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SYSTEMATIC APPROACH TO CRITICAL PHENOMENA BY THE EXTENDED VARIATIONAL METHOD AND COHERENT-ANOMALY METHOD

by

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Abstract

A general procedure to study critical phenomena of magnetic systems is discussed. It consists of systematic series of Landau-like approximations (Extended Variational Method) and the coherent-anomaly method (CAM). As for susceptibility, the present method is equivalent to the power-series CAM theory. On the other hand, the EVM gives a set of new approximants for other physical quantities. Applications to d-dimensional Ising ferromagnets are also described. The critical points and exponents are estimated with high accuracy.

KEYWORDS: mean-field approximation, critical phenomena, coherent anomaly method, extended variational method, d-dimensional Ising model

1. Introduction

In various physical problems in which cooperative effects play an essential role, the mean-field type approximation is the simplest approach. It is well-known, however, that these approximations have a serious defect that they only provide such classical values of critical exponents as do not in general coincide with the true ones. Of course, it is possible to improve the approximation in several ways, taking the fluctuation into account, enlarging the cluster size, treating many-body correlations correctly, and so on. Then the estimated value for the critical temperature gets closer to the exact value. In some cases it is even proved that there exists a systematic procedure of improvement which gives a series of approximate critical temperatures converging to the exact one. However, such procedures have been believed not to improve the values of critical exponents. Then, at a glance, such investigations based on mean-field approximations seem to be meaningless from a modern view-point of critical phenomena.

In order to overcome this difficulty, the coherent-anomaly method (CAM) was proposed by M. Suzuki [1] . Using this method it becomes possible to investigate the true critical behavior by a series of mean-field-type approximations. Various applications of the CAM theory based on cluster-mean-field approximations have already been reported. [3 $\sim$ 17] Instead of the ordinary cluster-mean-field approximations, we present here another type of approximations called the extended variational method (EVM). [2] This method consists in a cumulant expansion of the free energy and is related to the high temperature expansion(HTE). Each single approximant calculated by the EVM exhibits the same critical behavior as in the cluster-mean-field approximations. Then the EVM is expected to become canonical to obtain reasonable results about the true critical behavior by analyzing it using the CAM.

2. Extended Variational Method

The EVM was originally proposed by Tsallis and L.R.da Silva [2] to improve the standard variational method. The ordinary variational method is based on the following Bogoliubov inequality:

$$F_1 = F_0 + \langle \mathcal{H} - \mathcal{H}_0 \rangle_0 \geq F, \tag{1}$$

where

$$F \equiv -T \log \mathrm{Tr}[e^{-\beta \mathcal{H}}], \quad F_0 \equiv -T \log \mathrm{Tr}[e^{-\beta \mathcal{H}_0}], \quad \text{and} \quad \langle \cdots \rangle_0 \equiv \frac{\mathrm{Tr}[e^{-\beta \mathcal{H}_0} \cdots]}{\mathrm{Tr}[e^{-\beta \mathcal{H}_0}]}.$$

Here, $\mathcal{H}$ is the original hamiltonian and $\mathcal{H}_0$ is a conveniently chosen trial hamiltonian which depends on a set of variational parameters $\{\Lambda\}$. In determination of these parameters, it is quite natural to set these values so that the trial free energy F_1 may be the closest to the true one F. Thus we have the stationary condition

$$\frac{\partial F_1}{\partial \Lambda_i}\Big(\{x\}, \{\Lambda\}\Big) = 0 \qquad \text{(for all i)}. \tag{2}$$

For example, consider the Ising ferromagnets with the external magnetic field on a d-dimensional hypercubic lattice. It is well-known that Eq.(2) gives the same results as the Weiss molecular-field theory by Weiss if we take

$$\mathcal{H}_0 = -\Lambda \sum_i S_i$$

as the trial hamiltonian in Eq.(1). As to the critical exponents the EVM gives wrong estimations and especially in the one-dimensional case, it gives a finite critical temperature though it does not really exist. So we have to devise a general procedure to improve the approximation by which we can calculate any physical quantity with an arbitrary precision (at least in principle). The first procedure one may think of is to improve the trial hamiltonian. Since we have the exact result if we take $\mathcal{H}$ itself as a trial Hamiltonian, we can expect to get the desirable series of approximations by considering a series of trial hamiltonians $\{\mathcal{H}_0^{(n)}\}$ which converge to $\mathcal{H}$. The cluster-variational method (CVM) proposed by R.Kikuchi [18] is one of the procedures in this direction. In fact it has already been reported [17] that the combination of the CVM and the CAM works well for Ising ferromagnets on a simple cubic lattice. In the CVM, we consider not only an effective field imposed on a single spin but also many other kinds of effective fields. Because of this variety of the mean fields, the CVM gives very accurate prediction for the various physical quantities. The calculation, however, becomes extremely complicated when the cluster size is increased, because the number of mean fields which we must treat grows very rapidly. For this reason we can treat only small clusters in this scheme. Instead of the improvement of the trial hamiltonian, we can improve the Bogoliubov inequality Eq.(1) itself to get a canonical series of approximations. The EVM is a strategy of calculation in this direction.

