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A GENERALIZED YANG-MILLS THEORY I. GENERAL ASPECTS OF THE CLASSICAL THEORY

by

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ABSTRACT

A generalized Yang-Mills theory which is the non-Abelian version of the generalized electrodynamics proposed by Podolsky is analized both in the Lagrangian and Hamiltonian formulation. A simple class of solutions to the Euler-Lagrange equations is presented and the structure of the Hamiltonian constraints is studied in details.

Key-words: Classical field theory; Gauge fields; Constrained systems.

1 INTRODUCTION

Field theories with Lagrangians containing higher order derivatives has been the subject of considerable interest in several occasions. The earliest suggestions for using higher order theories seems to be those by Weyl and Eddington [1] in attempts to set up a geometric framework for the electromagnetic field. Subsequently, higher order curvature corrections to Einstein's Lagrangian have been proposed in attempts to obtain a more suitable theory to describe gravitation at the quantum level. (See [2] and references therein). Second order corrections to Maxwell Lagrangian have been proposed by Bopp [3] and Podolsky [4] with the purpose to remove the infinities of the theory. More recently higher order theories have been used as a mechanism of regulation of supersymmetric theories [5], and also as corrections to Nambu action in string theory [6] just to mention two examples.

Classical theories with higher order derivatives contains massive excitations which give negative contributions to the energy density of the system and can (possibly) violate causality. On the quantum level those ghost type fields show up as negative norm states so that unitarity could be violated. In spite of these undesirable aspects such theories are endowed with attractive properties including the improved convergence of the Feynman diagrams [8]. (For the general aspects of higher order quantum field theory see the paper by Pais and Uhlenbech [7].)

We feel it is worth to understand the general properties of these theories and in particular the problem of ghost states in higher order gauge theories. As our first step in this direction , this paper is devoted to an anlysis of the (classical) non-Abelian version of the generalized electrodynamics as proposed by Podolsky [4]. (The canonical structure of the Abelian model has been presented in ref. [9].)

The paper is organized as follows. In the second section the Lagrangian formalism is developed and in section III a simple solution of Euler-Lagrange equations is presented. Section IV is devoted to the Hamiltonian formulation: We obtain the constraints and construct the first class Hamiltonian; the generator of gauge transformation is constructed and some comments are made on the gauge fixing problem. Some useful expressions and calculations tools are presented in the appendices.

2 THE LAGRANGIAN EQUATIONS OF MOTION AND CONSERVATION LAWS

The generalized Yang-Mills Lagrangian we are going to consider is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - a^2 D^\mu F^a_{\mu\alpha} D_\nu F^{\nu\alpha}_a + J^a_\mu A^\mu_a \tag{2.1}$$

where $j^a_\mu(x)$ is an external current and $\underline{a}$ is a constant with dimension of length. From (2.1) we obtained the following equations of motion:

$$D_\nu F^{\nu\alpha}_a + 2a^2 \left[D^2 D_\lambda F^{\lambda\alpha}_a - 2D^\beta D^\alpha D^\lambda F^a_{\lambda\beta} \right] = J^\alpha_a \tag{2.2}$$

where we used the notation $D^2 = D^\beta D_\beta$. It follows from the above equations that the external currents is conserved in the covariant sense, $D_\alpha J^\alpha_a = 0$. Unless otherwise stated we shall only consider the free field, $J^\alpha_a = 0$.

In terms of the chromoelectric and chromomagnetic fields

$$E^k_a = F^{ok}_a \quad , \quad B^k_a = -\frac{1}{2}\,\varepsilon^{kij}F^a_{ij} \tag{2.3}$$

we can express the Lagrangian and equations of motion as

$$\mathcal{L} = \frac{1}{2}(\vec{E}^{k\,2} - \vec{B}^{k\,2}) - a^2\left[(D_k\vec{E}^k)^2 + (D_o\vec{E}^k + (D\times\vec{B})^k)^2\right] , \tag{2.4}$$

$$(1 + 2a^2D^2)D_k\vec{E}^k + 2a^2\left[\vec{E}^k, D_o\vec{E}^k + (D\times B)^k\right] = 0, \tag{2.5a}$$

$$(1 + 2a^2D^2)(D_o\vec{E}^k + (D\times\vec{B})^k + 4a^2\left[\vec{E}^k, D_j\vec{E}^i\right]$$

$$- 2a^2\left[D^j, D^k\right](D_o\vec{E}_j + (D\times\vec{B})_j) = 0 \tag{2.5b}$$

where $(D\times\vec{B})^k = \varepsilon^k_{ij}D_iB^a_j$.

