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SPINNING FLUIDS IN GENERAL RELATIVITY: A VARIATIONAL FORMULATION

by

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ABSTRACT

In this paper we present a variational formulation for spinning fluids in General Relativity. In our model each volume element of the fluid has rigid microstructure. We deduce a symmetrical energy-moment tensor where there is an explicit contribution of kinetic spin energy to the total energy.

Key-words: Spin; Variational formulation; Rigid microstructure.

1 INTRODUCTION

The variational formulations to describe relativistic fluids with internal spin has received a lot of attention by many authors in special relativity and in relativistic theories of gravitation (general relativity and Einstein-Cartan theory) [1-7]. The more notable were the generalizations of Halbwachs' [9] treatment of spinning fluids made by Ray and Smalley [1,2] and by Obukhov and Korotky [3]. The model in the ref. [2] is different from the original Weyssenhoff fluid [8], while in the Obukhov's work it's shown that, after some refinement and generalization of the Ray and Smalley action, it's possible to obtain a lagrangean theory to describe the Weyssenhoff spinning fluid. In this work we propose a lagrangean that differs a little from those already mentioned. The basic distinction is in the way we introduce the spin kinetic energy density and the Gibbs equation used to describe thermodynamical properties of the fluid. In the model we have introduced a symmetric second-order tensor $J^{\alpha\beta}$ that generalise the notion of rotational inertia. This tensor is used to express the condition of rigid microstructure of the fluid [10,11,12] that is adopted here. These notions can be found in Maugin's works [11,12] and in references therein. In order to generalize this scenario to general relativity we add the gravitational lagrangean density and couple the gravitational field minimally with the continuous media in consideration.

The structure of the paper is as follows. In the second section we present the basic results of the description of a simple fluid with rigid microstructure and give the Lagrangean of such fluid. We also discuss the question of including the spin tensor like a thermodynamical variable. In the third section we obtain the gravitational field equations and its source, the energy-momentum tensor of the fluid. This tensor coincide with the phenomenological energy-momentum tensor obtained by Maugin [12] when dissipation and eletromagnetic field are absent. The important fact with consequences on the dynamics of the gravitational field in cosmology and astrophysics is the enlarged total energy density ρ now including the kinetic energy of the fluid. Finally, in the conclusion we discuss some of generic consequences of the theory obtained.

2 RELATIVISTIC FLUIDS WITH RIGID MICROSTRUCTURE: VARIATIONAL PRINCIPLE

Let's introduce an orthonormal tetrad of vectors $e^{(A)}{}_{\mu}(x)$, where, in our notation, $A = 0,1,2,3$ label the tetrad vectors and $\mu = 0,1,2,3$ label the components. As usual these vectors satisfy:

$$e^{(A)}{}_{\mu} e_{(A)\nu} = g_{\mu\nu} \tag{2.1}$$

$$e^{(A)}{}_{\mu} e^{\mu(B)} = \eta^{AB} \tag{2.2}$$

where $g_{\mu\nu}$ is the spacetime metric and η_{AB} is the Minkowski metric diag(+1,-1,-1,-1). The following choice for the vector $e^{(0)}{}_{\mu}$ is done:

$$e^{(0)}{}_{\mu} = \frac{U_{\mu}}{c} \tag{2.3}$$

where U_μ is the hydrodynamical field of velocity of the fluid and c is the light velocity.

The angular velocity of the spin, $\tilde{\Omega}_{\alpha\beta}$, is given by [11]

$$\tilde{\Omega}_{\alpha\beta} = \hat{\dot{e}}^{(A)}{}_{[\alpha}\, e_{\beta](A)} \qquad (2.4)$$

In this expression the bracket indicates the usual antissimetrization and the dot denotes differentiation along the fluid flow

$$\dot{e}^{(A)}{}_{\mu} = U^{\alpha}\, \nabla_{\alpha} e^{(A)}{}_{\mu} \qquad (2.5)$$

