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COMPACT UNIONS OF CLOSED SUBSETS ARE CLOSED AND COMPACT INTERSECTIONS OF OPEN SUBSETS ARE OPEN*

by

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ABSTRACT

A general sufficient condition is given for unions of closed sets to be closed and intersections of open sets to be open. Examples of applications are offered.

Key-words: Compact; Graph; Equicontinuity.

Among the defining features of a topological space X we have that the union of a finite family of closed subsets of X is closed in X and that the intersection of a finite family of open subsets of X is open in X. These properties no longer remain valid if finite is dropped. They may remain true if additional conditions are assumed in place of finiteness. One of the simplest generalizations of a finite set is the concept of a compact space (for convenience a compact space here is not necessarily a Hausdorff space). To discover how to reach the goal of replacing finiteness by compactness aimed at the title of this text, let $(F_\lambda)_{\lambda\in\Lambda}$ be a family of closed subsets of X. Assume that we have fixed a compact topology on Λ. Every set Λ may be given some (even Hausdorff) compact topology. It will not follow from this mere assumption that the union $\bigcup_{\lambda\in\Lambda} F_\lambda$ is necessarily closed in X. To research a sufficient condition implying that such a union is indeed closed in X, let us assume for simplicity that X and Λ are metrizable spaces (as then we may use sequential reasoning). If a belongs to the closure of $\bigcup_{\lambda\in\Lambda} F_\lambda$ in X, there are $\lambda_n \in \Lambda$ and $a_n \in F_{\lambda_n}$ $(n \in \mathbb{N})$ such that $a_n \to a$ (because X is metrizable). By passing to subsequences, we may assume that there is $\mu \in \Lambda$ such that $\lambda_n \to \mu$ (since Λ is metrizable and compact). Conversely, if there are $\lambda_n \in \Lambda, \mu \in \Lambda, a \in X$ and $a_n \in F_{\lambda_n}$ $(n \in \mathbb{N})$ such that $a_n \to a$ and $\lambda_n \to \mu$, then a belongs to the closure of $\bigcup_{\lambda\in\Lambda} F_\lambda$ in X. A natural sufficient condition that will allow us to conclude that, then a belongs to $\bigcup_{\lambda\in\Lambda} F_\lambda$ (hence that this union is closed in X is the following: whenever $\lambda_n \in \Lambda$, $\mu \in \Lambda, a \in X$ and $a_n \in F_{\lambda_n}$ $(n \in \mathbb{N})$ are such that $a_n \to a$ and $\lambda_n \to \mu$, then it follows that $a \in F_\mu$ (hence $a \in \bigcup_{\lambda\in\Lambda} F_\lambda$). Such a sufficient condition is clearly equivalent to requiring that the subset $\{(x,\lambda) \in X \times \Lambda; x \in F_\lambda\}$ be closed in $X \times \Lambda$. In this new (non-sequential) formulation that sufficient condition is meaningful whether X and Λ are metrizable or not. We are thus led naturally to conjecture Proposition 2 below. Actually, the preceding motivation is a proof of Proposition 2 when X and Λ are metrizable (and the proof by filters of Proposition 2 is a translation of the preceding sequential proof to the general case).

Definition 1. *The subset of $X \times \Lambda$*

$$G = \{(x, \lambda) \in X \times \Lambda;\ x \in F_\lambda\} = \bigcup_{\lambda \in \Lambda} (F_\lambda \times \lambda)$$

is called the graph of the family $(F_\lambda)_{\lambda\in\Lambda}$ in $X \times \Lambda$. The complement $(X \times \Lambda) - G$ is the graph of the family of complements $(X - F_\lambda)_{\lambda\in\Lambda}$ in $X \times \Lambda$.