The symmetric gauge invariant energy-momentun tensor is given by

$$T_{\mu\nu} = -F^{a\beta}_\mu F^a_{\nu\beta} + \frac{1}{2}\,\eta_{\mu\nu}F^a_{\alpha\beta}F^{\alpha\beta}_a$$

$$- a^2\eta_{\mu\nu}\left(2F^{\alpha\beta}_a D_\alpha D_\lambda F^\lambda_{a\beta} - D_\alpha F^{\alpha\beta}_a D^\lambda F^a_{\lambda\beta}\right)$$

$$- 2a^2\left(2D_\nu F^a_{\mu\beta}D_\lambda F^{\lambda\beta}_a + 2D_\mu F^a_{\nu\beta}D_\lambda F^{\lambda\beta}_a - D^\alpha F^a_{\alpha\mu}D^\lambda F^a_{\lambda\nu}\right) +$$

$$+ 2a^2F^a_{\mu\alpha}\left(D_\nu D_\lambda F_a^{\lambda\alpha} + D^\alpha D^\lambda F^a_{\lambda\nu}\right) + 2a^2F^a_{\nu\alpha}\left(D_\mu D_\lambda F_a^{\lambda\alpha} + D^\alpha D^\lambda F^a_{\lambda\mu}\right) \tag{2.6}$$

which is conserved,

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.7}$$

and not traceless. (When external currents are present one obtains $\partial_\mu T^{\mu\nu} = j^a_\mu F^{\mu\nu}_a$ as expected). The energy density for the gauge fields, $E = T_{oo}$, is

$$\begin{aligned} E = &- F^{o\beta}_a F^a_{o\beta} + \frac{1}{2} F^{\alpha\beta}_a F^a_{\alpha\beta} \\ &- a^2\left(2F^{\alpha\beta}_a D_\alpha D_\lambda F^\lambda_{a\beta} - D_\alpha F^{\alpha\beta}_a D^\lambda F^a_{\lambda\beta}\right) \\ &- 2a^2\left(4D_o F^a_{ok} D_\lambda F^{\lambda k} - D^k F^a_{ko} D^j F^a_{jo}\right) \\ &+ 4a^2F^a_{oj}\left(D_o D_\lambda F_a^{\lambda j} + D^j D^k F^a_{ko}\right). \end{aligned} \tag{2.8}$$

From the above expression one cannot conclude that the total energy of the gauge fields is positive even if one assumes the standard behaviour of the fields at spatial infinity and static configurations.

Finally, for the total current we obtained

$$j^\mu_a = \partial_\nu\left[F^{\mu\nu}_a + a^2(D^\mu D_\alpha F^{\alpha\nu}_a - D^\nu D_\alpha F^{\alpha\mu}_a)\right] \tag{2.9}$$

where we have used equations (2.2). It is clear that this current is conserved,

$$\partial_\mu j^\mu_a = 0 \tag{2.10}$$

3 A CLASS OF PLANE WAVES SOLUTIONS

The equations of motion (2.2) admit a very simple class of solutions which are the generalization of the non-Abelian plane waves obtained by Coleman [10]. The general Ansatz for the gauge potentials is

$$A^a_\mu = (\delta^a_1 + \delta^a_3)\ell_\mu \phi(\ell.x\ ;\overset{(1)}{n}.x\ ,\overset{(2)}{n}.x)$$

$$\equiv (\delta^a_1 + \delta^a_3)\ell_\mu \phi(w;n_1,n_2)\quad . \tag{3.1}$$

In the above expression ℓ is a light-like vector while $\overset{(1)}{n}$ and $\overset{(2)}{n}$ are space-like, satisfying $\ell.\overset{(i)}{n} = 0$, $\overset{(i)}{n}.\overset{(j)}{n} = 0$. Chosing $\ell_\mu = \delta^0_\mu - \delta^3_\mu$ and $\overset{(2)}{n} = \delta^1_\mu$, $\overset{(2)}{n} = \delta^2_\mu$ the gauge potentials assume the form

$$A^a_\mu = (\delta^a_1 + \delta^a_3)(\delta^0_\mu - \delta^3_\mu)\psi(x^0 - x^3\ ; x^1\ , x^2). \tag{3.2}$$

