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ON TACHYON QUANTIZATION

by

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ABSTRACT

There are two different cases in $\vec{k}$-space for tachyon quantization, namely: when $|\vec{k}|^2 > m^2$ and when $|k^2| < m^2$. We concentrate our discussion on the second case, for which we write the hamiltonian, which is different from the usual harmonic oscillator one, and solve the corresponding eigenvalue equation. The spectrum is continuous and we find the energy creation and destruction operators.

Key-words: Quantum field theory; Particle theory.

1 INTRODUCTION

When one generalizes in a straghtforward way the supersymmetric Wess-Zumino model[1] to spaces of dimension $d > 4$, one is led to equations of motion of higher order.

For the free field case these equations have the form[2]

$$(\Box^{\frac{\omega}{2}} - m^{\omega})\phi = 0 \quad , \quad \omega = 2^{\frac{d}{2} - 1} \quad , \tag{1}$$

with metric $(-,+,+,+,...)$.

For any $d > 4$ these equations have a tachyonic component, i.e., we can always factorize a "tachyonic operator":$(\Box + m^2)$, with all the implied intrinsic difficulties: imaginary mass, lack of unitarity, causality, etc.

When quantizing the field that obeys eq. (1), the presence of the above mentioned tachyonic operator gives rise to some problems which we want to discuss in this paper.

2 LAGRANGIAN AND HAMILTONIAN

For the specific case of six dimensions, equation (1) takes the form

$$(\Box^2 - m^4)\phi = 0 \tag{2}$$

which can be derived from the second order Lagrangian

$$\mathcal{L} = \frac{1}{2}\Box\phi\Box\phi - \frac{1}{2}m^4\phi^2 \tag{3}$$

through the Euler equation:

$$\frac{\partial \mathfrak{L}}{\partial \phi} - \partial_\alpha \frac{\partial \mathfrak{L}}{\partial \partial_\alpha \phi} + \partial_\alpha \partial_\beta \frac{\partial \mathfrak{L}}{\partial \partial_\alpha \partial_\beta \phi} = 0 \tag{4}$$

The energy-momentum tensor is[3]

$$-T^{\mu\nu} = \partial^\mu \partial^\alpha \phi \partial^\nu \partial_\alpha \phi - 2\, \phi \partial^\mu \partial^\nu \phi - \partial^\alpha \phi \partial_\alpha \partial^\mu \partial^\nu \phi$$

$$+\partial^\mu \phi \partial^\nu \phi + \partial^\nu \phi \partial^\mu \phi + \frac{1}{2} \eta^{\mu\nu} (\Box \phi \Box \phi - m^4 \phi^2) \tag{5}$$

In particular

$$T^{00} = -\frac{1}{2} \ddot{\phi}\ddot{\phi} - \dot{\phi}\Delta\dot{\phi} + \dot{\phi}\dddot{\phi} + \frac{1}{2} \Delta\phi\Delta\phi - \frac{1}{2} m^4 \phi^2 \tag{6}$$

The solutions of eq. (3) can be written in the form:

$$\phi(x) = \int dk \left\{ a_k^1 e^{-ikx} + a_k^2 e^{ikx} + (b_k^1 e^{-i\bar{k}x} + b_k^2 e^{i\bar{k}x})\, \theta(\vec{k}^2 - m^2) + \right.$$

$$\left. + (c_K e^{-\bar{\omega}t} e^{i\vec{k}.\vec{r}} + d_k e^{+\bar{\omega}t} e^{-i\vec{k}.\vec{r}}) \theta(m^2 - \vec{k}^2) \right\} \tag{7}$$

Where $k^0 = \omega = \sqrt{\vec{k}^2 + m^2}$; $\bar{k}^0 = \bar{\omega} = \left| \sqrt{\vec{k}^2 - m^2} \right|$ and $\theta(x)$ is the Heaviside step function.

