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ABSTRACT

The phase diagram and thermal and crossover exponents of the Z(6) ferromagnet on the square lattice are calculated within a real-space renormalization group (RG) scheme. The obtained phase diagram (exhibiting 4 phases corresponding to Z(6), Z(3), Z(2) and completely broken symmetries) contains all exactly known critical points, and possibly is an excellent approximation eve-ry-where except in the high-temperature region where a soft phase is expected to appear. However, all the obtained results are exact in the Wheatstone-bridge hierarchical lattice. In addition to these results, we present an operational procedure (the "break-collapse method") which considerably sim-plifies the *exact* calculation of two-spin correlation functions for arbitrary Z(6) two-rooted clusters (frequent in RG approaches).

Key-words: Z(6) ferromagnet; Criticality; Real space Renormali-zation group; Correlation function.

1 INTRODUCTION

The Z(N) model contains as particular cases a considerable number of important statistical models, e.g. bond percolation, spin 1/2 Ising model, Potts model, discrete spin cubic model, clock model, the symmetric Ashkin-Teller model [1] and the classical XY model. It is identical to the N-state Potts and the symmetric Ashkin-Teller models for $N \leq 3$ and $N = 4$ respectively. Its critical behaviour and phase diagram, with three or more phases for $N \geq 4$, have attracted [2-11] the attention of physicists for a long time, especially in the square lattice where a simplification arises due to self-duality. In the study of the criticality of the Z(N) model, different techniques were used such as perturbation theory and Pade' extrapolants [3], variational method and the infinitesimal Migdal renormalization group (RG) transformations [6], finite-lattice approach [7], mean field [2], generalised duality transformations [4,5], finite-size scaling and conformal invariance [11], break-collapse method and real space renormalization group [9, 10,12,13].

The phase diagram of the Z(6) ferromagnet in the square lattice is known to have 5 phases [5,6], namely: the paramagnetic [P; Z(6) symmetry], the intermediate 1 [I1; the Z(6) symmetry has been partially broken into 3 sets, each of which has a Z(2) symmetry]; the intermediate 2 [I2; the Z(6) symmetry has been partially broken into 2 sets, each of which has a Z(3) symmetry], the ferromagnetic [F; completely broken symmetry] and the soft [S; with power law decaying correlations] ones. The P-I1, P-I2, I1-F, I2-F critical surfaces join at a segment of a self-dual line completely determined by duality arguments. This critical segment finishes on the P-F critical surface at the 6-state Potts critical point P_6. The S-F and S-P critical surfaces are expected [5] to intersect the continuation of the self-dual segment at a point corresponding to a temperature higher than the one of P_6. All the above mentioned critical surfaces are not exactly known, except for the self-dual critical segment. Herein we calculate these surfaces through a real-space renormalization group (RG) based on the two-rooted self-dual Wheatstone-bridge graph (Fig. 1a). Similarly to [9], we also propose a break-collapse method for the Z(6) model which allows the RG recursive relations to be calculated without doing any trace over spin configurations.

The paper is organized as follows: in section 2 we define the model and describe the RG formalism. In section 3, we present the break-collapse method. Sections 4 and 5 contain our results and conclusions respectively. Finally we present in the Appendix a method [14] for calculating the equivalent vector transmissivities [8] of terminal graphs G_{pr} whose bonds are all precollapsed.

2 MODEL AND RG FORMALISM

The Z(6) ferromagnet can be described by the following dimensionless Hamiltonian [5]:

$$\beta\mathcal{H} = \sum_{\langle i,j\rangle} h(n_i - n_j) \equiv \sum_{\langle i,j\rangle} \left\{ K_1 - \sum_{\beta=1}^{3} 2K_\beta \cos\left[\frac{\pi\beta}{3}(n_i - n_j)\right] \right\} \quad (\beta \equiv 1/k_B T) \qquad (1)$$

where to each site i we associate a random variable n_i which takes on the 6 integer values 0,1,...,5. In eq. (1) the sum is over all first-neighboring pairs of sites on a square lattice. In order that the ground state be ferromagnetic, the dimensionless couplings constants $K_i \equiv \beta J_i$ $(i = 1,2,3)$ must satisfy the following inequalities:

$$K_1 + K_2 \geq 0 \qquad (2a)$$

$$K_1 + K_3 \geq 0 \qquad (2b)$$

and

$$K_1 + 3K_2 + 4K_3 \geq 0 \qquad (2c)$$

which were obtained by imposing that $h(0) \leq h(\ell)$ $(\ell = 1,2,3)$.

Let us introduce the operationally convenient variable (vector transmissivity [8]) $\vec{t} \equiv (1,t_1,t_2,t_3,t_4,t_5)$ through

$$t_1 = t_5 \equiv \left[1 + e^{-(K_1+3K_2+4K_3)} - e^{-3(K_1+K_2)} - e^{-4(K_1+K_3)}\right]/D_0 \qquad (3a)$$

$$t_2 = t_4 \equiv \left[1 - e^{-(K_1+3K_2+4K_3)} - e^{-3(K_1+K_2)} + e^{-4(K_1+K_3)}\right]/D_0 \qquad (3b)$$

$$t_3 \equiv \left[1 - 2e^{-(K_1+3K_2+4K_3)} + 2e^{-3(K_1+K_2)} - e^{-4(K_1+K_3)}\right]/D_0 \qquad (3c)$$

where

$$D_Q = 1 + 2\,e^{-(K_1+3K_2+4K_3)} + 2e^{-3(K_1+K_2)} + e^{-4(K_1+K_3)} \quad . \tag{3d}$$

This vector transmissivity generalises the scalar one used in [15] for the Potts model.

The equivalent vector transmissivity $\vec{t}^{(s)}$ ($\vec{t}^{(p)}$) corresponding to a series (parallel) array of two bonds with respective vector transmissivities $\vec{t}^{(1)}$ and $\vec{t}^{(2)}$ is given by [8]:

$$t_r^{(s)} = t_r^{(1)} t_r^{(2)} \qquad (r = 1,2,3) \qquad \text{(series)} \tag{4}$$

and

$$t_1^{(p)} = \frac{t_1^{(1)} + t_1^{(2)} + t_1^{(1)}t_2^{(2)} + t_2^{(1)}t_1^{(2)} + t_2^{(1)}t_3^{(2)} + t_3^{(1)}t_2^{(2)}}{1 + 2t_1^{(1)}t_1^{(2)} + 2t_2^{(1)}t_2^{(2)} + t_3^{(1)}t_3^{(2)}} \qquad \text{(parallel)} \tag{5a}$$

$$t_2^{(p)} = \frac{t_2^{(1)} + t_2^{(2)} + t_1^{(1)}t_1^{(2)} + t_2^{(1)}t_2^{(2)} + t_1^{(1)}t_3^{(2)} + t_3^{(1)}t_1^{(2)}}{1 + 2t_1^{(1)}t_1^{(2)} + 2t_2^{(1)}t_2^{(2)} + t_3^{(1)}t_3^{(2)}} \qquad \text{(parallel)} \tag{5b}$$

$$t_3^{(p)} = \frac{t_3^{(1)} + t_3^{(2)} + 2t_1^{(1)}t_2^{(2)} + 2t_2^{(1)}t_1^{(2)}}{1 + 2t_1^{(1)}t_1^{(2)} + 2t_2^{(1)}t_2^{(2)} + t_3^{(1)}t_3^{(2)}} \quad . \qquad \text{(parallel)} \tag{5c}$$

Equations (5) can be conveniently-rewritten as follows:

$$t_r^{(p)D} = t_r^{(1)D} t_r^{(2)D} \qquad (r = 1,2,3) \tag{6}$$

where the *dual* vector transmissivity $\vec{t}^{D}$ [8] is given by

$$t_1^D = t_5^D \equiv \frac{1+t_1-t_2-t_3}{1+2t_1+2t_2+t_3} \tag{7a}$$

$$t_2^D = t_4^D \equiv \frac{1-t_1-t_2+t_3}{1+2t_1+2t_2+t_3} \tag{7b}$$

and

$$t_3^D \equiv \frac{1-2t_1+2t_2-t_3}{1+2t_1+2t_2+t_3} \quad . \tag{7c}$$

The relationships between t_r^D and the coupling constants are the following:

$$t_1^D = x_1 \equiv e^{-(K_1+3K_2+4K_3)} \tag{8a}$$

$$t_2^D = x_2 \equiv e^{-3(K_1+K_2)} \tag{8b}$$

and

$$t_3^D = x_3 \equiv e^{-4(K_1+K_3)} \tag{8c}$$

The above variables $x_r (r=1,2,3)$ are precisely the respective variables z, x_α and x_β used by Domany and Riedel [16] in the (3,2) model (which is equivalent to the Z(6) model). In the description of the phase diagram of the Z(6) model we will use the variables x_r rather than t_r since the physical parameter space is in the cube $0 \leq x_r \leq 1$ in the ferromagnetic case.