The above Ansatz leads to $\left[\vec{A}_\alpha , \vec{A}_\beta\right] = 0$, $\left[\vec{A}_\alpha\ , \vec{F}_{\mu\nu}\right] = 0$ so that the non-linear terms do not appear in the equations of motion. For the chromoelectric and chromomagnetic fields one finds

$$E^k_a = (\delta^1_a + \delta^3_a)(\delta^k_1 \frac{\partial\psi}{\partial x^1} + \delta^k_2 \frac{\partial\psi}{\partial x^2}) \tag{3.3a}$$

$$B^k_a = (\delta^1_a + \delta^3_a)(\delta^k_2 \frac{\partial\psi}{\partial x^1} - \delta^k_1 \frac{\partial\psi}{\partial x^2}) \quad , \tag{3.3b}$$

Hence, $\vec{E}_a$ and $\vec{B}_a$ are equal in magnitude and orthogonal to each other and to ℓ:

$$\vec{E}^2_a = \vec{B}^2_a = 2\left[(\frac{\partial\psi}{\partial x^1})^2 + (\frac{\partial\psi}{\partial x^2})^2\right] \quad ,$$

$$\vec{E}_a\cdot\vec{B}_a = 0 \quad , \quad E^k_a\ell_k = B^k_a\ell_k = 0 \quad ,$$

which are desirable properties for a plane wave solution. The energy density corresponding to (2.2) is given by

$$\mathcal{E} = 2\left[(\frac{\partial\psi}{\partial x^1})^2 + (\frac{\partial\psi}{\partial x^2})^2\right] + 4a^2(\frac{\partial^2\psi}{\partial {x^1}^2} + \frac{\partial^2\psi}{\partial {x^1}^2})^2$$

$$+ 8a^2\left[\frac{\partial\psi}{\partial x^1}\frac{\partial}{\partial x^1}(\frac{\partial^2\psi}{\partial {x^1}^2} + \frac{\partial^2\psi}{\partial {x^2}^2}) + \frac{\partial\psi}{\partial x^2}\frac{\partial}{\partial x^2}(\frac{\partial^2\psi}{\partial {x^1}^2} + \frac{\partial^2\psi}{\partial {x^2}^2})\right] \tag{3.4}$$

Now, using (3.1) the equations of motion lead to the following equation for the function ψ:

$$(1 + 2a^2\nabla^2)\nabla^2\psi = 0 \tag{3.5}$$

where $\nabla^2 = \partial^2_1 + \partial^2_2$. Taking into account that the above equation leaves the $x^0 - x^3$ dependence of ψ arbitrary, one can chose arbitrary bounded functions $A(x^0 - x^3)$, $B(x^0 - x^3)$ and $C(x^0 - x^3)$ so as to express its solution as

$$\psi(x^0-x^3;x^1,x^2) = A(x^0-x^3)x^1 + B(x^0-x^3)x^2 + C(x^0-x^3)\phi(x^1,x^2) \tag{3.6}$$

where

$$\phi(x^1,x^2) = \int d^2k f(k) e^{i\vec{k}.\vec{x}} + \int d^2\bar{k} g(\bar{k}) e^{i\vec{\bar{k}}.\vec{x}} \tag{3.7}$$

with

$$k^2 = k_1^2 + k_2^2 = 0 \quad , \quad \bar{k}^2 = \bar{k}_1^2 + \bar{k}_2^2 = \frac{1}{2a^2} \quad . \tag{3.8}$$

4 THE HAMILTONIAN FORMULATION

The Hamiltonian approach to higher derivatives theories was first developed by Ostrogrodskii [11] for non-singular systems(*), and his method consists in defining one more pair of canonical variables and so doubling the dimension of the phase space. For singular higher derivatives systems one can in an almost straightforward way generalize Dirac's [12] theory for constrained systems to combine it Ostrogradskii approach. However, the resulting framework is rather cumbersome when applied to a complex Lagrangian like the one given by (2.1) and it is particularly so when the fields under consideration are gauge fields. One can easily be convinced that the source of difficulties relies on the non-gauge covariance of the definition of the new canonical variables (see eq. (4.3) below), a

(*) By a singular system we mean one for which the generalized Hessian matrix is singular. In the present case we have $\det(\partial^2\mathcal{L}/\partial A^a_\mu \partial A^b_\nu) = 0$.

question which we are presently investigating.