When the field is real ($\phi(x) = \phi^*(x)$), we have:

$$a_k^{1*} = a_k^2 \quad , \quad b_k^{1*} = b_k^2 \quad ; \quad c_k^* = c_{-k} \quad , \quad d_k^* = d_{-k} \tag{8}$$

With (6) and (7) we get the Hamiltonian:

$$H = \int d\vec{k}\ T^{00} = m^2 \int d\vec{k} \Big\{ (a_k^* a_k + a_k a_k^*)\omega^2 - (b_k^* b_k + b_k b_k^*)\bar{\omega}^2 \theta(\vec{k}^2 - \omega^2) + $$

$$+ (c_k d_k + d_k c_k)\bar{\omega}^2 \theta(m^2 - \vec{k}^2) \Big\} \qquad (9)$$

The last term in eq. (9) is the contribution of tachyons having $|\vec{k}| < m$ to the energy of the field. This sphere in $\vec{k}$-space is left out of consideration in the work of reference [4].

3 COMMUTATION RELATIONS

Heisenberg equation of motion

$$\Big[H, \phi\Big] = i\dot{\phi} \qquad (10)$$

together with eq. (7) and eq. (9), gives for a, a^*,

$$m^2 \int dk \int dk^1 [(a_k^* a_k + a_k a_k^*),\ a_{k'}]\omega^2 e^{-ik'x} = \int dk\ a_k\ \omega\ e^{-ikx} \qquad (11)$$

From which it follows:

$$[a_k^*, a_{k'}] = -\frac{\delta(\vec{k} - \vec{k}')}{2m^2\ \omega} \qquad (12)$$

In an analogous way, we deduce:

$$[b_k^*, b_{k'}] = \frac{\delta(\vec{k}-k')}{2m^2\bar{\omega}} \tag{13}$$

and

$$[c_k, d_{k'}] = \frac{i\delta(\vec{k}-\vec{k}')}{2m^2\,\bar{\omega}} \tag{14}$$

In the Hamiltonian (9), the "a" and "b" terms have the usual form of the harmonic oscillator (up to a sign for the "b terms") and the same happens with the commutation rules (12) and (13), so we shall not discuss them here.

We take the "c d" terms and consider the following Hamiltonian

$$\mathcal{H} = m^2\int dk\ (c_k d_k + d_k c_k)\ \bar{\omega}^2\theta(m^2-\vec{k}^2) \tag{15}$$

which can be taken to represent a superposition of systems having for each degree of freedom $\vec{k}$, the Hamiltonian:

$$H_k = \frac{1}{2}\ (p_k q_k + q_k p_k) \tag{16}$$

where we have defined:

$$c_k = \frac{q_k}{m\sqrt{2\bar{\omega}}} \quad , \quad d_k = \frac{p_k}{m\sqrt{2\bar{\omega}}} \tag{17}$$

with $[q_k, p_k] = i$

in a "discretized" $\vec{k}$-space.

Note that according to (8), c_k and d_k are not hermitian, so, strictly speaking we should write in (17), Re c_k (or Im c_k) and Re d_k (or Im d_k), but for the sake of simplicity we assume c_k and d_k to be real. We shall also suppress the subindex k.

4 EIGENVALUE EQUATION

We have then the problem of solving the eigenvalue equation for the operator (16); i.e.:

$$\frac{1}{2}\,(pq+qp)\,\psi = E\psi \tag{18}$$

or, in the usual q-representation

$$-\frac{i}{2}\left(\frac{d}{dq}\,q + q\,\frac{d}{dq}\right)\psi = E\psi$$

leading to the differential equation:

$$q\,\frac{d}{dq}\,\psi = (iE - \frac{1}{2})\psi \tag{19}$$

whose formal solution is

$$\psi = A\,q^{iE - \frac{1}{2}} \tag{20}$$

However we have to be careful as (20) is not well defined at the origin. In order to overcome this difficulty we follow reference [5] and define:

$$\psi_+ = Aq_+^{iE-\frac{1}{2}} \tag{21}$$

and

$$\psi_- = Bq_-^{iE-\frac{1}{2}} \tag{22}$$

where

$$\left.\begin{array}{lll} q_+^\lambda = q^\lambda & & \text{for } q > 0 \\ q_+^\lambda = 0 & & \text{for } q \le 0 \\ q_-^\lambda = |q|^\lambda & & \text{for } q < 0 \\ q_-^\lambda = 0 & & \text{for } q \ge 0 \end{array}\right\} \tag{23}$$

Both (21) and (22) are linear independent solutions of (19) for any value of E. What about normalization and orthogonality?