In order to study the criticality of the Z(6) ferromagnet we use a RG based on the self-dual two-rooted graph (see Fig. 1a), which has proved to be very convenient for the square lattice (e.g., bond percolation [17], Potts model [15], Z(4) model

[9], etc). Following along the same lines of the Z(4) model treatment of Mariz et al. [9], we obtained the following RG recursive relations which preserve the correlation function:

$$t'_r = \frac{N_r}{D} \qquad (r = 1,2,3) \tag{9a}$$

with

$$N_1 = 2\{t_1^2 + t_1^3 + 3t_1^2t_2^2 + t_2^2t_3^2 + 2t_1^3t_2 + 2t_1t_2^2t_3 + 2t_1t_2t_3^2 + 3t_1^2t_2^2t_3 + \\ + 2t_1t_2^3t_3 + t_1t_2^2t_3^2\} \tag{9b}$$

$$N_2 = 2t_2^2 + 2t_2^3 + t_1^4 + 5t_2^4 + 2t_1^2t_3^2 + 4t_1^3t_2 + 4t_1^2t_2t_3 + 4t_1t_2t_3^2 + \\ + 2t_1^2t_2t_3^2 + 6t_1^2t_2^2t_3 + 4t_1^3t_2t_3 \tag{9c}$$

$$N_3 = 2\{t_3^2 + t_3^3 + 2t_1^2t_2^2 + 4t_1^2t_2t_3 + 4t_1t_2^2t_3 + 2t_1^3t_2^2 + 2t_1^2t_2^3 + \\ + 2t_1^2t_2^2t_3\} \tag{9d}$$

and

$$D = 1 + 4t_1^3 + 4t_2^3 + 2t_3^3 + 2t_1^4 + 2t_2^4 + t_3^4 + 4t_1^3t_2^2 + 4t_1t_2^2t_3^2 + 2t_1^4t_2 + \\ + 2t_2^5 + 4t_1^2t_2t_3^2 + 4t_1^2t_2^2t_3 \; . \tag{9e}$$

$\{t'_r\}$ $(r = 1,2,3)$ are the components of the vector transmissivity $\vec{t}'$ of the renormalized bond (Fig. 1b) whose coupling constants K'_r were obtained through the trace over n_3 and n_4, i.e. by imposing that

$$C\, e^{-\beta \mathcal{H}'_{12}(K'_r)} = \mathop{\mathrm{Tr}}_{3,4}\, e^{-\beta \mathcal{H}_{1234}(K_r)} \qquad (10)$$

where the constant C arises from the renormalization of the zero of energies. We call $\vec{t}'$ the equivalent vector transmissivity $\vec{t}^{eq}(G)$ between the roots 1 and 2 of the graph G shown in Fig. 1a.

3 BREAK-COLLAPSE METHOD

Before analysing the results coming from eqs. (9), let us describe a particularly simple way [*break-collapse method* (BCM)] of deriving these recursive relations without the examination of configurations, and, more generally speaking, the equivalent vector transmissivity $\vec{t}^{eq}(G)$ between the roots 1 and 2 of an *arbitrary* graph G associated with a Z(6) cluster. $\vec{t}^{eq}(G)$ is determined by $t_r^{eq}(\{\vec{t}^{(\ell)}\}) = N_r(\{\vec{t}^{(\ell)}\}/D(\{\vec{t}^{(\ell)}\})$ $(\ell = 1,2,3)$ where $\{\vec{t}^{(\ell)}\}$ denotes the set of vector transmissivities, each of which is associated with a bond of the graph. $N_r(\{\vec{t}^{(\ell)}\})$ and $D(\{t^{(\ell)}\})$ are *multilinear* polynomials of the form $A + B\,t_1^{(\ell)} + C\,t_2^{(\ell)} + E\,t_3^{(\ell)}$ for an *arbitrary* ℓ th bond where the coefficients A,B,C and E depend on the set of vector transmissivities (which we denote $\{t^{(\ell)}\}'$) of the remaining bonds. These coefficients can be determined by performing four different operations on the ℓ th bond, namely, the "break" $(t_1^{(\ell)} = t_2^{(\ell)} = t_3^{(\ell)} = 0)$, the "collapse" $(t_1^{(\ell)} = t_2^{(\ell)} = t_3^{(\ell)} = 1)$, the "precollapse of type 2" $(t_1^{(\ell)} = t_3^{(\ell)} = 0;\ t_2^{(\ell)} = 1)$ and the "precollapse of type 3" $(t_1^{(\ell)} = t_2^{(\ell)} = 0;\ t_3^{(\ell)} = 1)$. It follows that:

$$N_r(\{\vec{t}^{(\ell)}\}) = (1+t_1^{(\ell)}-t_2^{(\ell)}-t_3^{(\ell)})N_r^{bbb}(\{t^{(\ell)}\}') + t_1^{(\ell)}N_r^{ccc}(\{t^{(\ell)}\}')$$

$$+ (t_3^{(\ell)}-t_1^{(\ell)})N_r^{bbc}(\{t^{(\ell)}\}') + (t_2^{(\ell)}-t_1^{(\ell)})N_r^{bcb}(\{t^{(\ell)}\}') \quad (r=1,2,3) \tag{11a}$$

and

$$D(\{\vec{t}^{(\ell)}\}) = (1+t_1^{(\ell)}-t_2^{(\ell)}-t_3^{(\ell)})D^{bbb}(\{t^{(\ell)}\}' + t_1^{(\ell)}D^{ccc}(\{t^{(\ell)}\}') +$$

$$+ (t_3^{(\ell)}-t_1^{(\ell)})D^{bbc}(\{t^{(\ell)}\}') + (t_2^{(\ell)}-t_1^{(\ell)})D^{bcb}(\{t^{(\ell)}\}') \tag{11b}$$

where $N_r^{bbb},\ldots,D^{bcb}$ are the respective numerators and denominators of the "broken graph" (bbb), "collapsed graph" (ccc), "precollapsed graph of type 3" (bbc) and "precollapsed graph of type 2" (bcb). The BCM consists in applying recursively the break-collapse equation (eqs.(11)) together with the series and parallel algorithms (eqs.(4) and (5))until graphs *exclusively* made by precollapsed bonds (which we will denote by G_{pr}) are ultimately obtained. The numerators $N_r(G_{pr})$ $(r=1,2,3)$ and the denominator $D(G_{pr})$ of the equivalent vector transmissivity of a terminal graph G_{pr} have no simple formulae and their evaluation involves the counting of certain mod-6 flows [14] on G_{pr} (see Appendix).

Let us illustrate the BCM on the graph of Fig. (1a). Applying eqs. (11) to the ℓ th bond between sites 3 and 4 we obtain the broken graph, collapsed graph, precollapsed graph of type 2 and precollapsed graph of type 3 shown in Figs. (2a), (2b), (2c) and (2d) respectively. Their associated expressions for N_r $(r=1,2,3)$ and D are given by:

$$N_1^{bbb} = 2(t_1^2 + t_1^2 t_2^2 + t_2^2 t_3^2) \quad (12a)$$

$$N_2^{bbb} = 2(t_2^2 + t_1^2 t_3^2) + t_1^4 + t_2^4 \quad (12b)$$

$$N_3^{bbb} = 2(t_3^2 + 2t_1^2 t_2^2) \quad (12c)$$

$$D^{bbb} = 1 + 2(t_1^4 + t_2^4) + t_3^4 \quad (12d)$$

$$N_1^{ccc} = 4(t_1 + t_1 t_2 + t_2 t_3)^2 \quad (12e)$$

$$N_2^{ccc} = (2t_2 + t_1^2 + t_2^2 + 2t_1 t_3)^2 \quad (12f)$$

$$N_3^{ccc} = 4(t_3 + 2t_1 t_2)^2 \quad (12g)$$

$$D^{ccc} = (1 + 2t_1^2 + 2t_2^2 + t_3^2)^2 \quad (12h)$$

$$N_1^{bcb} = 2(t_1^2 + 3t_1^2 t_2^2 + t_2^2 t_3^2 + 2t_1 t_2^2 t_3 + 2t_1 t_2^3 t_3) \quad (12i)$$

$$N_2^{bcb} = 2t_2^2 + 2t_2^3 + 2t_1^2 t_3^2 + t_1^4 + 5t_2^4 + 2t_1^2 t_2 t_3^2 + 4t_1^3 t_2 t_3 \quad (12j)$$

$$N_3^{bcb} = 2(t_3^2 + 2t_1^2 t_2^2 + 2t_1^2 t_2^3 + 4t_1 t_2^2 t_3) \quad (12k)$$

$$D^{bcb} = 1 + 4t_2^3 + 2t_1^4 + 2t_2^4 + t_3^4 + 4t_1^2 t_2 t_3^2 + 2t_1^4 t_2 + 2t_2^5 \quad (12\ell)$$

$$N_1^{bbc} = 2(t_1^2 + t_1^2 t_2^2 + t_2^2 t_3^2 + t_1^2 t_2^2 t_3 + 2t_1 t_2 t_3^2) \quad (12m)$$

$$N_2^{bbc} = 2t_2^2 + 2t_1^2 t_3^2 + t_1^4 + t_2^4 + 4t_1 t_2 t_3^2 + 2t_1^2 t_2^2 t_3 \quad (12n)$$

$$N_3^{bbc} = 2(t_3^2 + t_3^3 + 2t_1^2 t_2^2 + 2t_1^2 t_2^2 t_3) \qquad (12o)$$

$$D^{bbc} = 1 + 2t_3^3 + 2t_1^4 + 2t_2^4 + t_3^4 + 4t_1^2 t_2^2 t_3 \qquad (12p)$$

Expressions (12a)-(12h) were obtained through exclusive use of eqs. (4) and (5), while the remaining ones required also eqs. (11). Substituting expressions (12) into eqs. (11) we recover eqs. (9) *without performing the traditional time consuming trace operations*. This type of producedure has proved to be very useful in other problems, such as the Potts model [15,18], resistor network [19], directed percolation [20], Z(4) model [9], among others.