We start by expressing the Lagrangian as

$$\mathcal{L} = -\frac{1}{4} F^a_{ij} F^{ij}_a - \frac{1}{2} F^a_{oi} F^{oi}_a$$

$$- a^2 \Big(D_i F^{oi}_a D_j F^{oj}_a + D_i F^{ik}_a D^j F^a_{jk} + D_o F^{ok}_a D_o F^a_{ok}$$

$$+ 2D_o F^{ok}_a D^j F^a_{jk} \Big) \quad . \tag{4.1}$$

The system will now be described by two pairs of canonical variables, (A^α_a, p^a_a) and $(B^\alpha_a \equiv \dot{A}^\alpha_a, \Pi^a_\alpha)$, with the canonical momenta defined by

$$\Pi^b_\alpha = \frac{\partial \mathcal{L}}{\partial \ddot{A}^\alpha_b} = -2a^2 (D_\lambda F^{o\lambda}_b \delta^o_\alpha - D^\lambda D^b_{\alpha\lambda}) \tag{4.2}$$

$$p^b_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{A}^\alpha_b} - 2\, \partial_k \left(\frac{\partial \mathcal{L}}{\partial(\partial_o \partial_k A^\alpha_b)} \right) - \partial_o \Pi^b_\alpha =$$

$$= - F^b_{o\alpha} + 2a^2 \left[\partial_k (D_\lambda F^{k\lambda}_b) \delta^o_\alpha + \partial_k (D_j F^{oj}_k) \delta^b_\alpha \right]$$

$$- 2a^2 C_{abc} \Big(2A^c_o D^\lambda F^a_{\alpha\lambda} - A^\mu_c D^\lambda F^a_{\mu\lambda} \delta^o_\alpha - A^c_\alpha D^k F^a_{ok} \Big) - \dot{\Pi}^b_\alpha \tag{4.3}$$

From the above expressions we obtained the following primary constraints

$$\phi^b_{(1)} = \Pi^b_o \approx 0 \quad , \tag{4.4a}$$

$$\phi^b_{(2)} = p^b_o - D^k \Pi^b_k \approx 0. \tag{4.4b}$$

Following the usual procedure we now eliminate $\dot{B}^k_a$ from the canonical Hamiltonian

$$H_c = \int d^3x \left[p^a_\alpha \dot{A}^\alpha_a + \Pi^a_\alpha \dot{B}^\alpha_a - \mathcal{L} \right] .$$

From (3.2) we get

$$\dot{B}^b_k = -\frac{1}{2a^2} \Pi^b_k + D^j F^b_{kj} + \partial_k B^b_o$$

$$-C_{bcd}\left[A^c_o F^d_{ko} + \partial_o (A^c_k A^d_o) \right]$$

which we substitute in the expression for H_c to obtain

$$H_c = \int d^3x \left[p^a_\alpha B^\alpha_a - \frac{1}{4a^2} \Pi^a_j \Pi^j_a + \Pi^j_a D^i F^a_{ji} \right.$$

$$+ \frac{1}{4} F^a_{ij} F^{ij}_a + \frac{1}{2} F^a_{oj} F^{oj}_a + a^2 D_i F^{oi}_a D_j F^{oj}_a$$

$$\left. + C_{adc}(-\Pi^j_a A^d_o \partial_j A^c_o + 2B^c_j \Pi^j_a A^d_o + C_{cef} A^j_e A^o_f A^d_o \Pi^a_j) \right] \qquad (4.5)$$

Introducing the fundamental Poisson brackets

$$\{p^a_\alpha(x) \ , A^\beta_b(x')\} = -\delta^a_b \delta^\beta_\alpha \delta^{(3)}(\vec{x}-\vec{x}') ,$$

$$\{\Pi^a_\alpha(x) , \ B^\beta_b(x')\} = -\ \delta^a_b \delta^\beta_\alpha \delta^{(x)}(\vec{x}-\vec{x}') \qquad (3.6)$$

one can show that the consistency conditions for the constraints (4.4) lead to the secondary constraint

$$\phi^b_{(3)} = D^k(p^a_k + C_{abc}\Pi^b_k A^c_o) - C_{abc}\Pi^k_b F^c_{ko} \approx 0 \qquad (3.7)$$