It is evident that the ψ_+ functions are orthogonal to the ψ_- functions. Let us take two ψ_+ eigenfunctions.

$$<\psi_+^1|\psi_+^2> = A_1A_2^*\int_0 dq\ q^{i(E_1-E_2)-1} \tag{24}$$

Changing variables to $y = \ell nq$, we obtain

$$<\psi_+^1|\psi_+^2> = A_1A_2^*\int_0^\infty dy\ e^{i(E_1-E_2)y} = 2\pi\ A_1A_2^*\ \delta(E_1-E_2) \tag{25}$$

So that for δ-function normalization we take $A = \frac{1}{\sqrt{2\pi}}$ and then eq. (25) gives the orthonormality relations.

To prove completeness let us consider

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dE\ x_+^{-iE-\frac{1}{2}}\ y_+^{iE-\frac{1}{2}} = \frac{1}{2\pi\sqrt{x_+y_+}}\int_{-\infty}^{+\infty} dE\ e^{iE(\ell ny_+-\ell nx_+)} =$$

$$= \frac{1}{\sqrt{x_+y_+}}\ \delta(\ell ny_+-\ell nx_+) = \delta(y_+-x_+) \tag{26}$$

Of course, we have formulae analogous to (25) and (26) for the ψ_- (eq. (22)) eigenfunctions.

As an example we shall consider the expansion of a plane wave in the eigenfunctions (21).

The Fourier transform of q_+^λ is (See ref. [5])

$$\int dq\ q_+^\lambda\ e^{ipq} = i\ e^{i\frac{\pi}{2}\lambda}\ \Gamma(-iE+\tfrac{1}{2})(p+io)^{iE-\frac{1}{2}} \tag{27}$$

$$\lambda = -iE - \frac{1}{2}$$

So, we can write the plane wave as a superposition of ψ_+ states

$$e^{ipq} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dE\ e^{i\frac{\pi}{2}(-iE+\frac{1}{2})}\Gamma(-iE+\tfrac{1}{2})(p+io)^{iE-\frac{1}{2}}\ q_+^{iE-\frac{1}{2}} \tag{28}$$

This formula is valid for $q > 0$ (zero otherwise). An analogous formula can be written for $q < 0$.

The expansion (28) can be verified. Taking into account Stirling's formula

$$\Gamma(z) = z^{z-\frac{1}{2}} e^{-z} \sqrt{2\pi} \ (1+0(\tfrac{1}{z}))$$

one sees that it is possible to close the contour of integration (in (28)) in the lower half plane by a semicircle at ∞. The only enclosed poles are those of $\Gamma(-iE+\frac{1}{2})$ which occur at

$$E = -\ i(n+\tfrac{1}{2}) \text{ with residues } (-1)^n \frac{1}{n!}$$

We then obtain for the right hand side of (28)

$$\sum \frac{(ipq_+)^n}{n!} = e^{ipq} \text{ (for } q > 0)$$

5 CREATION AND DESTRUCTION OPERATORS

It is interesting to observe that one can define "creation" and "destruction" operators for the energy.

In fact, due to the form of the Hamimltonian eq. (18) and the commutation rules, we see that:

$$[H,q_+^{iE}] = \frac{1}{2}\ [p,q_+^{iE}]q + \frac{1}{2}\ q\ [p,q_+^{iE}]$$

$$[H,q_+^{iE}] = -\ iq \frac{d}{dq}\ q_+^{iE} = E\ q_+^{iE} \tag{29}$$

So that q_+^{iE} can be considered to be an *energy* creation operator of amount E. It should be noted however that, if E is negative the operator turns out to be an annihilation operator.

Analogously, if we take $(p+io)^{-iE}$ and evaluate its commutator with H, we find:

$$[H,(p+io)^{-iE}] = ip \frac{d}{dp} (p+io)^{-iE} = E(p+io)^{-iE} \qquad (30)$$

Which shows that for positive E, $(p+io)^{-iE}$ is also an energy creation operator of amount E.

If we take an eigenfunction $q_+^{iE-\frac{1}{2}}$ of H with eigenvalue E and applied to it the operator $q_+^{i\varepsilon}$, we obviously obtain another eigenfunction $q_+^{i(E+\varepsilon)-\frac{1}{2}}$ with eigenvalue $E+\varepsilon$.