The decomposition of $\tilde{\Omega}_{\alpha\beta}$ in terms of its irredutibles parts has the form

$$\tilde{\Omega}_{\alpha\beta} = \Omega_{\alpha\beta} + \frac{2}{c^2}\, \dot{U}_{[\alpha}\, U_{\beta]} \qquad (2.6)$$

with $\Omega_{\alpha\beta}\, U^{\beta} = 0$

The spin tensor of fluid is related to angular velocity by:

$$S^{\alpha\beta} = \frac{1}{2}\, J^{\mu[\alpha}\, \Omega^{\beta]}{}_{\mu} \qquad (2.7)$$

where $J^{\mu\nu} = J^{\nu\mu}$ is the rotational inertia tensor and satisfy the relation

$$J^{\mu\nu}\, U_{\nu} = 0 \qquad (2.8)$$

The expression (2.7) can be write in the vector form as:

$$S^{\alpha} = J^{\alpha\beta}\,\Omega_{\beta} \tag{2.9}$$

with the axials vectors S^{α} and Ω_{β} defined in the usual way using the skew-symmetric tensor $\eta_{\alpha\beta\mu\nu}$.

The rotational inertia tensor is conserved along the fluid flow

$$\left(J^{\alpha\beta}\, e^{(A)}_{\alpha}\, e^{(B)}_{\beta}\right)^{\cdot} = \left(J^{AB}\right)^{\cdot} = 0 \tag{2.10}$$

traducing the rigidity of the structured microscopic constituents of the fluid. Finally, we write the density of kinetic energy of rotation:

$$\rho_s = \frac{1}{2}\, n\, S^{\alpha\beta}\Omega_{\alpha\beta} = \frac{1}{4}\, nJ^{\mu\alpha}\,\Omega^{\beta}{}_{\mu}\Omega_{\alpha\beta} = +\frac{1}{4}\, nJ^{ki} e^{\mu}_{(k)} e^{\alpha}_{(j)} \dot{e}_{\alpha(i)} \dot{e}^{(j)}{}_{\mu} \tag{2.11}$$

where n is particle density number.

With all this elements we can now write the following Lagrangean density for the spinning fluid as:

$$\mathcal{L}_f = -\sqrt{-g}\,\Big[F(n,s) + \frac{1}{4}\, nJ^{ki} e^{\mu}_{(k)} e_{\alpha(i)} \dot{e}^{\alpha}_{(j)} \dot{e}^{(j)}_{\mu} + \lambda_1 \nabla_{\alpha}(nU^{\alpha}) + \\ + \lambda_2 U^{\alpha}\partial_{\alpha} s + \lambda_3 U^{\alpha}\partial_{\alpha} X + \lambda^{AB}(g_{\mu\nu} e^{\mu}_{(A)} e^{\nu}_{(B)} - \eta_{AB})\Big] \tag{2.12}$$

In this expression s is the entropy per particle, F(n,s) is the usual energy density of the fluid, X the particle identity variable and λ_1, λ_2, λ_3, λ_{AB} are lagrange multipliers associated with the various constraints. The energy density can be write in terms of the internal energy density, $n\varepsilon$, and the mass density of constituents as

$$F(n,s) = n(ac^2+\varepsilon) = \rho \tag{2.13}$$

Before going to the variations with respect to the field variables we would like to call attention that our lagrangean is very similar to the others already mentioned in the Introduction[1,2,3,6,7]. The distinct points are the introduction of rotational inertia tensor and the form of density kinetic energy given by (2.11). In our approach we don't use any particular relation to orient the axial vector S^{α} as colinear to any $e_{\mu}^{(A)}$, neither the spin density tensor $S^{\alpha\beta}$ is considered a independent variable. Starting of field theory definition of spin density tensor and applying to the lagrangean (2.12) we have

$$\sqrt{-g}\, n\, S_{\alpha\beta} \equiv e_{[\beta(a)} \frac{\partial \mathcal{L}_f}{\partial \dot{e}^{\alpha]}_{(a)}} = \frac{\sqrt{-g}}{2}\, n\, J^{\mu}{}_{[\alpha}\Omega_{\beta]\mu} \tag{2.14}$$

that is in agreement with our earlier definition given by (2.7). The final remark is the fact of not considering $S^{\alpha\beta}$ as a thermodynamical variable. This is consistent with the theory of polar fluids[10], and due to the mechanical character of the spin tensor $S^{\alpha\beta}$, that wouldn't contribute to the thermodynamical state of the system.

