CNPq

CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Notas de Física

CBPF-NF-037/92

LIE ALGEBRAS FOR THE DIRAC-CLIFFORD RING

by

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CBPF-NF-037/92

Abstract

We show in general that the Dirac–Clifford ring formed by the Dirac matrices and all their products, for all even and odd spacetime dimensions D, span the commutation algebras $su(2^{D/2})$ for even D and $su(2^{(D-1)/2}) \oplus su(2^{(D-1)/2})$ for odd D. We discuss some physical consequences of these results.

Key-words: Dirac equation; Clifford algebra.

1 Introduction

In recent years, we have shown that there is a close relationship for space-time dimensions $D = 2$, 3 and 4 between the ring of Dirac–Clifford matrices and unitary algebras (respectively, $su(2)$, $su(2) \oplus su(2)$ and $su(4)$) [1,2].

We conjectured at the time on the existence of a general framework for the ring formed by anticommuting Dirac–Clifford generators and their products and that it is precisely related to definite unitary groups. We are now able to prove this in the present article.

The original evidence on this connection was the treatment by Becher and Joos [3] for the Dirac–Kähler equation, and its extension to the lattice. Becher and Joos used differential forms endowed with a Clifford product, introduced previously by Kähler [4], which allowed Graf [5] to prove the isomorphism, for all dimensions, of the differential forms with this product and Dirac gamma matrices.

Our proof is purely based on the algebra of commutators built from the generators of the Clifford algebra and including all products of the generators that produce the set known as the Dirac ring (in $D = 4$), a concept valid for all dimensions. This is far from trivial for an odd number of dimensions, where the usual representations for the Dirac matrices obtained from a finite group characterization [6] for the relevant Clifford algebra introduce an artificial restriction on the ring.

Because of the Graf isomorphism, our proofs apply equally to the differential forms with Clifford product. In fact, as it will be evident from the treatment of the subject, it is valid for any object having a finite number of indices and properties for exchange as any Clifford algebra.

Some interesting physical questions will be referred to in the text below, which is organized as follows. In the next section, we introduce the main concepts and notations, as known in the literature, which will be used in the following.

Section 3 is the beginning of the core of the article, where the proof is given for even dimensions of the relation between the Dirac ring formed with hermitian matrices (or, in general, objects with definite properties under transposition and conjugation) and the Lie algebra $su(2^{D/2})$. The proof is based on the existence of a subset forming a Cartan subalgebra under commutation.

Section 4 deals with the case of odd-dimensional spacetimes, for which we need the previous result and new ones regarding properties under commutation of the objects with the complementary number of indices with respect to the Clifford generators. We show there that the relevant Lie algebra is $su(2^{(D-1)/2}) \oplus su(2^{(D-1)/2})$.

Finally, we discuss our results, compare them, for odd-dimensional spacetimes, with the representation commonly used, and consider physical applications in the framework of calculations with particles with spin 1/2.

2 Fundamentals

Let us consider a flat D-dimensional spacetime manifold in which a quadratic bilinear diagonal form (metric tensor), g, is defined. Clifford algebras are formed out of a set of D objects, Γ^k, $k = 1, \ldots, D$, which we call generators, related to the set of spacetime indices (one for each generator) that, given a product of two generators, satisfy

$$\Gamma^{k_1} \cdot \Gamma^{k_2} + \Gamma^{k_2} \cdot \Gamma^{k_1} = 2g^{k_1 k_2}. \tag{1}$$

Particularly interesting objects with this property are the Dirac matrices, appearing in the description of relativistic spin-1/2 particles, γ^{k_1}; closely related are differential forms, if one purposedly introduces a Clifford product between two 1-forms in the following way:

$$dx^{k_1} \vee dx^{k_2} = g^{k_1 k_2} + dx^{k_1} \wedge dx^{k_2}, \tag{2}$$

where $\vee$ is the symbol for Clifford product, and the wedge $\wedge$ denotes the usual Grassmann exterior product. This set was introduced by Kähler [4] precisely to treat relativistic spin-1/2 particles with differential forms. In fact, the set of Dirac matrices and the one of differential forms are isomorphic [5]. See also [1,3].