Let us consider the hamiltonian $\mathcal{H}$ and the trial hamiltonian $\mathcal{H}_0(\{\Lambda\})$. If $\mathcal{H}_0$ commutes with $\mathcal{H}$, we can write the free energy as

$$\begin{aligned} F \equiv -T\log \mathrm{Tr}[e^{-\beta\mathcal{H}}] &= -T\log \mathrm{Tr}[e^{-\beta\mathcal{H}_0}e^{-\beta\Delta\mathcal{H}}] \\ &= -T\log \mathrm{Tr}[e^{-\beta\mathcal{H}_0}\sum_{m=0}^{\infty}\frac{(-\beta)^m}{m!}(\Delta\mathcal{H})^m] \\ &= F_0 + N\sum_{m=1}^{\infty}\frac{(-\beta)^{m-1}}{m!}\kappa_m. \end{aligned} \tag{3}$$

where

$$\Delta\mathcal{H} \equiv \mathcal{H} - \mathcal{H}_0, \quad F_0 \equiv -T\log \mathrm{Tr}[e^{-\beta\mathcal{H}_0}], \quad N \equiv \text{the size of the system}$$

and κ_m is the m-th cumulant. For example,

$$\begin{aligned} \kappa_1 &\equiv \frac{1}{N}\Big\{\langle\Delta\mathcal{H}\rangle\Big\}, \\ \kappa_2 &\equiv \frac{1}{N}\Big\{\langle(\Delta\mathcal{H})^2\rangle - \langle\Delta\mathcal{H}\rangle^2\Big\}, \\ \kappa_3 &\equiv \frac{1}{N}\Big\{\langle(\Delta\mathcal{H})^3\rangle - 3\langle(\Delta\mathcal{H})^2\rangle\langle\Delta\mathcal{H}\rangle + 2\langle\Delta\mathcal{H}\rangle^3\Big\}, \quad \text{etc.} \end{aligned}$$

Equation (3) is an identity and holds for any trial hamiltonian commutative with $\mathcal{H}$ and of course F does not depend on $\mathcal{H}_0$. Next, if we terminate the expansion series Eq.(3) at $m = n$ we have the n-th approximant of the free energy [1]

$$F_n = F_0 + N \sum_{m=1}^{n} \frac{(-\beta)^{m-1}}{m!} \kappa_m. \tag{4}$$

Here F_n depends on $\mathcal{H}_0$, but we can expect that

$$\lim_{n \to \infty} F_n(\{\Lambda\}) = F, \tag{5}$$

when we fix the variational parameters $\{\Lambda\}$. For determination of Λ's it is quite natural to require the stationary condition [2]

$$F_n = \text{stationary}, \quad \frac{\partial F_n}{\partial \Lambda_i} = 0 \qquad \text{(for all i)}. \tag{6}$$

In the case of $n = 1$, this condition coincides with the ordinary stationary condition Eq.(2), and consequently it is supported by the Bogoliubov inequality. For n's larger than unity, there is no inequality which supports Eq.(6). Furthermore, since Λ's determined in such a way depend on n, Eq.(5) should read

$$\lim_{n \to \infty} F_n(\{\Lambda_n\}) = F, \tag{7}$$

which is not necessarily satisfied. So far, we have no mathematical proof of convergence of physical quantities calculated by this scheme. However one can find in Ref.2 a pedagogical example. Non-interacting classical unharmonic oscillators are treated there in this scheme. The convergence of various physical quantities such as specific heat and susceptibility has been confirmed numerically.

Now let us see what happens when we apply this scheme to the model of ferromagnets

$$\tilde{\mathcal{H}} = \mathcal{H} + V = -J \sum_{(i,j)} \mathbf{S}_i \cdot \mathbf{S}_j - H \sum_{j} S_i, \tag{8}$$

In this case, the most natural choice of our trial hamiltonian may be the the following similar to the external field term:

$$\mathcal{H}_0 = -\Lambda M = -\Lambda \sum_i S_i^z. \tag{9}$$

According to the number of components of spin ($\equiv D$) we have the Ising model ($D = 1$), the XY model ($D = 2$), and the Heisenberg model ($D = 3$). Here, Λ is not a real external field a the variational parameter. Since $\mathcal{H}_0$ commutates with $\mathcal{H}$, all the formulas given below are applicable both to quantum systems and classical systems.

The quantities in which we are interested are the critical amplitudes of the susceptibility, the spontaneous magnetization and the critical magnetization. For this purpose we need only the first few terms of the free energy expanded with respect to H and Λ. All the coefficients of this expansion can be expressed in a compact form with HTE coefficients as follows. The relevant part of βf_n becomes

$$\beta f_n^{(relevant)} = \frac{\lambda^2}{2}\Phi(\beta) - \lambda h \Psi(\beta) - \frac{h^2}{2}\Gamma(\beta) + \frac{\lambda^4}{4}\Theta(\beta), \tag{10}$$

where

$$\lambda \equiv \beta\Lambda, \qquad h \equiv \beta H,$$

$$\Phi(\beta) \equiv a_{n-1}^{(2)}\beta^{n-1} - a_n^{(2)}\beta^n,$$

$$\Psi(\beta) \equiv a_{n-1}^{(2)}\beta^{n-1},$$

$$\Gamma(\beta) \equiv C_{n-2}^{(2)} = \sum_{k=0}^{n-2} a_k^{(2)}\beta^k,$$

$$\Theta(\beta) \equiv \frac{1}{6}(a_n^{(4)}\beta^n - 3a_{n-1}^{(4)}\beta^{n-1} + 3a_{n-2}^{(4)}\beta^{n-2} - a_{n-3}^{(4)}\beta^{n-3}),$$

and $a_n^{(2)}$ and $a_n^{(4)}$ are the HTE coefficients of the second and forth cumulants of magnetization, respectively. Equation (10) is a Landau-like expansion of the free energy in our scheme. The derivation of Eq.(10) is given in Appendix. In the derivation of this equation, we use only the facts that the $\mathcal{H}$ commutes with V

and that $\mathcal{H}_0$ has the same form as V has. Thus this formula can be used for various other systems as well as for the Ising ferromagnets.