Now, we outline the results of varying the lagrangean density (2.12) with respect to the field variables λ_1, λ_2, λ_3 and λ_{AB}. These variations lead to the following equations:

$$\nabla_{\alpha}(nU^{\alpha}) = 0 \tag{2.15}$$

$$\dot{s} = 0 \tag{2.16}$$

$$\dot{X} = 0 \tag{2.17}$$

$$g_{\mu\nu}e^{\mu}_{(A)}e^{\nu}_{(B)} - \eta_{AB} = 0 \tag{2.18}$$

Those equations express, respectively, the conservation of the number particle, the conservation of entropy per particle, the conservation of particle identity and the condition of orthonormalization of the tetrads. Next, we present the equations coming from variations with respect to the n, s, X, U^{ν} and $e^{\tau}_{(a)}$, respectively:

$$\dot{\lambda}_1 = \frac{\partial F}{\partial n} + \frac{1}{4} J^{ki}e^{\mu}_{(k)}e_{\alpha(i)}\dot{e}^{\alpha}_{(j)}\dot{e}^{(j)}_{\mu} \tag{2.19}$$

$$\frac{\partial F}{\partial s} - \nabla_{\mu}(\lambda_3 U^{\mu}) = 0 \tag{2.20}$$

$$\nabla_{\mu}(U^{\mu}\lambda_4) = 0 \tag{2.21}$$

$$\frac{1}{4} n J^{ki} e^{\mu}_{(k)}e_{\alpha(i)}\left(\dot{e}^{(j)}_{\mu}\nabla_{\nu}e^{\alpha}_{(j)} + \dot{e}^{\alpha}_{(j)}\nabla_{\nu}e^{(j)}_{\mu}\right) + \lambda_1\partial_{\nu}n + \\ + \lambda_2\partial_{\nu}s + 2\frac{\lambda_o}{c}^{B} e_{\nu(B)} + \lambda_3\partial_{\nu}X - \partial_{\nu}(n\lambda_1) = 0 \tag{2.22}$$

$$\frac{1}{2} n J^{ka} e^{\mu}_{(k)}\dot{e}_{\tau(j)}\dot{e}^{(j)}_{\mu} + 2\lambda^{Aa} e_{\tau(A)} - \frac{1}{2} n J^{ki}\left(e^{\mu}_{(k)}e_{\tau(i)}\dot{e}^{(a)}_{\mu}\right)^{\bullet} = 0 \tag{2.23}$$

where i, j, k, a = 1, 2, 3.

The equations (2.22) and (2.23) can be used to obtain the explicit form of multipliers λ^{oo}, λ^{oi} and λ^{ab}. Then, contracting (2.22) with U^{ν} and in view of (2.15)-(2.19), we get

$$2\lambda_{oo} = n\,\frac{\partial F}{\partial n} - \frac{1}{4}\,n\,J^{ki}\,e^{\mu}_{(k)}e_{\alpha(i)}\dot{e}^{\alpha}_{(j)}\,\dot{e}^{(j)}_{\mu} \qquad (2.24)$$

Now contracting (2.23) with U^{τ} one obtain:

$$2\lambda_{o}^{\;a} = \frac{1}{2c}\,nU^{\tau}\left(J^{ki}\dot{e}_{\tau(i)}e^{\mu}_{(k)}\dot{e}^{(a)}_{\mu} - J^{ka}\dot{e}_{\tau(k)}e^{\mu}_{(k)}\dot{e}^{(j)}_{\mu}\right) \qquad (2.25)$$

In order to determine λ^{ab} we contract (2.23) with $e^{\tau(a)}$ and obtain:

$$\frac{1}{2}\,n\,J^{ka}e^{\mu}_{(k)}\dot{e}_{\tau(j)}\dot{e}^{(j)}_{\mu}e^{\tau(b)} + 2\lambda^{ab} - \frac{1}{2}\,n\,J^{ki}e^{\tau(b)}\left(e^{\mu}_{(k)}\,e_{\tau(i)}\,\dot{e}^{(a)}_{\mu}\right)^{\bullet} = 0 \qquad (2.26)$$

By symmetrization and antisymmetrization of this equation in indices a, b, we get respectively

$$2\lambda^{ab} = \frac{1}{2}\,nJ^{ki}e^{((b)}_{\tau}\left(\dot{e}_{\mu}^{\;(a))}e^{\tau}_{(i)}e^{\mu}_{(k)}\right)^{\bullet} - \frac{1}{2}\,nJ^{k(a}e^{(b))}_{\tau}\,e^{\mu}_{(k)}\dot{e}^{\tau}_{(j)}\dot{e}^{(j)}_{\mu} \qquad (2.27)$$

$$0 = J^{k[a}\,e^{(b)]}{}_{\tau}\;e^{\mu}_{(k)}\dot{e}^{\tau}_{(j)}\dot{e}^{(j)}_{\mu} - J^{ki}\,e^{[(b)}{}_{\tau}\left(e_{\mu}^{\;(a)]}\,e^{\mu}_{(k)}e^{\tau}_{(i)}\right)^{\bullet} \qquad (2.28)$$