Proposition 2. *(Compact unions of closed subsets are closed.) Let X be a topological space, Λ be a compact space and $(F_\lambda)_{\lambda\in\Lambda}$ be a family of subsets of X. Assume that the graph G of this family is closed in $X \times \Lambda$. Then $\bigcup_{\lambda\in\Lambda} F_\lambda$ is closed in X.*

Proof by Covers: Assume that $a \in X$ and $a \notin \bigcup_{\lambda\in\Lambda} F_\lambda$. Then $(a, \lambda) \notin G$ for every $\lambda \in \Lambda$. There are a neighborhood U of a in X and an open neighborhood V of λ in Λ such that $(U \times V) \cap G = \phi$ for every $\lambda \in \Lambda$. We then get an open cover of Λ. Since Λ is compact, we may extract a finite subcover of Λ of that cover and consider the finite intersection U of the corresponding neighborhoods of a in X. We then get a neighborhood U of a in X such that $(U \times \Lambda) \cap G = \phi$. Then $U \cap (\bigcup_{\lambda\in\Lambda} F_\lambda) = \phi$. It follows that a does not belong to the closure of $\bigcup_{\lambda\in\Lambda} F_\lambda$ in X. Hence $\bigcup_{\lambda\in\Lambda} F_\lambda$ is closed in X. QED

Proof by Filters: Assume that $a \in X$ belongs to the closure of $\bigcup_{\lambda\in\Lambda} F_\lambda$ in X. For every neighborhood U of a in X, there is some $\lambda \in \Lambda$ such that $U \cap F_\lambda \neq \phi$ and thus $\Lambda_U = \{\lambda \in \Lambda; U \cap F_\lambda \neq \phi\}$ is nonvoid. The subsets Λ_U as U varies over the filter of all neighborhoods of a in X form a base for a filter Φ in Λ. By compactness of Λ, there is some $\mu \in \Lambda$ such that every neighborhood of μ in Λ meets all sets in Φ. Therefore (a, μ) belongs to the closure of G in $X \times \Lambda$. Hence $(a, \mu) \in G$ and $a \in F_\mu \subset \bigcup_{\lambda\in\Lambda} F_\lambda$. Therefore, $\bigcup_{\lambda\in\Lambda} F_\lambda$ is closed in X. QED

Proposition 3. *(Compact intersections of open subsets are open.) Let X be a topological space, Λ be a compact space and $(G_\lambda)_{\lambda\in\Lambda}$ be a family of subsets of X. Assume that the graph G of this family is open in $X \times \Lambda$. Then $\bigcap_{\lambda\in\Lambda} G_\lambda$ is open in X.*

PROOF: Propositions 2 and 3 are dual and equivalent by complementation. Once we are in the situation of Proposition 3, let F_λ be the complement $X - G_\lambda (\lambda \in \Lambda)$. The graph of $(F_\lambda)_{\lambda\in\Lambda}$ in $X \times \Lambda$ is the complement $(X \times \Lambda) - G$ and hence it is closed in $X \times \Lambda$. The assumptions in Proposition 3 lead to those in Proposition 2. We conclude that $\bigcup_{\lambda\in\Lambda} F_\lambda$ is closed in X. By complementation we get that $\bigcap_{\lambda\in\Lambda} G_\lambda$ is open in X. Thus Proposition 2 implies Proposition 3 (and a similar reasoning in the reversed order proves the converse implication). QED

Remark 4. We did not assume in Proposition 2 that the $F_\lambda(\lambda \in \Lambda)$ are closed in X. This is true as a consequence of the fact that G is closed in $X \times \Lambda$. In fact, $\overline{F}_\lambda \times \lambda \subset \overline{F_\lambda \times \lambda} \subset \overline{G} = G$ for any $\lambda \in \Lambda$. Hence $\overline{F}_\lambda \times \lambda \subset F_\lambda \times \lambda$ and $\overline{F}_\lambda \subset F_\lambda$ showing that F_λ is closed in X for any $\lambda \in \Lambda$. A similar remark applies to Proposition 3. See more generally Remark 8 (here Γ is reduced to one element).

COROLLARY 5. *Let X be a topological space and $\mathcal{F}$ be a collection of subsets of X. Assume that we have a compact topology on $\mathcal{F}$ and that the graph $\{(x, F) \in X \times \mathcal{F};\ x \in F\}$ is closed in $X \times \mathcal{F}$. Then $\bigcup \mathcal{F}$ is closed in X.*

COROLLARY 6. *Let X be a topological space and $\mathcal{G}$ be a collection of subsets of X. Assume that we have a compact topology on $\mathcal{G}$ and that the graph $\{(x, G) \in X \times \mathcal{G};\ x \in G\}$ is open in $X \times \mathcal{G}$. Then $\bigcap \mathcal{G}$ is open in X.*

Remark 7. Corollary 5 follows from Proposition 2 by taking $\Lambda = \mathcal{F}$ and $F_\lambda = F$ if $\lambda = F \in \mathcal{F} = \Lambda$. Conversely, Corollary 5 implies Proposition 2 by letting $\mathcal{F}$ be the image set of the mapping $\lambda \in \Lambda \mapsto F_\lambda \in 2^X$ and by using on $\mathcal{F}$ the compact quotient topology (via this mapping) of the compact topology on Λ. A similar remark applies to Corollary 6 and Proposition 3.

