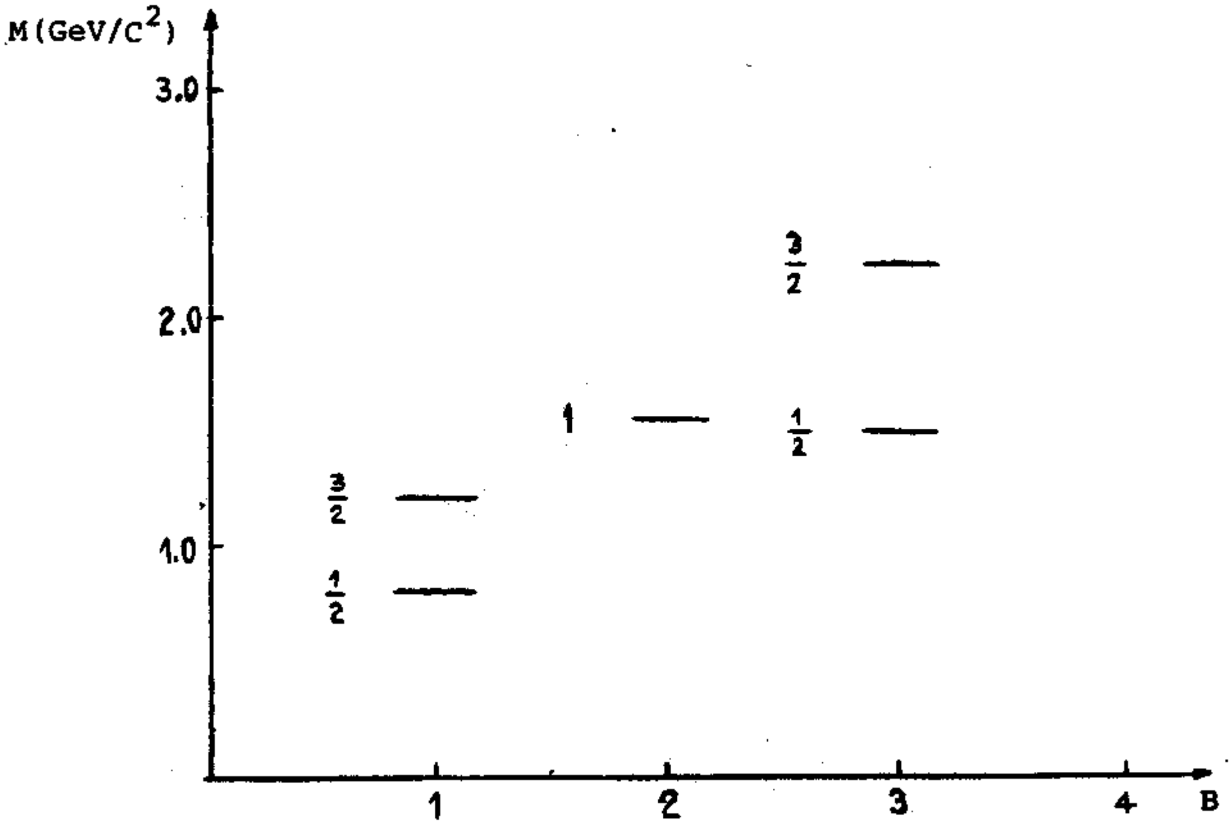

Fig.3

ϕ	B	$a_1(\phi)$	$b_1(\phi)$	$i=j$	e	$f_\pi(Gev)$	$F_1(Gev/c)$	$M(Gev/c)$
1.00376...	1	11.6	51.01	1/2	7.67	0.129	0.993	0.817
						0.186	1.421	1.178
	1			3/2	5.13	0.129	0.664	1.222
						0.186	0.950	1.762
1.9650...	2	32.22	219.5	1	11.16	0.129	2.83	1.560
						0.186	4.08	2.250
2.8882...	3	58.1	563.9	1/2	20.02	0.129	7.800	1.500
						0.186	11.2	2.164
				3/2	14.00	0.129	5.21	2.24
						0.186	7.52	3.24

Table 1: Results for B=1, 2 and 3

Quantitiy	*ANW* $f_\pi^* = 0.129\ Gev$	*This Work* $f_\pi = 0.129\ Gev$	*This Work* $f_\pi = 0.186\ Gev$	*Experiment*
M_N	*input*	0.817	1.18	0.939 *Gev*
M_Δ	*input*	1.222	1.76	1.232 *Gev*
e	5.45*	7.67	7.67	---
$< r^2 >^{1/2}_{I=0}$	0.59 f_m	0.422 f_m	0.293 f_m	0.72 f_m
$< r^2 >^{1/2}_{M,I=0}$	0.92 f_m	0.65 f_m	0.45 f_m	0.81 f_m
μ_p	1.87	0.84	0.84	2.79
μ_n	−1.31	−0.16	−0.16	−1.91
$\left\|\frac{\mu_p}{\mu_n}\right\|$	1.43	5.25	5.25	1.46
$g_{I=0}$	1.11	1.36	1.36	1.76
$g_{I=1}$	6.38	2.0	2.0	9.4
g_A	0.61	0.307	0.307	1.23
$g_{\pi NN}$	8.9	3.89	3.89	13.5
$g_{\pi ND}$	13.2	5.83	5.83	20.3
μ_{ND}	2.3	0.71	0.71	3.3
$F_1(Gev/c)$	0.7057	0.993	1.421	---
$K_2(Gev^{-2}/c^2)$	−34.95	−17.65	−8.489	---

Table 2: Results for the Nucleon Physical Parameters

* Obtained by fitting

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