The set of constraints $(\phi^a_{(1)},\phi^a_{(2)},\phi^a_{(3)})$ is first class as one can easely verify. We can now write the extended Hamiltonian as

$$H = \int d^3_x\Big[(p^a_k - C_{abc}A^o_b\Pi^c_k)B^k_a - \frac{1}{4a^2}\Pi^a_k\Pi^k_a + \Pi^k_a D^i F^a_{ki}$$

$$+ a^2 D_i F^{oi}_a D_j F^{oj}_a + \frac{1}{4}F^a_{ij}F^{ij}_a + \frac{1}{2}F^a_{oj}F^{oj}_a - C_{bcd}\Pi^k_b A^o_d F^c_{ko}\Big]$$

$$+ \int d^3x\Big[\overset{(1)}{C}_a(x)\Pi^a_o + \Big(\overset{(2)}{C}_a(x) + B^o_a\Big)\Big(p^a_o - D^k\Pi^a_k\Big)$$

$$+ \overset{(3)}{C}_a(x)\Big(D^k(p^a_k - C_{abc}A^o_b\Pi^c_k) - C_{abc}\Pi^k_b F^c_{ko}\Big)\Big] \tag{4.8}$$

where the $C^{(i)}_a(x)$, $i = 1,2,3$, are arbitrary functions. The equations of motion with full gauge freedom generated by the above Hamiltonian are

$$\dot{A}^a_o = B^a_o + \overset{(1)}{C}{}^a(x) \tag{4.9a}$$

$$\dot{A}^a_k = B^a_k - D_k\overset{(3)}{C}{}^a(x) \tag{4.9b}$$

$$\dot{p}^a_k = - D_k D_i\Pi^i_a + D^i D_i\Pi^a_k - D^i F^a_{ik}$$

$$- C_{acb}A^o_c F^b_{ok} + 2a^2 C_{acb}A^o_c D_k D_i F^{oi}_b + F^c_{ok}D_i F^{oi}_a \tag{4.9c}$$

$$- C_{bcd}C_{cfa}A^o_f A^o_d \Pi^b_k + \Big(C^{(2)}_c + B^o_c\Big)C_{cad}\Pi^d_k + \overset{(3)}{C}_d C_{dbc}C_{cfa}A^o_f\Pi^b_k$$

$$\dot{B}^a_o = \overset{(1)}{C}{}^a(x) \tag{4.9d}$$

$$\dot{B}_k^a = -\frac{1}{2a^2}\Pi_k^a + D^i F_{ik}^a - C_{abc}A_b^o B_k^o - C_{abc}A_c^o F_{ko}^b$$

$$+ D_k(C_a^{(2)} + B_a^o) - C_b^{(3)} C_{bac} F_{ko}^c + C_{abc} A_b^o D_k C_c^{(3)} \qquad (4.9c)$$

The equation for Π_k^a simply reproduces the definition (4.3), while for p_o^a we recover the consistency condition for the constraint (4.4b). Following the same steps as in the Abelian case [9] one can show that the system of equations (4.9) is equivalent to the Euler-Lagrange equations (2.2).

With the first class constraints (4.4a,b) and (4.7) we obtained the following generating functional of gauge transformations [13]:

$$G = \int d^3x (p_a^\mu - C_{abc}A_o^c \Pi_b^\mu) D_\mu \Delta^a(x) + \int d^3x \Pi_a^\mu \Omega_\mu + \int d^3x \left[D_\alpha, D_o\right] \Pi_a^\alpha \Delta^a \qquad (4.10)$$

where $\Delta^a(x)$ are arbitrary functions and $\Omega^a \equiv \dot{\Delta}^a$. It is easy to verify that it generates the correct transformations for $A_k^a(x)$ on $B_k^a(x)$, namely

$$\delta A_k^a = \{A_k^a, G\} = D_k \Delta^a ,$$

$$\delta B_k^a = \partial_o (D_k \Delta^a).$$

The generator G obeys the algebra

$$\{G_1(\Delta, \Omega),\ G(\Gamma,\omega)\} = G\left(\Delta\Gamma, (\omega\Delta - \Omega\Gamma)\right) \qquad (4.11)$$

where

$$G\left(\Delta\Gamma,(\omega\Delta-\Omega\Gamma)\right) = \int dx\,(p_a^\alpha - C_{cde}A_o^e\Pi_d^\alpha)D_\alpha(C_{abc}\Delta^b\Gamma^c)$$

$$+\int dx\,\Pi^\alpha D_\alpha\left[C_{abc}(\Omega^c\Gamma^b - \omega^c\Delta^b)\right]$$

$$+\int dx\left[D_\alpha,\ D_o\right]\Pi_a^\alpha(C_{abc}\Delta^b\Gamma^c). \qquad (4.12)$$