(1) Susceptibility χ

By differentiating Eq.(10) with respect to λ, we have

$$\lambda\Phi(\beta) - h\Psi(\beta) + \lambda^3\Theta(\beta) = 0. \tag{11}$$

This equation determines the relevant variational parameter. Then the zero-field susceptibility becomes

$$\begin{aligned}
\beta^{-1}\chi^{(n)} &= -\frac{\partial^2}{\partial h^2}(\beta f_n^{(relevant)})\Big|_{h\to 0} \\
&= -\Big(\frac{\partial\lambda}{\partial h}\Big)^2\Phi(\beta) + 2\Big(\frac{\partial\lambda}{\partial h}\Big)\Psi(\beta) + \Gamma(\beta)\Big|_{h\to 0} \\
&= \frac{\Psi(\beta)^2}{\Phi(\beta)} + \Gamma(\beta) \qquad (12a) \\
&= \Big\{\sum_{k=0}^{n-2} a_k^{(2)}\beta^k\Big\} + \frac{a_{n-1}^{(2)}\beta^{n-1}}{1-\frac{a_n^{(2)}}{a_{n-1}^{(2)}}\beta}. \qquad (12b)
\end{aligned}$$

Here the relation $\partial\lambda/\partial h|_{h\to 0} = \Psi/\Phi$ has been used. This is the n-th approximant for the susceptibility in the present scheme. One can easily see that $\beta^{-1}\chi^{(n)}$ is $[n-1,1]$- Padé approximant of the HTE. As is well known, the poles of $[n-1,1]$-Padé approximants are located at the critical point obtained by the ratio method. In fact $\chi^{(n)}$ has a single pole at

$$\beta_c^{(n)} = \frac{a_{n-1}^{(2)}}{a_n^{(2)}}. \tag{13}$$

Concerning the amplitude of singularity, we find, from Eq.(12b),

$$\bar{\chi}^{(n)} = \frac{(a_{n-1}^{(2)})^{n+1}}{(a_n^{(2)})^n}. \tag{14}$$

(2) Spontaneous Magnetization m_S

Differentiating Eq.(10) with respect to h and setting h equal to zero yields

$$\begin{aligned} m_S^{(n)}(\beta) &= -\Big(\frac{\partial h}{\partial \beta} f_n\Big)_{h=0} \\ &= -\lambda \frac{\partial \lambda}{\partial h}\Phi(\beta) - \lambda\Psi(\beta) - \lambda^3 \frac{\partial \lambda}{\partial h}\Theta(\beta). \end{aligned} \tag{15}$$

If we set $h = 0$ in Eq.(11) we get

$$\lambda = \Big(-\frac{\Phi(\beta)}{\Theta(\beta)}\Big)^{\frac{1}{2}}. \tag{16}$$

Differentiating Eq.(11) with respect to h yields

$$\frac{\partial \lambda}{\partial h}(\Phi(\beta) + 3\lambda^2\Theta(\beta)) = \Psi(\beta).$$

Substituting the right hand side of Eq.(16) into this equation, we have

$$\frac{\partial \lambda}{\partial h} = -\frac{\Psi(\beta)}{2\Phi(\beta)}. \tag{17}$$

Then, we can rewrite Eq.(15) using Eqs.(16) and (17) as

$$\begin{aligned} m_S^{(n)}(\beta) &= \lambda\Psi(\beta) = \Psi(\beta)\Big(-\frac{\Phi(\beta)}{\Theta(\beta)}\Big)^{\frac{1}{2}} \\ &\simeq \Big(\frac{\Psi(\beta_c^{(n)})^3}{\Theta(\beta_c^{(n)})}\Big)^{\frac{1}{2}}\Big(\frac{\beta - \beta^{(n)}}{\beta_c^{(n)}}\Big)^{\frac{1}{2}}. \end{aligned} \tag{18}$$

Thus we find

$$\bar{m}_S^{(n)} = \Big(\frac{\Psi(\beta_c^{(n)})^3}{\Theta(\beta_c^{(n)})}\Big)^{\frac{1}{2}}. \tag{19}$$

(3) Critical Magnetization m_c

Next, let us calculate the magnetization at the critical point in a finite external field H. At the critical point, $\Phi(\beta)$ is equal to zero. Consequently, from Eq.(11) we can find

$$\lambda \simeq \Big(\frac{\Psi(\beta_c^{(n)})}{\Theta(\beta_c^{(n)})} h\Big)^{\frac{1}{3}}. \tag{20}$$

Substituting this into Eq.(10) and setting Φ equal to zero, we obtain

$$\beta_c f_n^{(relevant)} \simeq -\frac{3}{4}\Big(\frac{\Psi(\beta_c^{(n)})^4}{\Theta(\beta_c^{(n)})}\Big)^{\frac{1}{3}} h^{\frac{4}{3}}.$$