Remark 8. Let the assumptions be those of Proposition 2. If $\Gamma \subset \Lambda$ is compact, then the subfamily $(F_\gamma)_{\gamma\in\Gamma}$ whose graph in $X \times \Gamma$ is the intersection of $X \times \Gamma$ with the graph of $(F_\lambda)_{\lambda\in\Lambda}$ is closed in $X \times \Gamma$. In applying Proposition 2, we see that the sufficient condition for $(F_\lambda)_{\lambda\in\Lambda}$ implies the sufficient condition for $(F_\gamma)_{\gamma\in\Gamma}$. Dually for Proposition 3.

Example 9. Let X, Y be topological spaces and $f : X \to Y$ be a mapping whose graph $G = \{(x, f(x)); x \in X\}$ is closed in $X \times Y$. Then f is continuous if Y is compact. In fact, we claim that $f^{-1}(F)$ is closed in X for every F closed in Y. To prove this claim via Proposition 2, take $\Lambda = F$ and $F_\lambda = f^{-1}(\lambda)$ for every $\lambda \in \Lambda$. Next notice that the graph in $X \times \Lambda$ of the family $(F_\lambda)_{\lambda\in\Lambda}$ is the intersection $(X \times \Lambda) \cap G$ which is closed in $X \times \Lambda$. Give Λ the compact topology induced by Y. Apply Proposition 2 to conclude that $\bigcup_{\lambda\in\Lambda} F_\lambda = f^{-1}(F)$ is closed in X.

Example 10. Let X be a topological group, A be closed in X and B be compact in X. We claim that AB is closed in X (and likewise BA is closed in X). We have $AB = \bigcup_{y\in B}(Ay)$. To apply Proposition 2, take $\Lambda = B$ and $F_\lambda = A\lambda (\lambda \in \Lambda)$. We assert that the graph $G = \{(x,\lambda) \in X \times \Lambda; x \in F_\lambda\} = \{(x, y) \in X \times B; xy^{-1} \in A\}$ of the family $(F_\lambda)_{\lambda\in\Lambda}$ is closed in $X \times \Lambda$. In fact, G is the inverse image of the closed subset A of X by the continuous mapping $(x, y) \in X \times B \mapsto xy^{-1} \in X$. Therefore, Proposition 2 gives that $\bigcup_{\lambda\in\Lambda} F_\lambda = AB$ is closed in X.

Example 11. Let X be a topological space which is also an ordered set such that the graph $G = \{(x, y) \in X^2; x \leq y\}$ of the order on X is closed in X^2. For every $A \subset X$ consider its decreasing hull $d(A)$ in X, which is the set of all $x \in X$ such that $x \leq y$ for some $y \in A$. We claim that $d(A)$ is closed in X for every A compact in X. We have $d(A) = \bigcup_{y\in A} d(y)$. To apply Proposition 2, take $\Lambda = A$ and $F_\lambda = d(\lambda) \quad (\lambda \in \Lambda)$. The graph $\{(x,\lambda) \in X \times \Lambda; x \in F_\lambda\} = (X \times \Lambda) \cap G$ of the family $(F_\lambda)_{\lambda\in\Lambda}$ is clearly closed in $X \times \Lambda$. Therefore, Proposition 2 gives that $\bigcup_{\lambda\in\Lambda} F_\lambda = d(A)$ is closed in X. Dually for the increasing hull $i(A)$ in X.