The product involved in eq. (1) should be associative,

$$(\Gamma^{k_1} \cdot \Gamma^{k_2}) \cdot \Gamma^{k_3} = \Gamma^{k_1} \cdot (\Gamma^{k_2} \cdot \Gamma^{k_3}) = \Gamma^{k_1} \cdot \Gamma^{k_2} \cdot \Gamma^{k_3}. \tag{3}$$

It is a simple exercise to show that the usual matrix product for the Dirac matrices and the Clifford product defined in (3) for differential forms satisfy this condition.

In terms of matrices, we take the generators to be hermitian:

$$(\Gamma^k)^\dagger = \Gamma^k, \qquad \text{for all } k, \tag{4}$$

or, equivalently,

$$(\Gamma^k)^2 = I.$$

This last condition is the convenient one for differential forms (with complex coefficients) [4,5].

Let us now consider the product of generators. From the definition, we know that at most D of them can be multiplied; they all form a set

$$\Gamma^1, \ldots, \Gamma^D, \Gamma^1 \cdot \Gamma^2, \Gamma^1 \cdot \Gamma^3, \ldots, \Gamma^{D-1} \cdot \Gamma^D, \Gamma^1 \cdot \Gamma^2 \cdot \Gamma^3, \ldots,$$
$$\Gamma^{D-2} \cdot \Gamma^{D-1} \cdot \Gamma^D, \ldots, \Gamma^{D+1}, \tag{5}$$

where all are taken with convenient complex numerical coefficients in order to become hermitian. For this, we need to take into account the commutation relations of the generators with products of generators. The main properties are:

1. For a given generator and a product of p generators, all different from the one given, we have

$$\Gamma^{k_j} \cdot (\Gamma^{l_1} \cdot \Gamma^{l_2} \cdot \ldots \cdot \Gamma^{l_p}) = (-1)^p (\Gamma^{l_1} \cdot \Gamma^{l_2} \cdot \ldots \cdot \Gamma^{l_p}) \cdot \Gamma^{k_j}.$$

2. For a given generator, the commutation with a product of p generators including it obeys

$$\Gamma^{l_r} \cdot (\Gamma^{l_1} \cdot \Gamma^{l_2} \cdot \ldots \cdot \Gamma^{l_{r-1}} \cdot \Gamma^{l_r} \cdot \Gamma^{l_{r+1}} \cdot \ldots \cdot \Gamma^{l_p})$$
$$= (-1)^{p-1} (\Gamma^{l_1} \cdot \Gamma^{l_2} \cdot \ldots \cdot \Gamma^{l_{r-1}} \cdot \Gamma^{l_r} \cdot \Gamma^{l_{r+1}} \cdot \ldots \cdot \Gamma^{l_p}) \cdot \Gamma^{l_r}.$$

These properties are proven just by taking into account the change of sign for commutation between different generators and comparing the results for both sides in the second case. As a corollary of these properties, we have for the square of a given product of p generators the result

3.

$$(\Gamma^{k_1} \cdot \Gamma^{k_2} \cdot \ldots \cdot \Gamma^{k_p})^2 = (-1)^{h(p)}$$

where

$$h(p) = p + 1 + \binom{p-1}{2}\theta(p-3).$$

In this expression, we have the combinatorial number which is valid for $p \geq 3$. With this result, we are able to write factors such that the square is always $+1$. Notice the independence from the number of generators, D.

Examples of hermitian generators are

$$\Gamma^k,\ i\Gamma^k \cdot \Gamma^l,\ i\Gamma^k \cdot \Gamma^l \cdot \Gamma^m, \ldots \qquad (k, l, m = 1, \ldots, D).$$

In the case of physical interest for $D = 4$, in a spacetime with metric $g^{00} = -g^{kk} = 1$, $k = 1, 2, 3$, $(\gamma^0)^2 = -(\gamma^k)^2 = I$,

$$\gamma^0,\ i\gamma^k,\ \gamma^0\gamma^k,\ i\gamma^k\gamma^l,\ i\gamma^0\gamma^k\gamma^l,\ \gamma^1\gamma^2\gamma^3,\ i\gamma^0\gamma^1\gamma^2\gamma^3.$$

From the algebraic point of view, these objects, together with a purely scalar and constant object, the unit under the multiplication used in eq. (1), constitute a ring, which for $D = 4$ and Dirac matrices is called the Dirac ring. We adopt the notation for all dimensions.

The important point is that *all* these objects are taken to be different, or, more precisely, the Dirac ring is an irreducible set.