For the choice $\Delta^a = \delta_b^a$ both A_k^a and B_k^a transform under the adjoint representation of the gauge group and in this case we have.

$$\{G_a, G_b\} = C_{abc}G_c \quad . \qquad (4.13)$$

The choice of suitable gauge conditions for this systen is obviously not a trivial task for it inherits all the difficulties of the usual Yang-Mills theory besides the peculiarities introduced by the second order term of the Lagrangian. Hence, in order to satisfy the initial data for the system the gauge conditions must involve the fields and its derivatives up to the third order. A good candidate for a generalized Coulomb gauge condition, for instance, could be $\Omega_a[A] = (1-2a^2\Box)\nabla.\vec{A}_a \approx 0$. Using this gauge condition one can follow the same steps as in Yang-Mills theory [14] but it leads to very complicated results with no practical use. In particular, canonical quantization of this system certainly poses unusually difficult problems. One way to circumvent this situation would be to reduce the order of the Lagrangian by introducing some auxiliary field interacting with the Yang-Mills fields such that on shell the original theory is recovered. Of course this is not possible in the general case but it does work in the case under consideration [15], and the resulting theory admits a more tractable Hamiltonian formulation.

APPENDIX A: Notation, convenctions and some calculations tools

In this paper the Minkowski space-time metric is $\eta_{\mu\nu}$ = diag (+1, -1,-1,-1), with $\mu,\nu = 0,1,2,3$. (Latin indices i,j, take the values 1,2,3.) The gauge group G is supposed to be semi-simple and compact. The generators of the corresponding Lie algebra $\{T_a\}$, $a = 1,\ldots,N$, satisfy the commutation relations $[T_a, T_b] = C_{abc}T_c$, $T_a^\dagger = -T_a$.

The field strength tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC_{abc}A^b_\mu A^c_\nu \tag{A.1}$$

(we set the coupling constant g equal to one) satisfy the following relations:

$$[D_\mu, D_\nu]\psi = [F_{\mu k}, \psi] \tag{A.2}$$

$$D_{[\mu} F_{\alpha\beta]} = 0 \tag{A.3}$$

For the variational calculations that lead to the equations of motion the following expressions are useful

$$\tilde{\delta}F^a_{\mu\nu} = D_\mu \tilde{\delta}A^a_\nu - D_\nu \tilde{\delta}A^a_\mu \quad , \tag{A.4a}$$

$$\tilde{\delta}D_\mu f^a(x) = - C_{abc}\delta A^c_\mu f^c(x) \quad . \tag{A.5b}$$

Now, concerning the calculation of the conserved quantities, a, direct application of Noether's theorem to the particular Lagrangian we studied in this paper leads to lengthy calculations and cumbersome non-gauge invariant results. However, one can avoid a lot of work by using gauge invariant variations as proposed by Jackiw and Manton [16] which automatically lead to gauge invariant conserved currents. Their procedure consists in the following. Denoting the space-time coordinate transformations by $\delta x^{\mu} = \omega^{i}\xi^{\mu}_{i}(x) \equiv \xi^{\mu}(x)$ where $\{\xi^{\mu}_{i}(x)\}$ are Killing vectors, one can write the corresponding variations of the gauge potentials as

$$\bar{\delta}A^{a}_{\mu} = \xi^{\alpha}F^{a}_{\alpha\mu} \quad . \tag{A.6}$$

For the field strength one obtains

$$\bar{\delta}F^{a}_{\mu\nu} = \xi^{\alpha}D_{\alpha}F^{a}_{\mu\nu} + (\partial_{\mu}\xi^{\alpha})F^{a}_{\alpha\nu} + (\partial_{\nu}\xi^{\alpha})F^{a}_{\mu\alpha} \tag{A.7}$$