Example 12. Let X be a Hausdorff topological vector space over the field $\mathbf{K}$ (namely $\mathbf{R}$ or $\mathbf{C}$), $n \geq 1$ and $m = 0, \ldots, n$ be fixed integers. Use the notation $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{K}^n$, $x = (x_1, \ldots, x_n) \in X^n$ and $\lambda x = \lambda_1 x_1 + \cdots + \lambda_n x_n \in X$. Denote by F^n_m the subset of X^n of all $x \in X^n$ such that the vector subspace generated by $x_1, \ldots, x_n$ in X has dimension at most equal to $n-m$. Equivalently, F^n_m is the subset of X^n of all $x \in X^n$ such that the vector subspace of $\mathbf{K}^n$ of all $\lambda \in \mathbf{K}^n$ satisfying $\lambda x = 0$ has dimension at least equal to m. Let G^n_m be the complement of F^n_m in X^n. We have $F^n_0 = X^n$, $F^n_n = 0$, $F^n_m \supset F^n_{m+1} (m = 0, \ldots, n-1)$ and F^n_1 is the set of all $x \in X^n$ such that $x_1, \ldots, x_n$ are linearly dependent. Moreover, $G^n_0 = \phi$, $G^n_n = X^n - 0$, $G^n_m \subset G^n_{m+1} (m = 0, \ldots, n-1)$ and G^n_1 is the set of all $x \in X^n$ such that $x_1, \ldots, x_n$ are linearly independent. We claim that F^n_m is closed in X^n and equivalently that G^n_m is open in X^n. To prove this claim we may assume that $m \geq 1$. Fix a scalar product on $\mathbf{K}^n$. Denote by Λ the sphere of center 0 and radius 1 for that scalar product. It is a compact subset of $\mathbf{K}^n$. Let Λ_m be the compact subset of Λ^m formed by all $\lambda = (\lambda^1, \ldots, \lambda^m) \in \Lambda^m$ such that $\lambda^1, \ldots, \lambda^m \in \Lambda$ are pairwise orthogonal. Use the notation $\lambda x = (\lambda^1 x, \ldots, \lambda^m x) \in X^m$ if $\lambda = (\lambda^1, \ldots, \lambda^m) \in (\mathbf{K}^n)^m$ and $x \in X^n$. Set $F_\lambda = \{x \in X^n; \lambda x = 0\} (\lambda \in \Lambda_m)$. Notice that $F^n_m = \bigcup_{\lambda \in \Lambda_m} F_\lambda$. To apply Proposition 2 in establishing that F^n_m is closed in X^n, notice that the graph $\{(x, \lambda) \in X^n \times \Lambda_m; x \in F_\lambda\}$ of the family $(F_\lambda)_{\lambda \in \Lambda_m}$ is closed in $X^n \times \Lambda_m$. In fact that graph is the set of points of $X^n \times \Lambda_m$ where the continuous mapping $(x, \lambda) \in X^n \times \Lambda_m \mapsto \lambda x \in X^m$ vanishes. Therefore, Proposition 2 implies that $\bigcup_{\lambda \in \Lambda_m} F_\lambda = F^n_m$ is closed in X^n.

Proposition 14 below is equivalent to Proposition 2. Hence Proposition 14 subsumes all the preceding examples (see Remark 15).

DEFINITION 13. *Let X and Y be sets. A binary relation between X and Y is a subset R of $X \times Y$. We also call R the graph of that binary relation. We write xRy to denote that $(x, y) \in R$. The inverse binary relation R^{-1} between Y and X is defined by $(y, x) \in R^{-1}$ if and only if $(x, y) \in R$. Hence, yR^{-1} is equivalent to xRy. Clearly R^{-1} is the image of R by the bijection $(x, y) \in X \times Y \leftrightarrow (y, x) \in Y \times X$. Hence $(R^{-1})^{-1} = R$. The direct image $R(A)$ of $A \subset X$ by R is the set of all $y \in Y$ such that there is some $x \in A$ for which xRy. The inverse image $R^{-1}(B)$ of $B \subset Y$ by R is the set of all $x \in X$ such* *that there is some $y \in B$ for which xRy. Hence $R^{-1}(B)$ is also the direct image of $B \subset Y$ by R^{-1}. We have $R(A) = \bigcup_{x \in A} R(x)$ and $R^{-1}(B) = \bigcup_{y \in B} R^{-1}(y)$. Assume that X and Y are topological spaces. We say that R is closed when R is a closed subset of $X \times Y$. Hence R is closed if and only if R^{-1} is closed.*