Another important property we recall for the members of the ring is what, for matrices, is the null trace. For differential forms, the equivalent operation was introduced in an article by the present authors and M.A.Rego Monteiro [7], and was called the scalar value of differential forms, and represented by the symbol \$. It is defined with the help of the contraction operation of a vector with differential forms, $e_\mu\rfloor$. Applied on a zero-form, the result of \$ is D, by convention. Then, it follows that

$$\$(dx^\mu) \equiv \$(dx^\mu \vee 1) = e^\mu\rfloor 1 = 0$$

$$(dx^\mu \vee dx^\nu) \equiv e^\mu\rfloor dx^\nu \$(1) = Dg^{\mu\nu}.$$

It follows from its definition that the scalar value of an odd-degree form is always zero. The scalar value of a Clifford product of an even number of differentials is a combination of metric tensors.

This completes the characterization of the Dirac ring: for any value for the spacetime dimension is the set of all objects formed by the generators of the Clifford algebra and their products, all being hermitian (in the generalized sense referred to above), and having null trace or scalar value.

3 Clifford algebras in even dimensions and unitary algebras

In this section, we prove the following

Theorem 1 *Given a Clifford algebra with complex coefficients and with an even number D of generators, the algebra of commutators for the members of the Dirac ring is isomorphic to the Lie algebra of the special unitary group of order $2^{D/2}$, denoted by $su(2^{D/2})$.*

Comment. In the preceding section, we have shown that the Dirac ring is formed by the unit element, the hermitian generators of the Clifford algebra and all their (hermitian) products. The heart of the proof, given the dimension of the ring, being 2^D, and the fact that the members of the ring are all hermitian and traceless (with the exception of the unit), is to find how many objects commute among them. It is enough to prove that there are $2^{D/2}-1$ commuting objects, since the algebra is determined by the number of hermitian, traceless elements (2^D-1) and the dimension of the Cartan subalgebra [8]. The resulting algebra is, then, $su(2^{D/2})$.

Proof. The proof is based on the results 1. to 3. of the preceding section. We must show that there exists one set of commuting objects, by construction.

Select two generators, for instance, Γ^{k_1} and Γ^{k_D}. One of them is a "spectator" for the commuting subalgebra to be built. The other one will be a member of the commuting set. Let Γ^{k_D} be chosen as the spectator.

Consider now the remaining generators, $\Gamma^{k_2},\ldots,\Gamma^{k_{D-1}}$. Take them by pairs in an ordered way, that is, define

$$\zeta^1=\Gamma^{k_2}\cdot\Gamma^{k_3},\ \zeta^2=\Gamma^{k_4}\cdot\Gamma^{k_5},\ldots,\zeta^{(D-2)/2}=\Gamma^{k_{D-2}}\cdot\Gamma^{k_{D-1}}.$$

By property 1. in the preceding section, they commute among themselves and are $(D-2)/2$ in number. Now, take all their products; they are commuting and form a set with a number of members which is the same as in a "Dirac ring" (excluding the unit) of dimension $(D-2)/2$, that is, $\mathrm{card}\{\zeta^k\}=$ number of commuting ζ^k and their products: $2^{D/2-1}-1$.

By property 1., $\Gamma^{k_1}\cdot\Gamma^l\cdot\Gamma^m$ also commute among themselves and with Γ^{k_1}. That is, we have another set with the same cardinality as the set $\{\zeta\}$. Then, we have

$$2(2^{D/2-1}-1)=2^{D/2}-2$$

commuting objects; adding Γ^{k_1}, which commutes with the other sets, we have

$$2^{D/2}-2+1=2^{D/2}-1$$

objects, as it was to be proved.

Another way of looking at the problem is to consider the generators of the commuting ring, being $(D-2)/2$ from ζ^l, and Γ^{k_1}. As a whole, we have $D/2$ and the resulting commuting ring is of dimension $2^{D/2}-1$.

Other sets of commuting objects are made out of $\Gamma^{D+1}=a\Gamma^{k_1}\cdots\Gamma^{k_D}$ (where a is chosen so that the product is hermitian) and all the ordered pairs $\xi^1=\Gamma^{k_1}\cdot\Gamma^{k_2},\ldots,\xi^{D/2}=\Gamma^{k_{D-1}}\cdot\Gamma^{k_D}$. Obviously, the dimension of any of these sets is, again, $2^{D/2}-1$. The sets are different because of the pairings. There are $2^{D/2}-1$ ways of having a diagonal matrix with an equal number of $+1$ and -1 eigenvalues.