4 RESULTS

Let us now study the criticality provided by eqs. (9). Since eqs. (2) lead to complicate inequalities in the t_r variables (in contrast with the simple conditions $0 \leq x_r \leq 1$), generating thus a phase diagram which is more difficult to visualize than the one expressed in the x_r variables, we will present our results in the x_r variables. Due to the self-duality of the two-rooted graphs drawn in Figs. (1a) and (1b), the RG recursion relations in the latter variables have the same functional form as the ones in the t_r variables (i.e., if we replace t_r by x_r in eqs. (9) we obtain $x_r' = N_r/D$ in terms of x_1, x_2 and x_3). These recursion relations lead to a phase diagram (see Fig. 3) whose structure is similar to the one discussed by Domany and Riedel [16] in their

(3,2) model. Our RG flow exhibits four completely stable fixed points, namely $H(x_1=1, x_2=1, x_3=1)$ (characterising the P phase), L(0,0,0) (characterising the F phase), L"(0,0,1) (characterising the I1 phase) and L'(0,1,0) (characterising the I2 phase). These four phases are separated by the two-dimensional critical surfaces I1-F, P-I2, F-I2, I1-P and P-F which are the domains of attraction of the respective singly unstable fixed points I_1, I_2, P_1, P_2 and S. The first four critical surfaces meet on the segment $\overline{MP}_6$ (see Fig. 3) of the self-dual line which contains the 6-state Potts critical point P_6 and the decoupled multicritical point D. The doubly unstable fixed point D (whose domain of attraction is $\overline{MP}_6$) belongs to the decoupling surface $x_1 = x_2x_3$ where the Z(6) model decouples into an Ising model and a 3-state Potts one. The completely unstable fixed point P_6 belongs to the line $x_1=x_2=x_3$ (and hence $K_1=K_2=2K_3$) where the Z(6) model reduces to the 6-state Potts model. Other particular cases of the Z(6) model correspond to the limits: (i) $K_2\to\infty$ ($x_1=x_2=0$) and $K_1=K_2=0$ ($x_2=1$, $x_1=x_3$) (Ising models with respective critical points I_1 and I_2); (ii) $K_3\to\infty$ ($x_1=x_3=0$) and $K_1=K_3=0$ ($x_3=1$, $x_1=x_2$) (3-state Potts models with respective critical points P_1 and P_2); (iii) $K_1=2K_3$ ($x_1=x_2$) (3-component cubic model with critical point C); (iv) $K_2=K_3=0$ (6-state vector Potts model with critical point S). The critical surface P-F intersects the surface F-I2 along the curve P_6CA that lies in the cubic model plane $x_1 = x_2$; this curve is the domain of attraction of the doubly unstable fixed point C. The other curve P_6C^DB, which is the intersection of the P-F and I1-F critical surfaces, lies in the dual plane($x_1 = x_3$) of $x_1 = x_2$ and all of its points flow to the doubly unstable fixed point C^D (the dual of C). Cuts of the full phase diagram for fixed values of x_1 are represented in Fig. 4. The RG flow is illustrated in Fig. 5 for the invariant space $x_1 = x_2$. We also present the phase diagram and some of its cuts in the variables $A \equiv k_B T/(J_1 + J_3)$, $B \equiv (J_1 + J_2)/(J_1 + J_3)$ and $C \equiv (J_1 + 3J_2 + 4J_3)/(J_1 + J_3)$ (see Figs. 6 and 7 respectively). The I1 phase appears only for $C > 4$. For any specific system, as we vary the temperature, the system will present one or two phase transitions depending on the ratios of the coupling constants. In the case of a single transition (P-F), the critical behaviour belongs to the 6-state vector Potts universality class, whereas in the other case the system undergoes an Ising plus a 3-state Potts transitions.

Similarly to the infinitesimal Migdal RG of [6], the present RG recovers the following exact results for the Z(6) ferromagnet in the square lattice:

(i) the self-dual line which contains the points D, P_6 and S

(ii) the location of the Ising (I_1 and I_2) and 3-state Potts (P_1 and P_2) critical points (see the Table)

All the above mentioned particular cases (except the vector Potts model) and the decoupling surface are preserved in the sense that they are invariant subspaces under our RG transformation.

The correlation length critical exponents ν and the crossover exponents ϕ can be obtained from the eigenvalues $\lambda_\ell > 1$ (ℓ = 1,2 or 3) of the Jacobian matrix $\partial(x_1', x_2', x_3')/\partial(x_1, x_2, x_3)$

on the unstable fixed points through the formulae:

$$\nu_\ell = \frac{\ell n\, b}{\ell n\, \lambda_\ell} \qquad (\text{where } \lambda_\ell \gg 1)$$

and

$$\phi = \frac{\ell n\, \lambda_\ell}{\ell n\, \lambda_j} \qquad (\ell, j = 1,2 \text{ or } 3;\ \ell \neq j)$$

where b is the linear expansion factor (b = 2 for Fig. 1). The approximate values of ν_ℓ and ϕ are shown in the Table.

5 CONCLUSIONS

The present RG based on a self-dual two-rooted graph yields to a phase diagram for the Z(6) ferromagnet which agrees with all *exact* results available for the square lattice, except questions related to the presence of a soft phase and to the first-order nature of the 6-state Potts transition. These type of questions are not solved by most of the approximate real space renormalization group methods. Possibly the obtained critical surfaces, with the exception of the para-feromagnetic one, are excellent approximations whereas the thermal critical exponents exhibit non-neglectable discrepancies with those corresponding to the square lattice. On the other hand *all* the present results are *exact* in the hierarchical lattice generated by the transformation shown in **Fig. 1**.

Similarly to the method proposed by Mariz et al [9] for the Z(4) model, we present a new technique for calculating the pair-correlation function in an *arbitrary* Z(6) cluster. Our method avoids the usual tracing procedure, and RG's calculations based in relatively large clusters become tractable through its use.

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APPENDIX

Equivalent vector transmissivities of terminal graphs G_{pr}.

The calculation of the numerator $N_\alpha(G_{pr})$ $(\alpha = 1,2,3)$ and denominator $D(G_{pr})$ of the α-component of the equivalent vector transmissivity $t_\alpha^{eq}(G_{pr})$ of any terminal graph G_{pr} whose bonds are *all* precollapsed (of type 2 and/or 3) involves the counting of certain mod-6 flows on G_{pr}. A mod-N flow $\vec{\phi}$ on any graph G is defined as follows (see, e.g., [23]). First, consider an arbitrary directing of the ε edges $e_1, e_2, \ldots, e_\varepsilon$ of G and then associate to each edge e_i a value ϕ_i which takes on the N values 0, 1,...,N-1. Define an incidence matrix S for each vertex j and edge e by

$$S_{je} = \begin{cases} +1 & \text{if e is directed into j} \\ -1 & \text{if e is directed out of j} \\ 0 & \text{if j is not a vertex of e.} \end{cases}$$

We say that $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_\varepsilon)$ is a *mod-N flow* on G if for each vertex j there is a conservation condition, namely:

$$\partial\phi_j \equiv \sum_{\ell=1}^{\varepsilon} S_{je}\phi_e = 0 \text{ mod } N.$$

If the above condition holds only for unrooted vertices and if in addition, there is an external flow α entering at root 1 and leaving at root 2, i.e.

$$\partial\phi_j = \begin{cases} -\alpha & \text{if } j = 1 \\ +\alpha & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

then $\vec{\phi}$ is a *rooted mod-N* α *flow* on G. Notice that a mod-N flow corresponds to the particular case of $\alpha = 0$.

Now we can go back to out specific problem. It has been shown [14] that $N_\alpha(G_{pr})$ $(\alpha = 1,2,3)$ is the number of rooted mod-6 α flows with the constraint that the flow on any precollapsed edge of type 2 must be 0,2 or 4 and the flow on any precollapsed edge of type 3 must be 0 or 3. $D(G_{pr})$ corresponds to the particular case of zero external flow, i.e., $D(G_{pr}) = N_o(G_{pr})$. In Fig. 8 is shown an example of a graph G_{pr} constituted of 2 precollapsed edges of type and 3 precollapsed edges of type 3. In this example there is only one mod-6 α flow for a fixed external flow $\alpha(\alpha = 0,1,2,3)$.