Differentiating this with respect to h, we get

$$\begin{aligned} m_c(H) &= -\frac{\partial}{\partial h}(\beta f_n^{(relevant)}) \\ &= \Big(\beta\frac{\Psi(\beta_c^{(n)})^4}{\Theta(\beta_c^{(n)})}\Big)^{\frac{1}{3}} H^{\frac{1}{3}} \end{aligned} \tag{21}$$

Hence,

$$\bar{m}_c^{(n)} = \Big(\beta_c^{(n)}\frac{\Phi(\beta_c^{(n)})^4}{\Theta(\beta_c^{(n)})}\Big)^{\frac{1}{3}}. \tag{22}$$

We have now derived all the formulas necessary for the CAM analysis on the basis of Eq.(10). We can obtain the critical exponents γ,β and δ by using these equations together with the coefficients of the high-temperature expansion, and using the coherent-anomaly relations to be briefly reviewed in the following section.

3. Coherent-Anomaly Method

The coherent-anomaly method (CAM) is recently proposed by one of the present authors (M.S.) in order to estimate the true critical exponents from a series of approximants with classical singularities. The general formulation is given in Ref.1 and Ref.3 and various applications have already been reported to confirme the validity of the CAM. In the case of Weiss-like approximations for the Ising ferromagnet, it has been proved [2] that the approximate critical temperatures converge to the exact one when the cluster size goes to infinity. In the same case, the relation between the CAM and the finite size scaling theory [19, 20] is also clarified [2] . The essence of the CAM is the following coherent-anomaly relation:

$$\bar{\chi}^{(n)} \propto \Big|\frac{\beta_c^*}{\beta_c^* - \beta_c^{(n)}}\Big|^{\gamma-\phi}. \tag{23}$$

Here $\bar{\chi}^{(n)}$ is the critical amplitude of the n-th approximant defined by

$$\chi^{(n)}(\beta) \approx \bar{\chi}^{(n)} \left| \frac{\beta_c^{(n)}}{\beta_c^{(n)} - \beta} \right|^{\phi} \qquad (\beta \approx \beta_c^{(n)})$$

where $\chi^{(n)}(\beta)$ is the n-th approximant of the susceptibility $\chi(\beta)$, and $\beta_c^{(n)}$ and β_c^* are the approximate and exact critical points, respectively. The index γ is the exact critical exponent and ϕ is the classical exponent of γ. Using Eq.(23) and several pairs of $\bar{\chi}^{(n)}$ and $\beta_c^{(n)}$, one can estimate β_c^* and γ by some regression analysis, say, the least square method. Moreover, in Ref.1, the coherent-anomaly relation for the critical amplitudes of the spontaneous magnetization m_s and the critical magnetization m_c are derived. That is,

$$\bar{m}_s^{(n)} \propto \left| \frac{\beta_c^*}{\beta_c^* - \beta_c^{(n)}} \right|^{\psi_s}, \qquad \bar{m}_c^{(n)} \propto \left| \frac{\beta_c^*}{\beta_c^* - \beta_c^{(n)}} \right|^{\psi_c}, \tag{24}$$

where

$$\psi_s \equiv \frac{1}{2} - \beta \quad \text{and} \quad \psi_c \equiv (\gamma + \beta)(\frac{1}{3} - \frac{1}{\delta}).$$

4. Application to the Ising Ferromagnets

The approximate critical points, and the critical amplitudes of singularities for the above three quantities are listed in Tab.1 for the Ising model on the simple quadratic lattice. In Fig.1 and Fig.2 the coherent anomalies for the magnetic susceptibility, the spontaneous magnetization, and the critical magnetization are shown. We assumed the following fitting function:

$$\bar{Q}^{(n)} = At^B(C + Dt + Et^2) \qquad (Q = \chi, m_s, m_c)$$

where $t \equiv \beta_c^* - \beta_c^{(n)}$ and A, B, C, D and E are the fitting parameters. Here we have used the exact value for β_c^*. The values of the critical temperature and the exponents obtained by the least-square fitting are

$$\gamma - 1 = 0.746(2)\ [0.75], \qquad \psi_s = 0.39(1)\ [0.375], \qquad \psi_c = 0.510(6)\ [0.5].$$

where the number in the square brackets are the exact values. These values result in

$$\gamma = 1.746(2), \qquad \beta = 0.11(1) \quad and \quad \delta = 17(1).$$

Here, the errors stand for one standard deviation if the data points obey the Gaussian distribution. In this case, however, it is not justified because the data points are not even stochastic. This is the reason of the slight under-estimation of the errors. Unfortunately, the effective method for error estimation of this type is not yet established. In Fig.3, we have shown the values of γ for various numbers of dimensions calculated using the HTE coefficients up to the 8th order. In this case, the fitting was done for the last 3 even points and the fitting curves are mere straight lines.

5. Summary and Discussions

We have presented a general procedure to investigate the critical behavior of magnetic systems and applied it to the Ising Ferromagnets. The relation to the HTE(β-expansion) has also been clarified. Especially, the approximant for the fluctuation of Q coincides with $[n-1,1]$- Padé approximant for the HTE of $< Q^2 >$ and consequently the approximate critical point is the same as that of ratio method. Thus, as for the susceptibility, the present procedure is equivalent to the power-series CAM theory proposed by one of the authors(M.S.). [10, 11] Moreover, as for other quantities, we have obtained new formulas consistent with the scaling law.