It is not so obvious what happens when we apply to the same eigenfunction, the operator $(p+io)^{-i\varepsilon}$. The result can be obtained by going over to the p-representation by Fourier transforming $q_+^{iE-\frac{1}{2}}$ (see eq.(27)). It is also possible to work in q-space if we note that, for arbitrary ε, $(p+io)^{-i\varepsilon}$ acts as a fractional derivative on functions of q: (cf. reference [5] chapter (1.5))

$$(p+io)^{-i\varepsilon}\, q^{iE-\frac{1}{2}} = \frac{1}{\Gamma(iE)} \int_0^{\bar{q}} (q-\bar{q})_+^{i\varepsilon-1}\, q^{iE-\frac{1}{2}}\, dq =$$

$$= \frac{1}{\Gamma(iE)}\, q^{i\varepsilon-1} * q^{iE-\frac{1}{2}} = \frac{\Gamma(iE+\frac{1}{2})\, q_+^{i(E+\varepsilon)-\frac{1}{2}}}{\Gamma(i(E+\varepsilon)+\frac{1}{2})} \qquad (31)$$

So that, acting on $q_+^{iE-\frac{1}{2}}$, the operator $(p+io)^{-i\varepsilon}$ produces the same effect as $q_+^{i\varepsilon}$. Of course the same happens with any linear combination of these two operators

$$[H,(\alpha\, q_+^{iE} + \beta(p+io)^{-iE})] = E\left\{\alpha\, q_+^{iE} + \beta(p+io)^{-iE}\right\}$$

Analogously, we can show that q_-^{iE} (defined in eq.(23)) is a creation or destruction operator, according to the sign of E. Acting on the q_--eigenfunctions (eq.(22)), it produces the same effect as $(p+io)^{-iE}$.

Note that we have the relations (cf ref. [5])

$$(q+io)^{\lambda} = q_+^{\lambda} + e^{i\lambda\pi}\, q_-^{\lambda}$$

$$(q-io)^{\lambda} = q_+^{\lambda} + e^{-i\lambda\pi}\, q_-^{\lambda}$$

which means that $(q\pm io)^{iE}$ can also be taken as creation (destruction) operators for the energy.

A word of warning is perhaps useful here. Although we can define several creation and annihilation operators for the energy, they should not be mistaken with the corresponding operators for the number of particles. As a matter of facts, a number operator with integer eigenvalues does not exist for tachyons with $|\vec{k}| < m$.

6 DISCUSSION

We have here considered some properties of the wave equation for tachyons having $|\vec{k}| < m$. This is the region of $\vec{k}$-space which was left out of consideration in reference [4].

The base for our discussion is contained in formulae (7), (8) and (9), and, for the specific case of the above mentioned sphere, in formulae (14), (15) and (16).

We observe that while the coefficients of the harmonic oscillator expansion appear when $|\vec{k}| \geq m$, and correspond to the usual creation and annihilation operators; for $|\vec{k}| < m$ those coefficients are to be compared with the ordinarily used canonical conjugate operators q and p (see eqs. (14) and (17). This is the origin of the eigenvalue equation (18) or (19), whose solutions are given by (21) and (22).

Note that the spectrum of the Hamiltonian (16) is continuous, and extends from - to + . The energy is not quantized, i.e. (16) has not discrete eigenvalues.

An operator for the "number of tachyons" does not exist for $|\vec{k}| < m$. What it does exist is a set of operators $q_{\pm}^{iE}$, $(q \pm io)^{iE}$, $(p \pm io)^{-iE}$, $p_{\pm}^{-i\,E}$ which "create" (or "destroy") excitations with energy E.

There is a striking difference with the usual harmonic oscillator case for which we have

$$a = \frac{1}{2}\,(p-iq) \qquad , \qquad a^* = \frac{1}{2}\,(p+iq)$$

and these are destruction and creation operators of a quantum (particle), whose number operator is a*a with integer eigenvalues.

In our case, an operator for the number of tachyons does not exist, i.e. it does not exist a particle which we could call a thachyon.

REFERENCES

[1] J. Wess and B. Zumino: Nuclear Physics B70(1974) 39.

[2] C.G. Bollini and J.J. Giambiagi: Phys. Rev. D32 (1985) 3316.

[3] C.G. Bollini and J.J. Giambiagi: Rev. Brasileira de Física 17 (1987) 14.

[4] J. Dhar and E.C.G. Sudarshan: Phys. Rev. 174 (1968) 1808.

[5] I.M. Guelfand, G.E. Chilov: "Les Distributions", Dunod-Paris, 1962 (ch. 1.3)