CAPTION FOR FIGURES AND TABLE

Fig. 1 - Two-rooted graphs associated with the RG recursive relations obtained by renormalizing cluster (a) into cluster (b). The rooted and unrooted vertices are represented by o and ● respectively.

Fig. 2 - (a) Broken graph, (b) collapsed graph, (c) precollapsed graph of type 2, and (d) precollapsed graph of type 3 obtained from that of Fig. 1(a), considering, respectively, $t_1^{(\ell)} = t_2^{(\ell)} = t_3^{(\ell)} = 0$; $t_1^{(\ell)} = t_2^{(\ell)} = t_3^{(\ell)} = 1$; $t_1^{(\ell)} = t_3^{(\ell)} = 0$, $t_2^{(\ell)} = 1$; $t_1^{(\ell)} = t_2^{(\ell)} = 0$, $t_3^{(\ell)} = 1$, where the ℓth bond links vertices 3 and 4.

Fig. 3 - Z(6) ferromagnet phase diagram in the (x_1, x_2, x_3) space. F,P,I1 and I2 denote the respective ferromagnetic paramagnetic, intermediate 1(Z(2) symmetry) and intermediate 2(Z(3) symmetry) phases whose sinks are L,H,L" and L' respectively. Stable fixed points are represented by ■, whereas singly, doubly and completely unstable fixed points are denoted by ●, ⊗ and o respectively. The critical points are: I_1, I_2 Ising; P_1, P_2 3-state Potts; S 6-state vector Potts, P_6 6-state Potts; C 3-component cubic; C^D dual of C; D Ising and 3-state Potts decoupled. The critical lines at $x_3 = 1$ and the self-dual line are represented by --- and -·-· respectively. The arrows indicate the RG flow directions.

Fig. 4 - Phase diagram cuts in the (x_2, x_3) space for the following fixed values of x_1: (a) $x_1 = x_1|_D \simeq 0.1516$; (b) $x_1 = x_1|_{P_6} \simeq 0.2898$; (C) $x_1 = x_2|_{P_2} \simeq 0.3660$; (d) $x_1 = x_1|_S \simeq 0.5860$.

Fig. 5 - RG flow in the invariant subspace $x_1 = x_2$ corresponding to the 3-component cubic model. The dashed line corresponds to the 6-state Potts model. L,L" and H are the sinks of the respective ferromagnetic (F), intermediate 1(I1) and paramagnetic (P) phases.

Fig. 6 - Z(6) ferromagnetic phase diagram in the (A,B,C) space, where $A \equiv k_B T/(J_1 + J_3)$, $B \equiv (J_1 + J_2)/(J_1 + J_3)$ and $C \equiv (J_1 + 3J_2 + 4J_3)/(J_1 + J_3)$. The I1 phase appears only for $C > 4$. The curve OP_6D does not lie in the plane $B = 0$.

Fig. 7 - Phase diagram cuts in the (A,B) space for the following fixed values of C: (a) $C=0$, (b) $C=4$; (c) $C=C|_D \simeq 8.5613$.

Fig. 8 - Example of a terminal graph G_{pr} where $D(G_{pr}) = N_1(G_{pr}) = N_2(G_{pr}) = N_3(G_{pr}) = 1$. The corresponding unrooted and rooted mod-6 α ($\alpha = 1,2,3$) flows on G_{pr} are shown in (a),(b),(c) and (d) respectively. The arrows indicate the arbitrarily chosen directions of the edges; the dashed and dotted lines refer to precollapsed edges of types 2 and 3 respectively. The value of the nonzero external flow entering at root 1 and leaving at root 2 is also shown.

TABLE - Fixed-point coordinates in the (x_1,x_2,x_3) space, correlation length critical exponents (ν) and crossover exponents (ϕ) calculated within our RG scheme. For comparison we list in the last column (ν_{lit}) the values of ν_2 (Ising universality class), ν_3(3-state Potts universality class), ν_c (3-component cubic universality class) and ν_v(6-state vector Potts universality class) available in the literature for the square lattice.

The eigenvectors $\vec{v}_i$ associated to the eigenvalues $\lambda_i=2^{1/\nu_i}$ are in the following directions: $\vec{v}_6$ $\overline{LH}$; $\vec{v}_D$ $\overline{P_6D}$; $\vec{v}_{ND}$ line $x_1=-0.57x_2=0.68x_3$; $\vec{v}_2\overline{P_1P_2}$; $\vec{v}_3$ $\overline{I_1I_2}$; $\vec{v}_c$ line $x_1=x_2=1.22x_3$; $\vec{v}_{NC}$ line $x_1=-x_2$, $x_3=0$; $\vec{v}_c^{(D)}$ line $x_1=x_3=1.07x_2$; $\vec{v}_{NC}^{(D)}$ $x_1=-0.5x_3$, $x_2=0$.

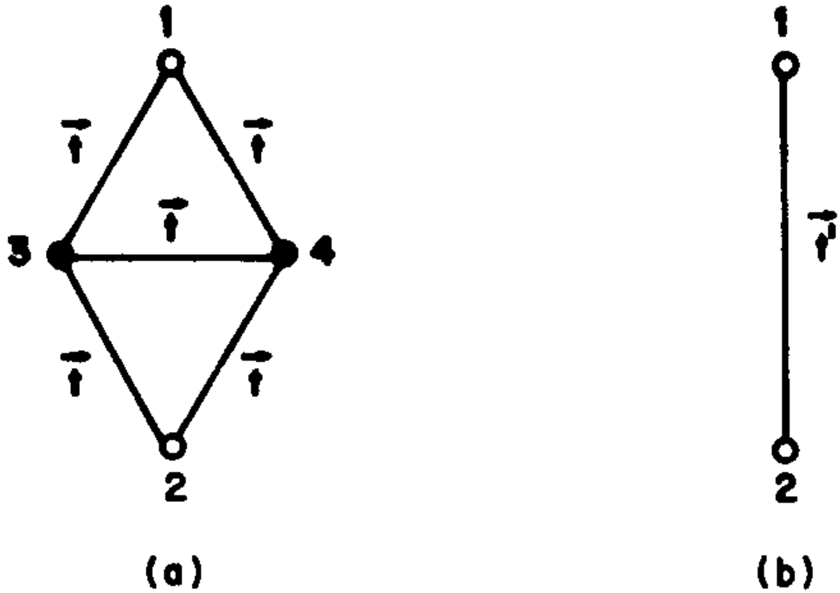
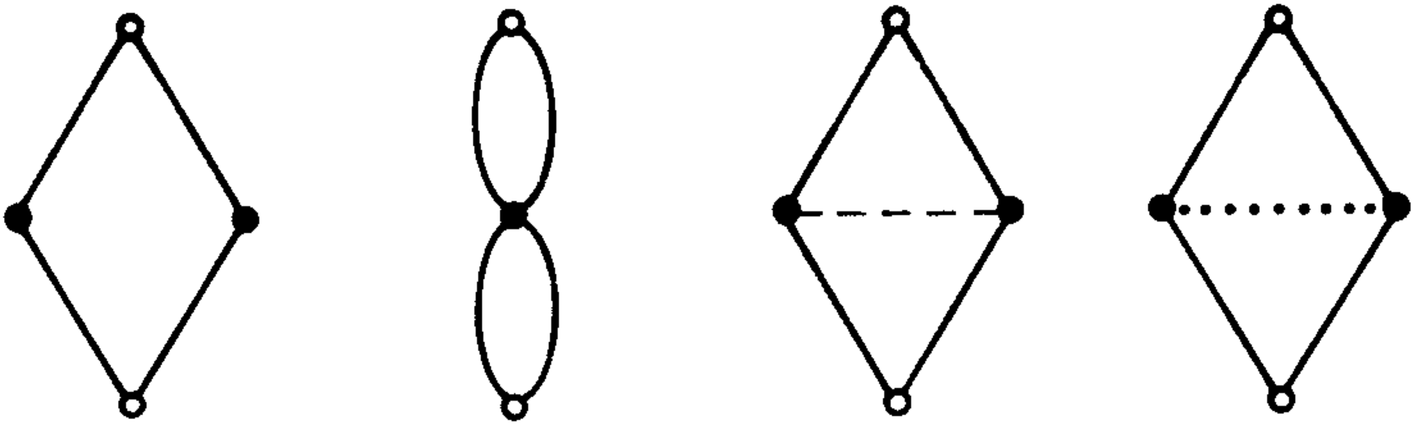


FIG.1



(a) (b) (c) (d)