Using (A.2),

$$(\bar{\delta}D_{\mu})f^{a} = \xi^{\lambda}\left[D_{\lambda},D_{\mu}\right]f^{a}$$

so that

$$\bar{\delta}(D_{\alpha}F^{a}_{\mu\nu}) = \xi^{\lambda}\left[D_{\mu},D_{\nu}\right]F^{a}_{\lambda\alpha} + \xi^{\lambda}D_{\alpha}D_{\lambda}F^{a}_{\mu\nu}$$

$$+ (\partial_{\alpha}\partial_{\mu}\xi^{\lambda})F^{a}_{\nu\lambda} - (\partial_{\alpha}\partial_{\nu}\xi^{\lambda})F^{a}_{\lambda\mu}$$

$$- (\partial^{\lambda}\xi_{\mu})D_{\alpha}F^{a}_{\lambda\nu} + (\partial^{\lambda}\xi_{\nu})D_{\alpha}F^{a}_{\lambda\mu} \quad . \tag{A.8}$$

The above expressions is all that is necessary to calculate the gauge invariant Noether currents. We remark that the application of this procedure is admissible since $\mathcal{L}$ is gauge invariant so that $\bar{\delta}A^a_\mu$ can be expressed as in (A-6).

Clearly, if one is interested only in the energy-momentum tensor $T_{\mu\nu}$ the easiest way is to apply Hilbert's prescription. which leads to the following general expression.

$$\frac{1}{2}\sqrt{-g}\; T_{\mu\nu} = \frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}\,\mathcal{L}) - \partial_\lambda\left(\sqrt{-g}\;\frac{\partial\mathcal{L}}{(\partial(\partial_\lambda g^{\mu\nu})}\right) \qquad (A.12)$$

APPENDIX B: Some useful Poisson brackets relations

$$\int d_x F^a_{oi}(x)\{p^b_o(z)\ ,F^{oi}_a(x)\} = -\ D_i F^{oi}_b$$

$$\int d_x D_i F^{oi}_a(x)\{p^b_o(z)\ ,D_k F^{ok}_a(x)\} = D^k D_k D_i F^{oi}_b$$

$$\int d_x D_k F^{oi}_a(x)\{p^b_o(z)\ ,D^k F^a_{oi}(x)\} = D_k D_i D^k F^{oi}_b$$

$$\int d_x D_i F^{oi}_a(x)\{\Pi^b_k(z)\ ,D_j F^{oj}_a(x)\} = D_k D_i F^{oi}_b$$

$$\int d_x D^j F^{oi}_a(x)\{\Pi^b_k(z)\ ,D_j F^a_{oi}(x)\} = D^j D_j F^b_{ok}$$

$$\int d_x F^a_{oi}(x)\{D_k \Pi^k_b(z)\ ,F^{oi}_a(x)\} = D_k F^{ok}_b$$

$$\int d_x D_i F^{oi}_a(x)\{D_k \Pi^k_b(z)\ ,D_j F^{oj}_a(x)\} = D^k D_k D_i F^{oi}_b$$

$$\int d_x D_j F^{oi}_a(x)\{D_k \Pi^k_b(z)\ ,D^j F^a_{oi}(x)\} = D^k D^j D_j F^b_{ok}$$

$$\int d_x D_i F^{oi}_a(x)\{p^b_k(z)\ ,D_j F^{oj}_a(x)\} = C_{bca}(A^o_c D_k D_i F^{oi}_a + F^c_{ok} D_i F^{oi}_a)$$

$$\int d_x D_j F^{oi}_a(x)\{p^b_k(z)\ ,D^j F^a_{oi}(x)\} = C_{bca}(A^o_c D_j D^j F^{ok}_a + F^{oi}_c D_k F^a_{oi})$$

$$\int d_x D_i F^{oi}_a(x)\{D_k p^k_b(z)\ ,D_j F^{oi}_a(x)\} = C_{bca} D^k (A^o_c D_k D_i F^{oi}_a + F^c_{ok} D_i F^{oi}_a)$$

$$\int d_x D_j F^{oi}_a(x)\{D_j p^k_b(z)\ ,D^j F^a_{oi}(x)\} = C_{bca} D^k (A^o_c D_j D^j F^i_{ok} + F^{oi}_c D_k F^a_{oi})$$

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