In the application of the present scheme to the Ising Ferromagnet we have confirmed the validity of it. We have obtained

$$\gamma = 1.746(2), \qquad \beta = 0.11(1), \qquad \delta = 17(1).$$

Although we have no mathematical proof of the convergence so far, the result is quite reasonable except for the slight under-estimations of the errors. In other words, the convergence of the present theory is almost equivalent to the convergence of the $[n-1,1]$- Padé approximant of the HTE.

Applications of the present formulation to various other systems with complicated order parameters such as the planar model are also interesting future problems.

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Appendix

First, let us write the original and the trial hamiltonian as

$$\tilde{\mathcal{H}} = \mathcal{H} - HQ, \tag{A1}$$

and

$$\mathcal{H}_0 = -\Lambda Q, \tag{A2}$$

respectively. Then, the density matrix is

$$e^{-\beta\mathcal{H}} = e^{-\beta\mathcal{H}_0} e^{-\beta(\tilde{\mathcal{H}}-\mathcal{H}_0)}. \tag{A3}$$

We have to expand the second term of the right hand side of this equation. In other words, we have to expand the whole density matrix with respect to the second β only. Thus it is convenient to distinguish two β's by rewriting the second β as $\bar{\beta}$. Then the resulting expression is slightly different from Eq.(A3):

$$e^{-\beta\tilde{\mathcal{H}}} = e^{-\beta\mathcal{H}_0} e^{-\bar{\beta}(\tilde{\mathcal{H}}-\mathcal{H}_0)}. \tag{A4}$$

Then our n-th approximant of the free energy becomes

$$\beta f_n = \frac{1}{N}[-\log \mathrm{Tr}\{e^{-\beta\mathcal{H}_0}e^{-\bar{\beta}(\tilde{\mathcal{H}}-\mathcal{H}_0)}\}]_n, \tag{A6}$$

where

$$[\sum_{k=0}^{\infty} p_k\bar{\beta}^k]_n \equiv \sum_{k=0}^{n} p_k\bar{\beta}^k. \tag{A7}$$

If we change the order of the exponents in Eq.(A6), we get

$$\beta f_n = \frac{1}{N}[-\log \mathrm{Tr}\{e^{-\bar{\beta}\mathcal{H}}e^{\Xi Q}\}]_n, \tag{A8}$$

where

$$\Xi = \beta\Lambda + \bar{\beta}(H - \Lambda).$$

It should be noted here that we need only the terms proportional to Λ^2,ΛH and H^2 in order to calculate the susceptibility χ, and that the terms proportional to Λ^2 and ΛH are sufficient for the amplitude of the singularity $\bar{\chi}$. In addition to these terms, for $\bar{m}_s$ (the amplitude of the singularity of the spontaneous magnetization), the term of order H^4 is needed. In any way, the full expansion series is not necessary. The important terms are the first few terms in the Ξ-expansion series. Keeping this in mind, we can rewrite Eq.(A8) as

$$\begin{aligned}\beta f_n &= \frac{1}{N}\Big[\Big[-\log \mathrm{Tr}\Big\{e^{-\beta\mathcal{H}}\Big(1+\Xi Q+\frac{1}{2!}\Xi^2Q^2+\frac{1}{3!}\Xi^3Q^3+\frac{1}{4!}\Xi^4Q^4\Big)\Big\}\Big]\Big]_n + o(\Xi^4)\\ &= \frac{1}{N}\Big[\Big[-\log\Big\{Z_0(\bar{\beta})\times\Big(1+\frac{\Xi^2}{2!}\langle Q^2\rangle_{\bar{\beta}}+\frac{\Xi^4}{4!}\langle Q^4\rangle_{\bar{\beta}}\Big)\Big\}\Big]\Big]_n + o(\Xi^4)\\ &= [\bar{\beta}f_0(\bar{\beta})]_n - [\frac{\Xi^2}{2!}C^{(2)}(\bar{\beta})]_n - [\frac{\Xi^4}{4!}C^{(4)}(\bar{\beta})]_n + o(\Xi^4),\end{aligned} \tag{A9}$$

where

$$\beta f_0(\beta) = \frac{1}{N}(-\log \mathrm{Tr}[e^{-\beta\mathcal{H}}]),$$

$$C^{(2)}(\beta) = \frac{1}{N}\langle Q^2\rangle_\beta, \tag{A10a}$$

$$C^{(4)}(\beta) = \frac{1}{N}\Big\{\langle Q^4\rangle_\beta - 3\langle Q^2\rangle_\beta^2\Big\}, \tag{A10b}$$

and

$$\langle \cdots \rangle_\beta \equiv \frac{\mathrm{Tr}(e^{-\beta\mathcal{H}}\ldots)}{\mathrm{Tr}(e^{-\beta\mathcal{H}})}.$$