The equation (2.28) is nothing else than the dynamical equation of the spin angular momentum:

$$\dot{S}^{ba} + S^{ja}\,\Omega^{b}{}_{j} + S^{bj}\,\Omega^{a}{}_{j} = 0 \qquad (2.29)$$

Using the equation (2.6) in a convenient way, the above equation can be write as

$$\dot{S}^{\alpha\beta} + S^{\alpha\mu} \frac{\dot{U}_\mu U^\beta}{c^2} - S^{\beta\mu} \frac{\dot{U}_\mu U^\alpha}{c^2} = 0 \tag{2.30}$$

telling us that the spin tensor $S^{\alpha\beta}$ is Fermi transported along of the world-line of each element of fluid.

3 GRAVITATIONAL FIELD EQUATIONS FOR A SPINNING FLUID IN GENERAL RELATIVITY

The total lagrangean density that describe the perfect fluid with gravitational interaction is given by:

$$\mathcal{L} = \frac{1}{2\chi} \sqrt{-g}\, R + \mathcal{L}_f \tag{3.1}$$

where R is the scalar curvature, $\chi = \frac{8\pi G}{c^4}$ and G is the gravitational constant. The variation of the action obtained with this lagrangean density with respect to $g_{\mu\nu}$ gives the following equations:

$$G^{\sigma\tau} = -\chi \Big[g^{\sigma\tau}\Big(F(n,s) - n \frac{\partial F}{\partial n}\Big) + \frac{U^\sigma U^\tau}{c^2} \Big(n \frac{\partial F}{\partial n} - \frac{1}{4} n J^{ki} e^\mu_{(k)} e_{\alpha(i)} \dot{e}^\alpha_{(j)} \dot{e}^{(j)}_\mu\Big) -$$

$$- 2\nabla_\mu (n S^{\mu(\sigma} U^{\tau)}) + \frac{2n}{c^2} \dot{U}_\mu S^{\mu(\tau} U^{\sigma)} \Big] \tag{3.2}$$

In view of equations (2.11), (2.13) and the Gibbs equation

$$T\, ds = d\varepsilon + p\, d\Big(\frac{1}{n}\Big) \tag{3.3}$$

we can rewrite (3.2) in the more familiar form

$$G^{\sigma\tau} = -\chi\left[(\rho + \frac{1}{2} n S^{\alpha\beta}\Omega_{\alpha\beta}) \frac{U^{\sigma}U^{\tau}}{c^2} - ph^{\sigma\tau} - 2\nabla_{\mu}(nS^{\mu(\sigma}U^{\tau)}) + \right.$$

$$\left. + \frac{2n}{c^2} \dot{U}_{\mu} S^{\mu(\tau}U^{\sigma)}\right] \tag{3.4}$$

In this equation $h^{\sigma\tau} = g^{\sigma\tau} - \frac{U^{\sigma}U^{\tau}}{c^2}$ is the projector, p is the thermodynamical pressure and the parentheses around the indices mean symmetrization, $A_{(\sigma\tau)} = \frac{1}{2}(A_{\sigma\tau} + A_{\tau\sigma})$.

The source of the gravitational field can be separated in two parts: a perfect fluid energy-momentum tensor, $T_f^{\sigma\tau}$, and the spin energy-momentum tensor $T_s^{\sigma\tau}$ as follows:

$$T_f^{\sigma\tau} = \rho \frac{U^{\sigma}U^{\tau}}{c^2} - ph^{\sigma\tau} \tag{3.5}$$

$$T_s^{\sigma\tau} = \frac{1}{2} nS^{\alpha\beta}\Omega_{\alpha\beta} \frac{U^{\sigma}U^{\tau}}{c^2} - 2\nabla_{\mu}(nS^{\mu(\sigma}U^{\tau)}) + \frac{2n}{c^2}\dot{U}_{\mu} S^{\mu(\sigma}U^{\tau)} \tag{3.6}$$

The distinction between the tensor we obtain and those obtained by Ray and Smalley [1,2] and Obhukov and Korotky [3] is the presence of the kinetic energy density in our model. Such result express the fact that a commoving observer with an arbitrary volume element of the fluid perceive its rest energy, internal energy and its internal rotating energy. The new energy contribution can change the dynamics of usuals cosmological models with a spinning perfect fluid as source. We also emphasize that the tensor $T^{\sigma\tau} = T_f^{\sigma\tau} + T_s^{\sigma\tau}$ is the same derived by Maugin [12] from a phenomenological approach when the fluid is in thermodynamical equilibrium and there is no electromagnetic interaction.