PROPOSITION 14. *Let X and Y be topological spaces, and R be a closed binary relation between X and Y. Then $R(A)$ is closed in Y for every A compact in X and $R^{-1}(B)$ is closed in X for every B compact in Y.*

PROOF: The two parts of the claim in this proposition are equivalent by inversion. Let us prove the second part. To apply Proposition 2, take $\Lambda = B$ and $F_\lambda = R^{-1}(\lambda)$ $(\lambda \in \Lambda)$. We have $R^{-1}(B) = \bigcup_{\lambda \in \Lambda} F_\lambda$. The graph $\{(x, \lambda) \in X \times \Lambda; x \in F_\lambda\} = (X \times \Lambda) \cap R$ of the family $(F_\lambda)_{\lambda \in \Lambda}$ is closed in $X \times \Lambda$. Therefore, $R^{-1}(B)$ is closed in X by Proposition 2 as wanted. Proposition 14 implies Proposition 2 by taking $Y = \Lambda$ and by using the graph G of Proposition 2 as the binary relation between X and Y. The two proofs we gave for Proposition 2 have corresponding direct proofs of Proposition 14. QED

Remark 15. We get Example 9 from Proposition 14 by defining the binary relation between X and Y whose graph is the graph of f. We get Example 10 from Proposition 14 by defining the binary relation R between X and X as follows: we set xRy if $x \in X$, $y \in X$ and $xy^{-1} \in A$. We get Example 11 from Proposition 14 by defining the binary relation between X and X as the order relation on X. We get Example 12 from Proposition 14 by defining the binary relation R between X^n and Λ_m as follows: we set $xR\lambda$ if $x \in X^n$, $\lambda \in \Lambda_m$ and $\lambda x = 0$.

Another important example is provided by the following result.

PROPOSITION 16. *Let X, Y and Λ be topological spaces where Λ is compact. Consider a family of mappings $f_\lambda : X \to Y (\lambda \in \Lambda)$ such that the joint mapping $(x, \lambda) \in X \times \Lambda \mapsto f_\lambda(x) \in Y$ is continuous (in particular every $f_\lambda : X \to Y$ is* *continuous for $\lambda \in \Lambda$). Then $\bigcup_{\lambda\in\Lambda} f_\lambda^{-1}(F)$ is closed in X for every F closed in Y and $\bigcap_{\lambda\in\Lambda} f_\lambda^{-1}(G)$ is open in X for every G open in Y.*

PROOF: To apply Proposition 2 we set $F_\lambda = f_\lambda^{-1}(F)(\lambda \in \Lambda)$. Notice that the graph $\{(x,\lambda) \in X \times \Lambda; x \in F_\lambda\} = \{(x,\lambda) \in X \times \Lambda; f_\lambda(x) \in F\}$ of the family $(F_\lambda)_{\lambda\in\Lambda}$ is the inverse image of the closed subset F of Y by the continuous mapping $(x,\lambda) \in X \times \Lambda \mapsto f_\lambda(x) \in Y$. Hence this graph is closed in $X \times \Lambda$. Proposition 2 implies that $\bigcup_{\lambda\in\Lambda} f_\lambda^{-1}(F)$ is closed in X. By complementation, we get that $\bigcap_{\lambda\in\Lambda} f_\lambda^{-1}(G)$ is open in X. QED

COROLLARY 17. *Let X, Y be topological spaces and $\mathcal{F}$ be a set of mappings from X to Y. Let $\mathcal{F}$ be endowed with a compact topology such that the mapping $(x, f) \in X \times \mathcal{F} \mapsto f(x) \in Y$ is continuous (in particular, every $f \in \mathcal{F}$ is continuous). Then $\bigcup_{f\in\mathcal{F}} f^{-1}(F)$ is closed in X for every F closed in Y and $\bigcap_{f\in\mathcal{F}} f^{-1}(G)$ is open in X for every G open in Y.*

Remark 18. As in Remark 7, we get Corollary 17 from Proposition 16 and conversely Corollary 17 implies Proposition 16.

Remark 19. In the spirit of Proposition 3 we can give a sufficient condition for a compact intersection of a family of neighborhoods of a point to be a neighborhood of that point. Equicontinuity in the Ascoli Theorem and the Banach-Steinhaus Theorem, deals with an intersection of a family of neighborhoods of a point being a neighborhood of that point but in a fashion different from the one treated here.