Furthermore, we have

$$[\Xi^2 C^{(2)}(\bar\beta)]_n = \beta^2\Lambda^2 C_n^{(2)} + 2\beta\bar\beta\Lambda(H-\Lambda)C_{n-1}^{(2)} + \bar\beta^2(H-\Lambda)^2 C_{n-2}^{(2)},$$

where

$$C_n^{(2)} \equiv [C^{(2)}(\bar\beta)]_n.$$

Now, there is no need to distinguish $\bar\beta$ from β any more. Thus we get

$$[\Xi^2 C^{(2)}]_n = \beta^2\{\Lambda^2(C_n^{(2)} - 2C_{n-1}^{(2)} + C_{n-2}^{(2)}) - \Lambda H(-2C_{n-1}^{(2)} + 2C_{n-2}^{(2)}) + H^2 C_{n-2}^{(2)}\}.$$

If we expand $C^{(\nu)}(\bar\beta)$ with respect to $\bar\beta$ as

$$C^{(\nu)}(\bar\beta) = (-1)^{\frac{\nu}{2}-1}\sum_{k=0}^{\infty} a_k^{(\nu)}\bar\beta^k, \tag{A11}$$

we find

$$[\Xi^2 C^{(2)}(\bar\beta)]_n = \beta^2\{\Lambda^2(a_n^{(2)}\beta^n - a_{n-1}^{(2)}\beta^{n-1}) + \Lambda H(2a_{n-1}^{(2)}\beta^{n-1}) + H^2 C_{n-2}^{(2)}\}. \tag{A12}$$

In a similar fashion, we get

$$\begin{aligned}[\Xi^4 C^{(4)}(\bar\beta)]_n = \beta^4\Lambda^4(-a_n^{(4)}\beta^n + 3a_{n-1}^{(4)}\beta^{n-1} - 3a_{n-2}^{(4)}\beta^{n-2} + a_{n-3}^{(4)}\beta^{n-3}) \\ + O(H^4, H^3\Lambda, H^2\Lambda^2, H\Lambda^3).\end{aligned} \tag{A13}$$

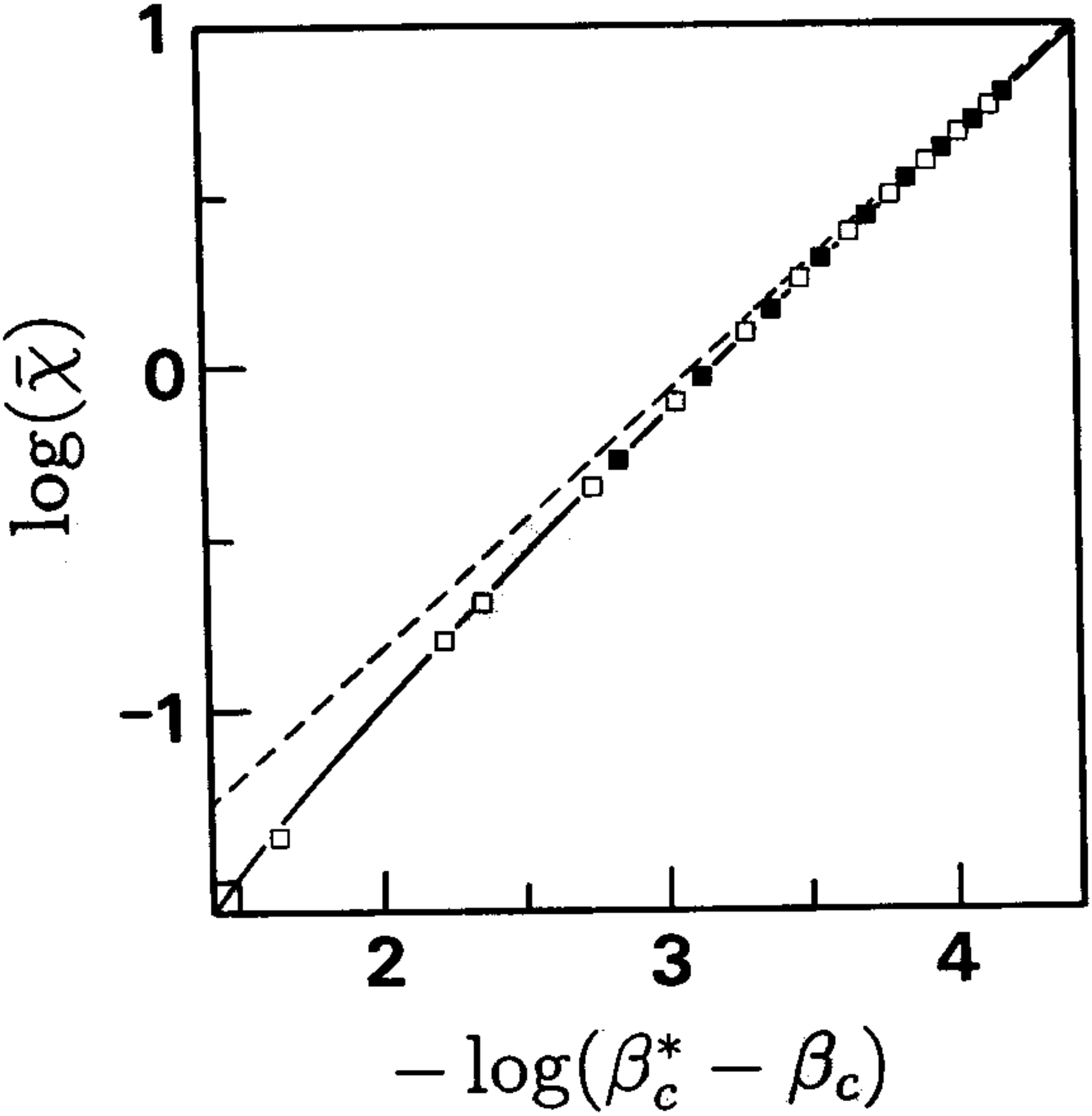
Substituting Eq.(A12) and Eq.(A13) into Eq.(A9), we get Eq.(10).