As a concluding remark we write the conservation laws of energy and momentum. So, from Einstein's equations and Bianchi's identities we have:

$$\nabla_\tau\left[n(ac^2+\varepsilon+\frac{1}{2}S^{\alpha\beta}\Omega_{\alpha\beta})\frac{U^\sigma U^\tau}{c^2}-ph^{\sigma\tau}+\frac{2n}{c^2}\dot{U}_\mu S^{\mu\sigma}U^\tau\right]+$$

$$+\,n\,R^\sigma{}_{\beta\mu\nu}S^{\nu\mu}U^\beta=0 \qquad (3.7)$$

The coupling between the curvature and the spin is know as Mathisson-Papapetron force. Projecting equation (3.7) parallel and perpendicular to U^μ we obtain respectively:

$$n\left[\dot{\varepsilon}+p\left(\frac{1}{n}\right)^{\cdot}\right]=0 \qquad (3.8)$$

$$n\,J_{\sigma\lambda}\dot{U}^\sigma+\frac{2n}{c^2}S_{\mu\lambda}(\ddot{U}^\mu)_\perp-(\nabla_\lambda p)_\perp+n\,h_{\sigma\lambda}R^\sigma{}_{\beta\mu\nu}S^{\nu\mu}U^\beta=0 \qquad (3.9)$$

In deducting equations (3.8) and (3.9) we use the balance law of spin kinetic energy [11] $\frac{1}{2}n(S^{\alpha\beta}\Omega_{\alpha\beta})^{\cdot}=0$. The spatial tensor $J_{\sigma\lambda}$ that is given by:

$$J_{\sigma\lambda}=\left[a+c^{-2}(\varepsilon+\frac{1}{2}S^{\alpha\beta}\Omega_{\alpha\beta}+\frac{p}{n})\right]h_{\sigma\lambda}+\frac{2}{c^2}(\dot{S}_{\sigma\lambda})_\perp \qquad (3.10)$$

can be interpreted as a generalization of inertia (see ref. 12). The second derivative $\ddot{U}^\mu$ is an usual phenomenon of self interaction, so we need three initial conditions to specify a unique solution of the equation even in flat space-time.

4 CONCLUSIONS

In this work we present a variational formulation for spinning perfect fluid where each element volume is considered to have rigid microstructure. The energy-momentum tensor is derived and we note the explicit contribution of the spin kinetic energy to the total energy. As stressed in text, our treatment is distinct of another formulations [1,2,3,6]: we not "tie" the spin density vector to the third axis of the tetrads, neither consider independents variations of S^i and $e^{\alpha}_{(i)}$. By the contrary we use a well established relation between the spin density tensor and the spin angular velocity [12].

The next step is to apply the results of this paper in cosmological models as well in astrophysics. In the first case, the galaxies (or clusters) would be the spinnings particles of the fluid, and in the second, localizated spinning fluids distributions with spherical and cylindrical symmetries are subjects of interest. In the mentioned problems, besides the usual equations (Einstein's equations, conservations laws of energy-momentum), it will be necessary to take into account equations of balance of spin angular momentum and spin kinetic energy. The last is given by [11]:

$$n\,\Omega_{\alpha\beta}(S^{\alpha\beta})^{\cdot}{}_{\perp} = \frac{1}{2}\,n\,(S^{\alpha\beta}\Omega_{\alpha\beta})^{\cdot} = 0 \qquad (4.1)$$

Some final remarks are important to emphasize. In our theory the Gibbs equations is maintained in its original form, that is, spin density tensor isn't considered a thermodynamical variable. In this way, we don't need to impose certain "consistency relations" [5] that are lacking of precise physical meaning, and can be violated in general case. Finally, the present theory is in agreement with the phenomenological approach done by Maugin[12] if we consider the case of thermodynamical equilibrium and absence of electromagnetic interaction.

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