FIG. 2

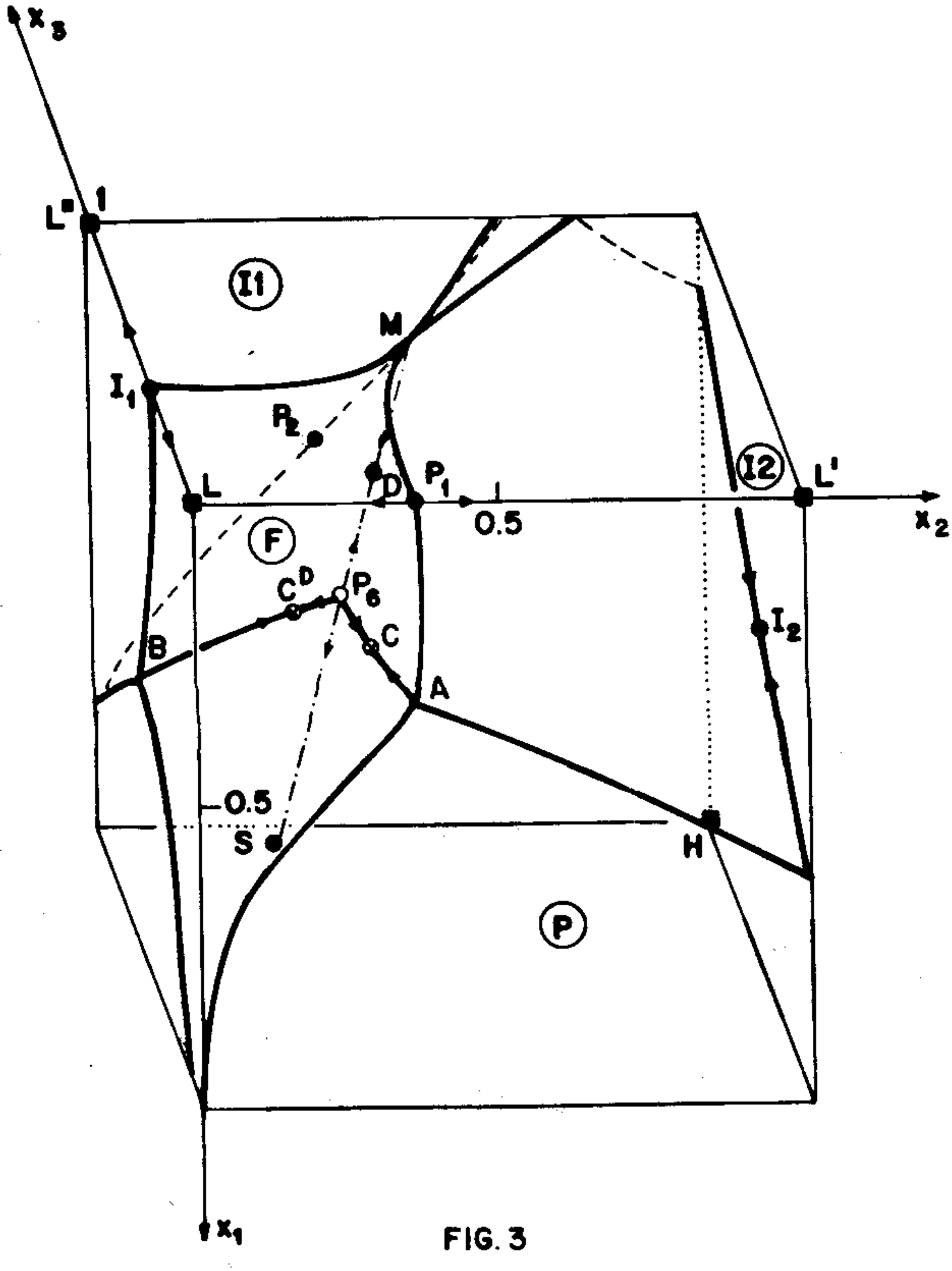


FIG. 3

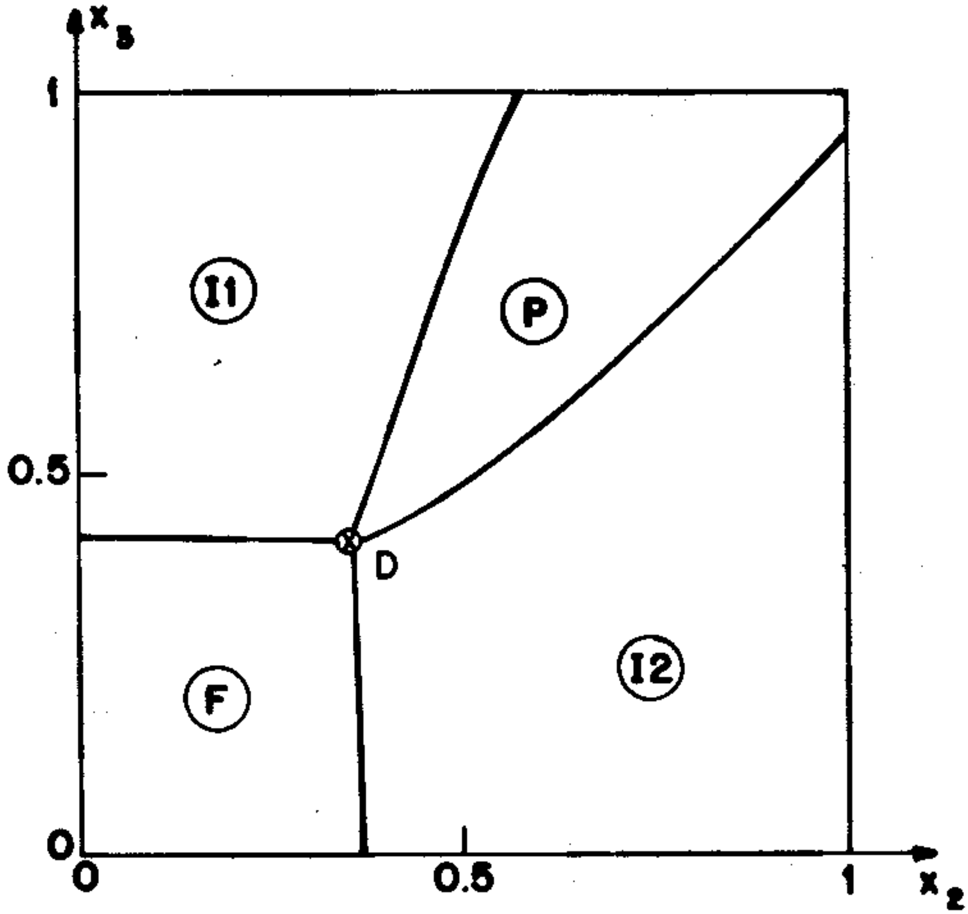


FIG. 4a

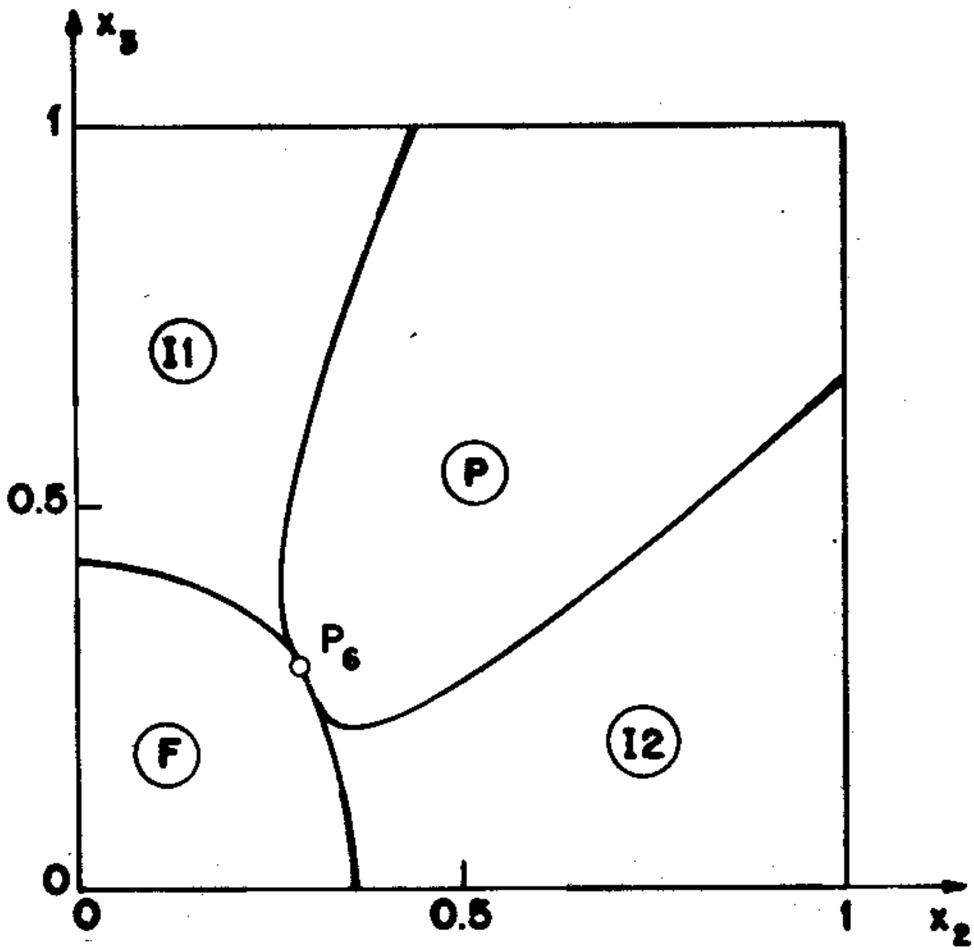


FIG.4b

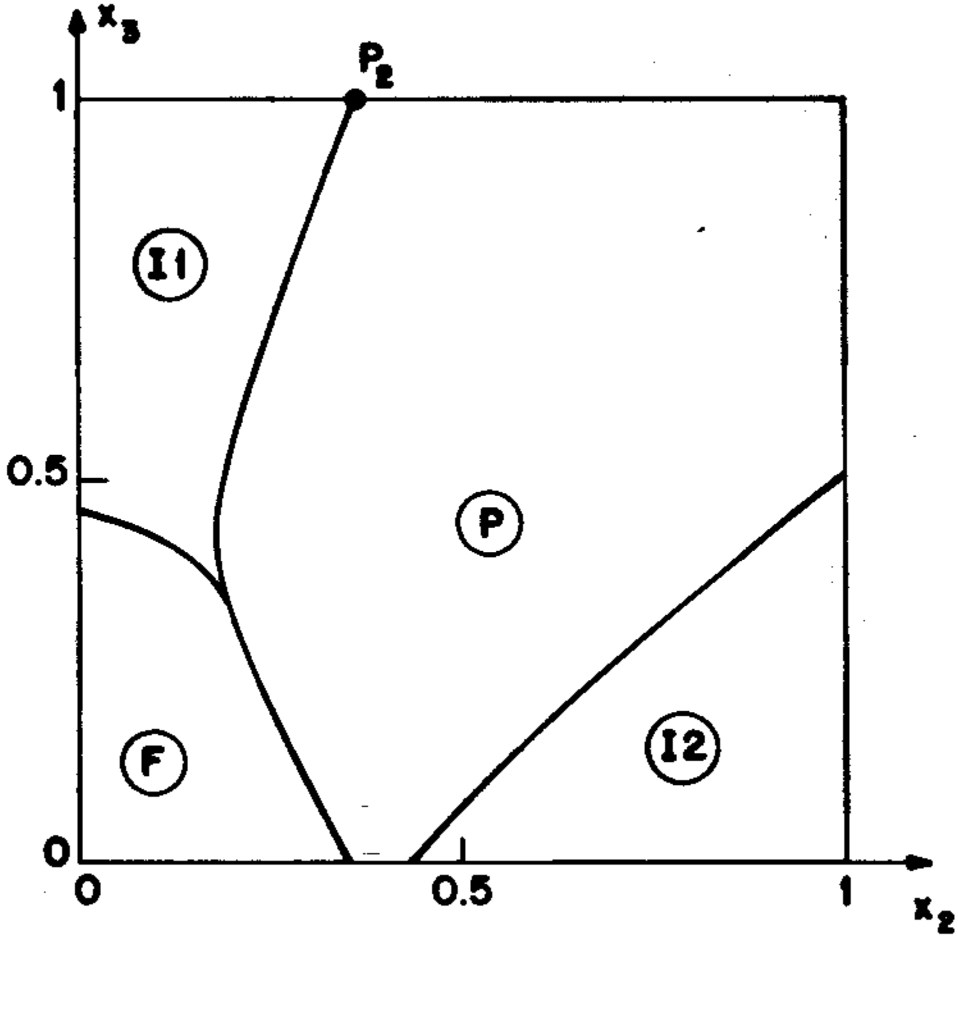


FIG. 4c

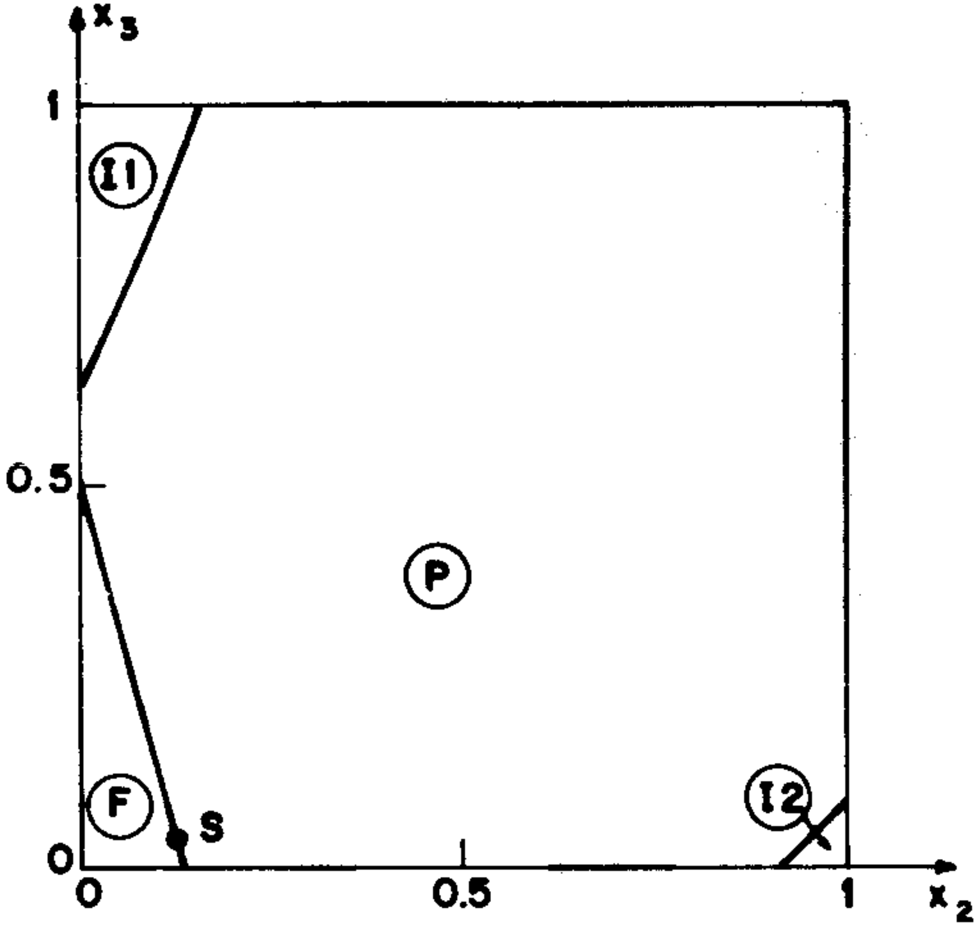


FIG.4d

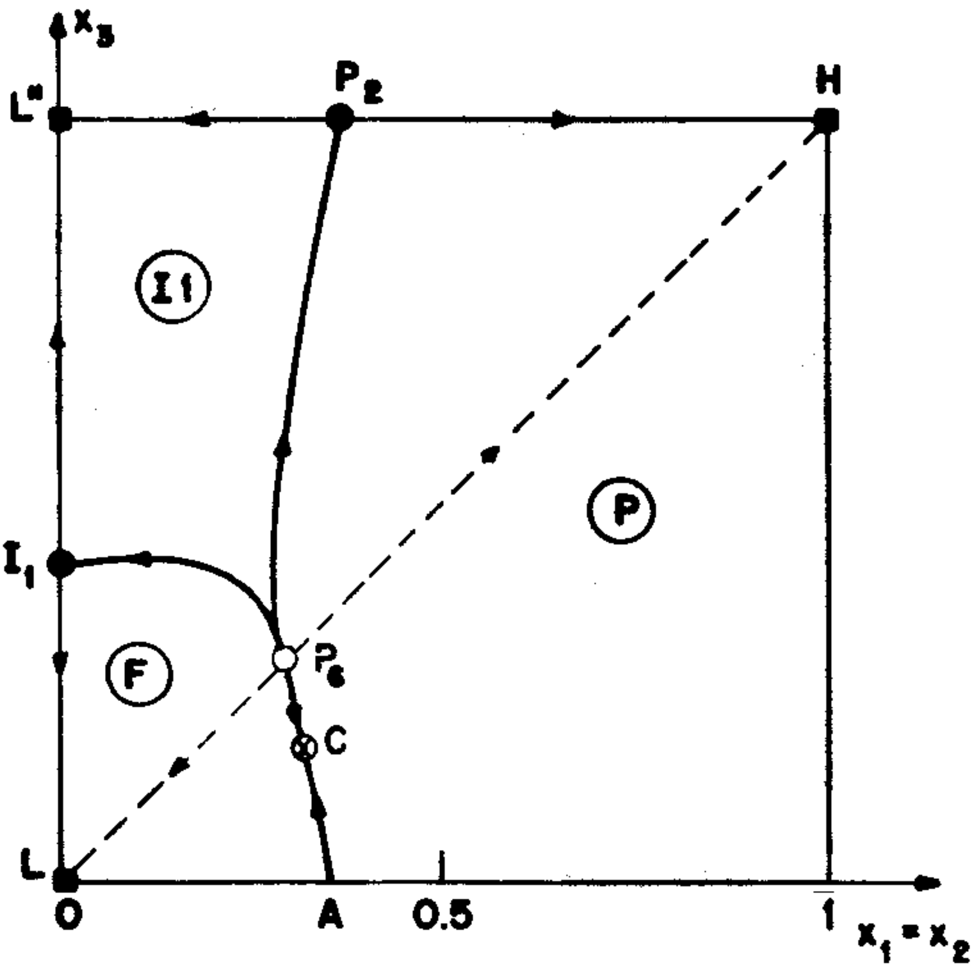


FIG. 5