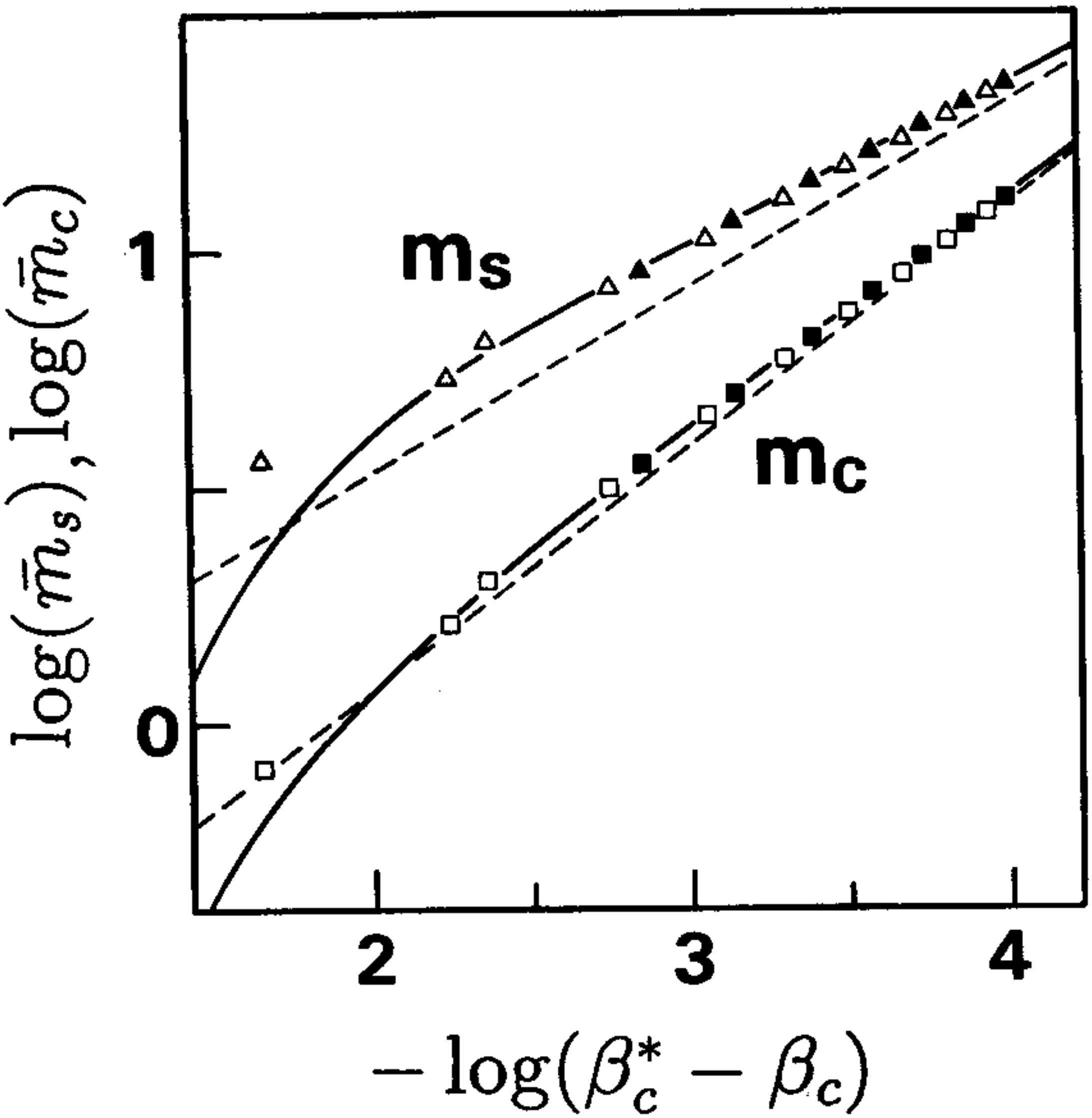
Figure Captions

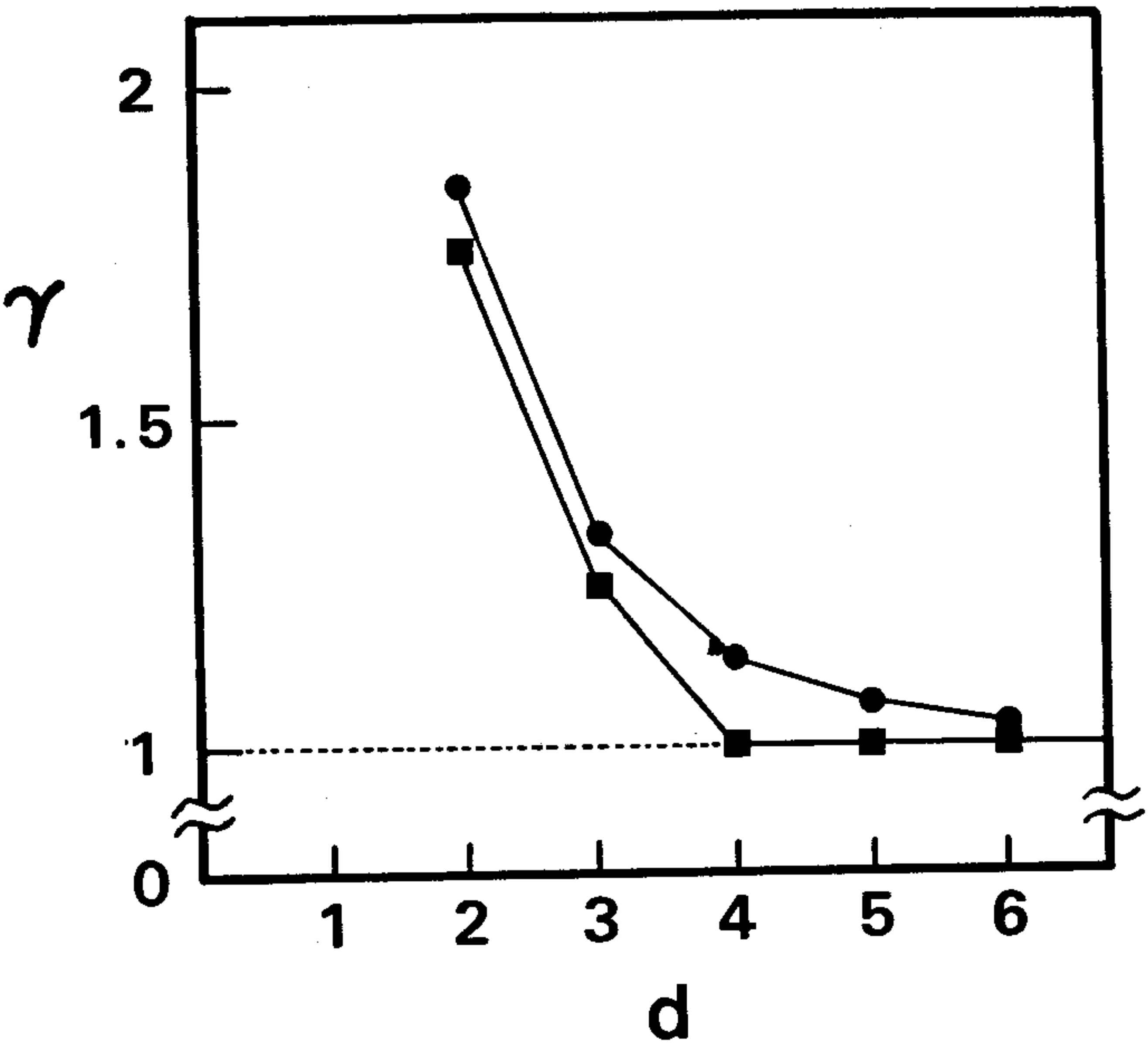
Fig.1 The CAM plot of the susceptibility χ obtained by the EVM for the Ising model. The closed squares are used in the least square analysis. The solid line is the fitting curve and the dashed line is its linear part.

Fig.2 The CAM plot of the spontaneous magnetization m_s and the critical magnetization m_c. The amplitudes of the singularities are calculated by use of the high-temperature expansion series of the forth cumulant of the magnetization. The closed squares(triangles) are used in the least square analysis.

Fig.3 The value of γ for $d = 2, 3, 4, 5$ and 6 calculated with the HTE coefficients up to the 8th order. Closed circles denotes the values obtained by the present method and closed squares are the exact values.


1
0
-1
2
3
4
$\log(\bar{\chi})$
$-\log(\beta_c^* - \beta_c)$


ms
mc
1
0
2
3
4
$\log(\bar{m}_s), \log(\bar{m}_c)$
$-\log(\beta_c^* - \beta_c)$


2
1.5
1
0
γ
1
2
3
4
5
6
d

n	$\beta_c^{(n)}$	$\bar{\chi}^{(n)}$	$\bar{m}_s$	$\bar{m}_c$
1	0.250000	0.250000	1.732051	0.908560