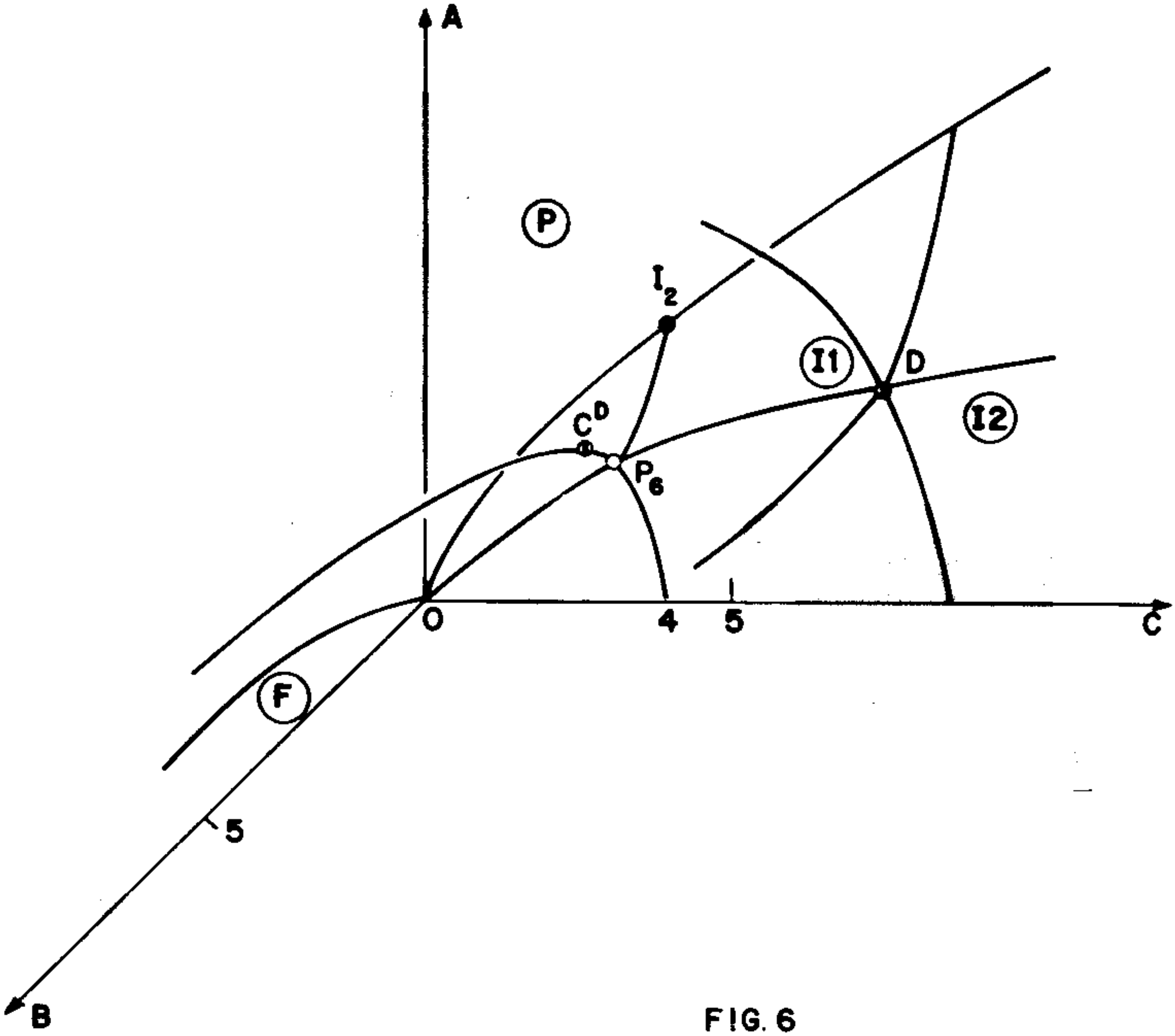


FIG. 6

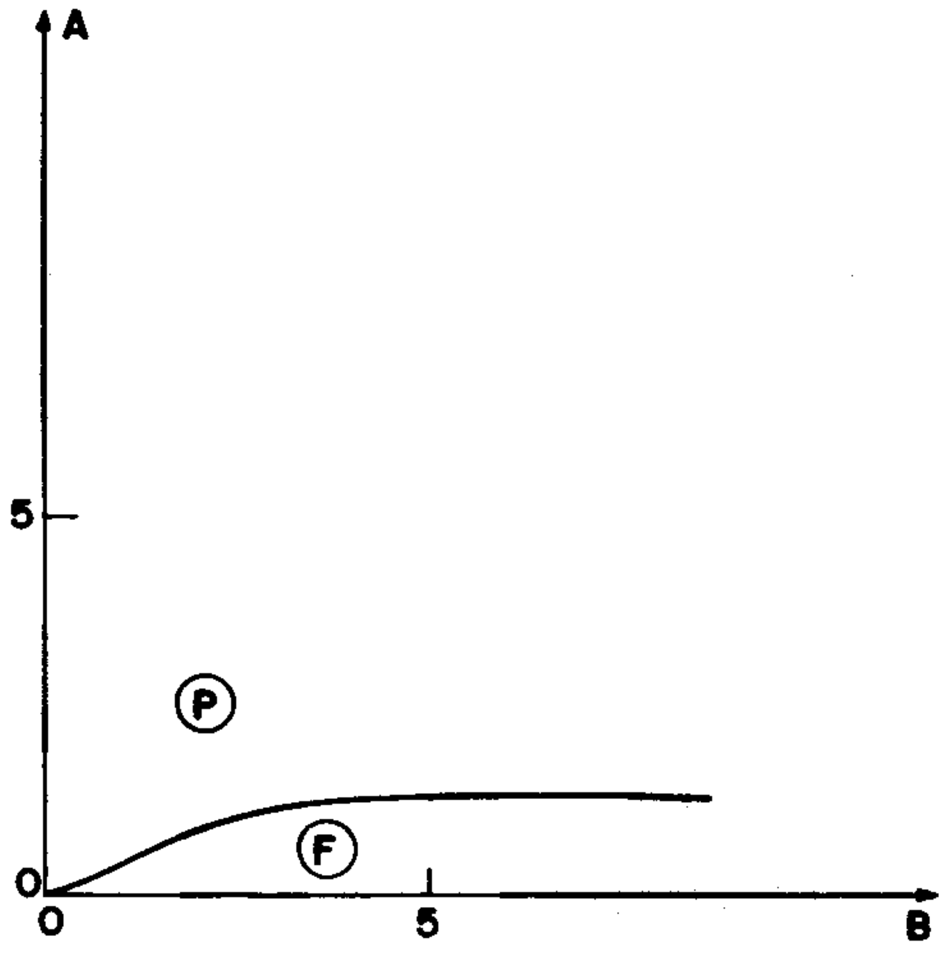


FIG. 7a

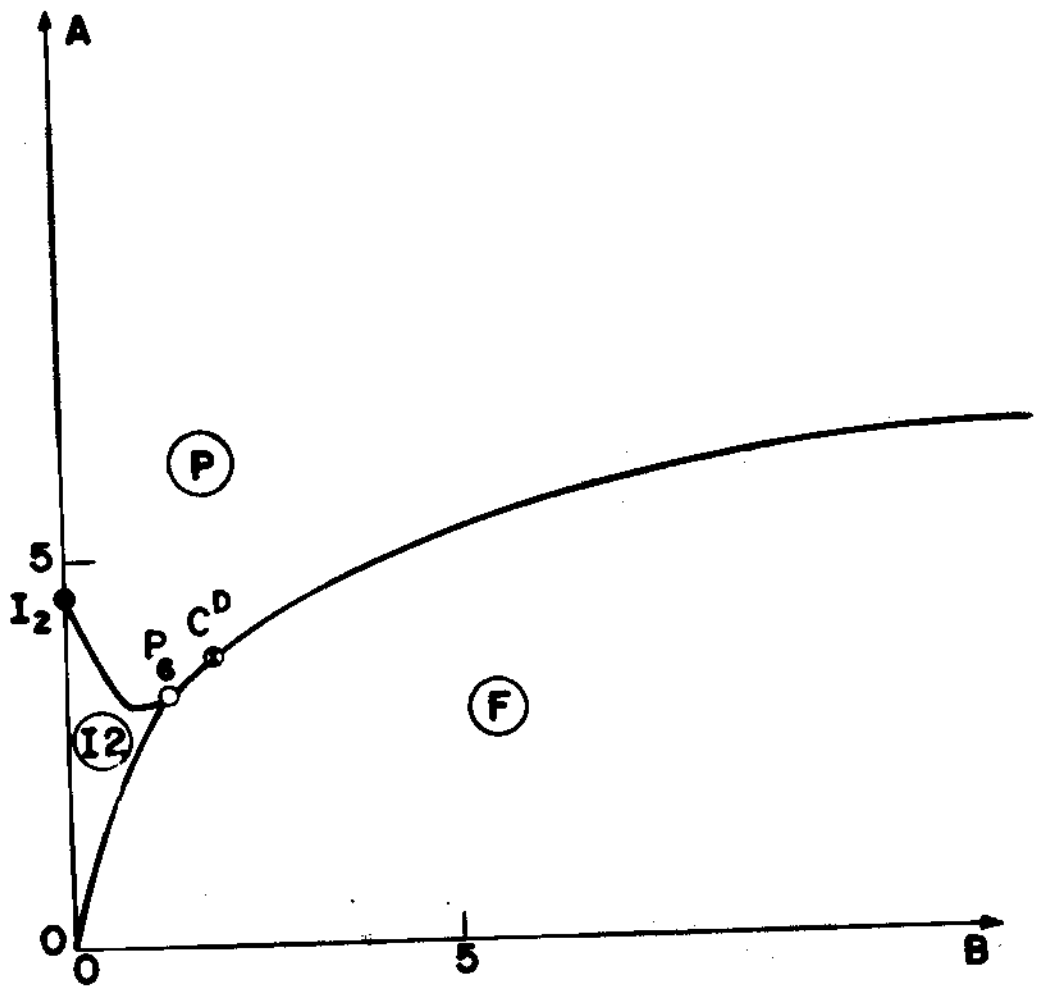


FIG. 7b

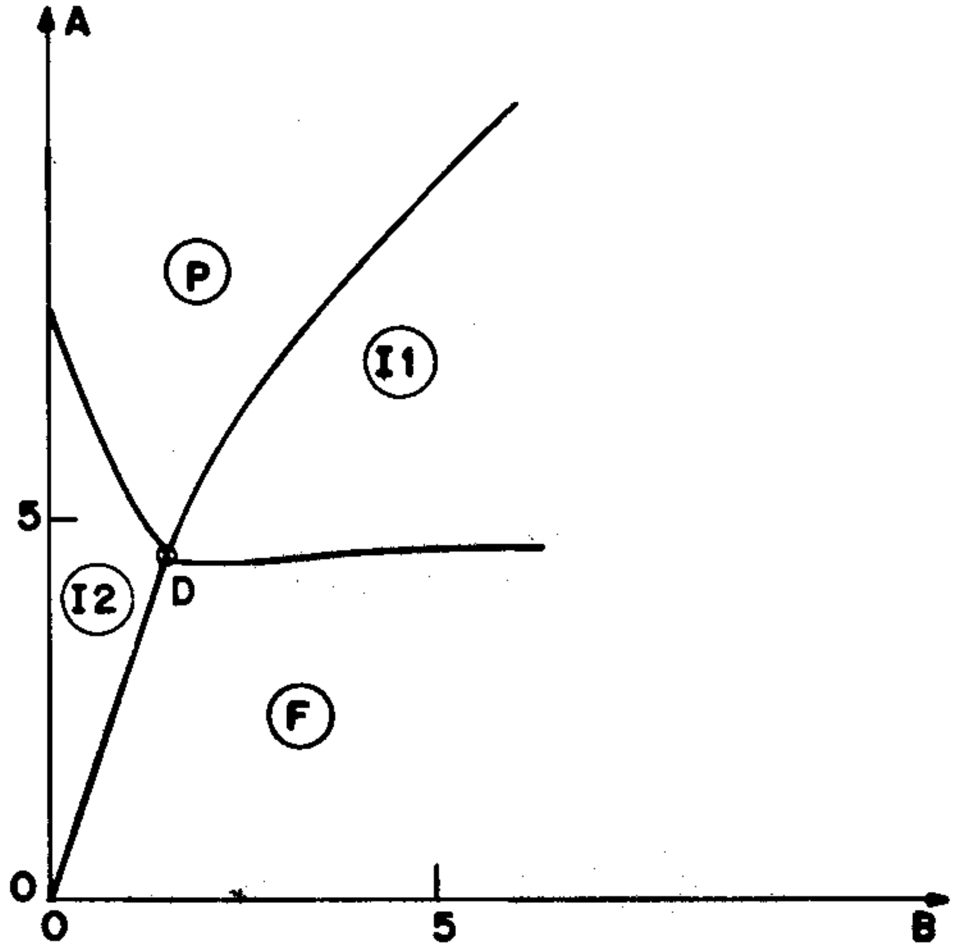


FIG. 7c

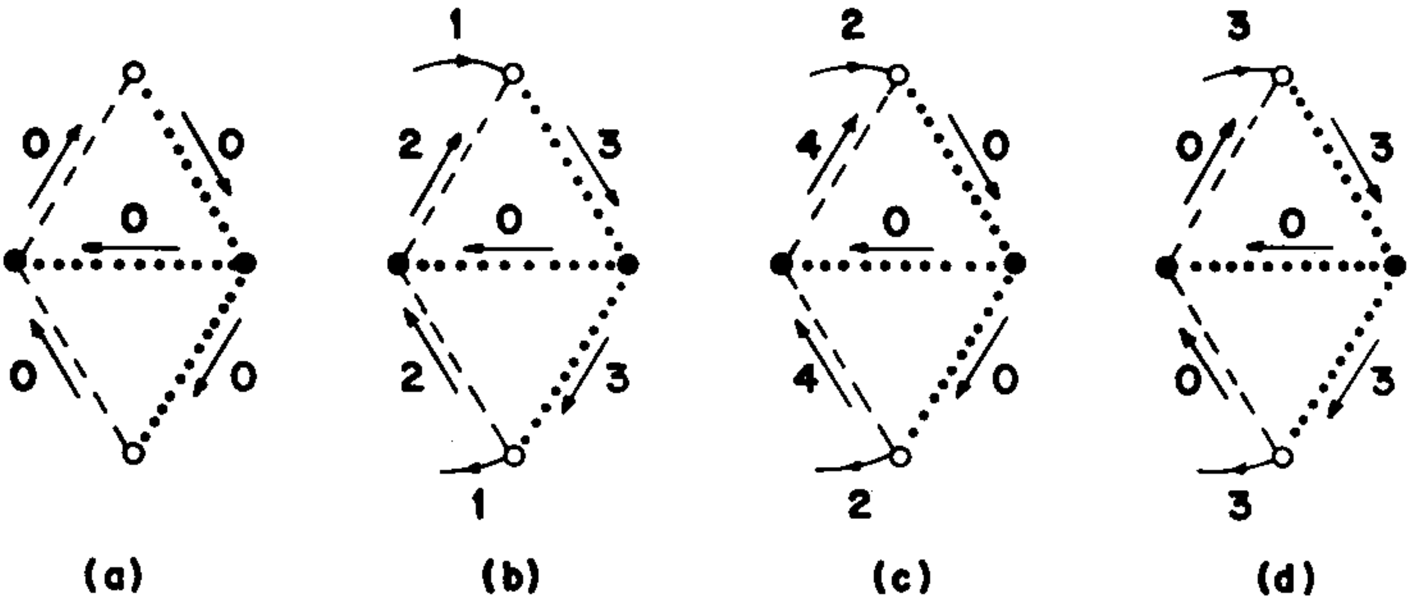


FIG.8

TABLE

Fixed Point (x_1,x_2,x_3)	ν	ν_{lit}
$I_1(0,0,\sqrt{2}-1)$	1.15	1 [21]
$I_2(\sqrt{2}-1,1,\sqrt{2}-1)$	1.15	1 [21]
$P_1(0,(\sqrt{3}-1)/2,0)$	1.02	5/6 [21]
$P_2(\sqrt{3}-1)/2,(\sqrt{3}-1)/2,1)$	1.02	5/5 [21]
S(0.58601,0.12688,0.02370)	$\nu_v = 6.47$	6.7 [22]
$P_6(\sqrt{6}-1)/5,(\sqrt{6}-1)/5,(\sqrt{6}-1)/5)$	$\nu_6 = 0.86$ $\nu_D = 3.68$ $\nu_{ND} = 3.68$ $\phi = 4.28$	-
$D(\sqrt{3}-1)(\sqrt{2}-1)/2,(\sqrt{3}-1)/2,(\sqrt{2}-1)$	$\nu_2 = 1.15$ $\nu_3 = 1.02$ $\phi = 1.13$	1 [21] 5/6 [21] 6/5
C(0.31829,0.31829,0.17343)	$\nu_c = 0.90$ $\nu_{NC} = 1.82$ $\phi = 2.02$	0.67 [11]
C^D(0.33785,0.21943,0.33785)	$\nu_c^{(D)} = 0.90$ $\nu_{NC}^{(D)} = 1.82$ $\phi = 2.02$	-

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