2	0.333333	0.444444	2.065591	1.237757
3	0.346154	0.497724	2.232278	1.353615
4	0.376812	0.698890	2.501090	1.635144
5	0.382483	0.753096	2.594958	1.718055
6	0.393543	0.893566	2.779137	1.903921
7	0.397352	0.955891	2.870876	1.989814
8	0.403761	1.086439	3.015059	2.145518
9	0.406810	1.162544	3.120271	2.245250
10	0.410357	1.267970	3.225047	2.362619
11	0.412541	1.344217	3.328487	2.460298
12	0.415088	1.447246	3.423927	2.569590
13	0.416639	1.519162	3.515527	2.657837
14	0.418506	1.617262	3.604082	2.759243
15	0.419713	1.688666	3.689207	2.843173
16	0.421117	1.781345	3.769659	2.936189
17	0.422083	1.852133	3.849885	3.016634
18	0.423184	1.941077		
19	0.423969	2.010568		
20	0.424858	2.096647		
21	0.425508	2.165017		
∞	0.440687			

Table 1. The approximate critical points and the amplitudes of singularities for the zero-field susceptibility, the spontaneous magnetization and the critical magnetization obtained by the EVM using the HTE coefficients of the second and forth cumulant. As to the HTE data, see the Ref.21 for the second cumulant and the Refs.22 and 23 for the forth cumulant.

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