Examples. (with Minkowski metric diag$(+-\cdots-)$)

- $D=2$: the commuting algebra is trivial and corresponds to choosing as σ_3 either γ^0, $-i\gamma^1$ or $-i\gamma^2$.

- $D=4$: Dirac–Pauli representations:

$$\begin{array}{cc}
\gamma^3:\ \text{spectator} & \gamma^2:\ \text{spectator} \\
\gamma^0,\ i\gamma^1\gamma^2,\ i\gamma^0\gamma^1\gamma^2 & \gamma^0,\ i\gamma^1\gamma^3,\ i\gamma^3\gamma^0\gamma^1 \\
-i\gamma^1,\ \gamma^0\gamma^2,\ -i\gamma^0\gamma^1\gamma^2 & i\gamma^1,\ \gamma^0\gamma^3,\ i\gamma^1\gamma^0\gamma^3 \\
i\gamma^2,\ \gamma^0\gamma^1,\ i\gamma^0\gamma^1\gamma^2 & i\gamma^3,\ \gamma^1\gamma^3,\ i\gamma^0\gamma^1\gamma^3.
\end{array}$$

Kramers–Weyl representations:

$$\gamma_5=\gamma^0\gamma^1\gamma^2\gamma^3$$
$$\gamma^0\gamma^3,\ i\gamma^1\gamma^2$$
$$\gamma^0\gamma^1,\ i\gamma^2\gamma^3$$
$$\gamma^0\gamma^2,\ -i\gamma^1\gamma^3.$$

- $D=6$: Dirac–Pauli representations:

$$\text{spectator: } \gamma^5;\ \text{generator: } \gamma^0$$
$$\gamma^0$$
$$i\gamma^1\gamma^2,\ i\gamma^3\gamma^4$$
$$i\gamma^0\gamma^1\gamma^2,\ i\gamma^0\gamma^3\gamma^4$$
$$\gamma^1\gamma^2\gamma^3\gamma^4$$

Kramers–Weyl representations:

$$\gamma_7 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5$$

$$\gamma^0\gamma^1,\, i\gamma^2\gamma^3,\, i\gamma^4\gamma^5$$

$$i\gamma^0\gamma^1\gamma^2\gamma^3,\, i\gamma^0\gamma^1\gamma^4\gamma^5, \gamma^2\gamma^3\gamma^4\gamma^5$$

Notice the irrelevant sign for γ_7, in order to suit the relation [9]

$$\gamma_{D+1} = -i^{D/2+1}\gamma^0 \cdots \gamma^{D-1}.$$

4 Clifford algebras in odd dimensions

The theorem for odd dimensions is stated as follows:

Theorem 2 *Given a Clifford algebra with complex coefficients, with an odd number of generators, D, the algebra of commutators for the members of the Dirac ring is isomorphic to the Lie algebra of the direct product of two special unitary groups of order* $(D-1)/2$*, i.e.,* $su(2^{(D-1)/2}) \oplus su(2^{(D-1)/2})$.

Comment. The proof is almost the same as that for even dimensions, but the argument needs also the result for even dimensions. The procedure is to show that two complementary commutator algebras, with generators X and Y, say (without entering on much detail upon indices for the time being), are available. The one for X is closed, but the commutator of two Y generators is always of the X type. The last point is crucial for the proof, the rest is standard manipulation to convert to generators $W^{\pm}$ that close, each set, an $su(2^{(D-1)/2})$ algebra.

Proof. Again, one should center on which are the members of a commuting set of the Dirac ring. Again, for obvious reasons, the unit element is not considered to establish the algebra of commutators.

For odd dimensions, the product of all generators commute with all the members of the Dirac ring. This has deep consequences, as we shall show.

Let us take all members of the Dirac ring which do not include a given operator; to be specific, let Γ^D be this generator. The set is then equivalent to the Dirac ring in dimension $D-1$ (even). The algebra of commutators, corresponding to the set, is, as shown in the preceding section, $su(2^{(D-1)/2})$. For the sake of precision, let us call, in general, a member of this set as X_k, $k = 1, \ldots, 2^{(D-1)/2} - 1$.

The remaining set of the Dirac ring is made out of the complementary members, in the sense that, given one X_k, there is always one this set, Y_k, such that

$$X_k \cdot Y_k = \Gamma^{D+1}, \qquad (6)$$

where Γ^{D+1} is a hermitian object.

The crucial point now is that any two members of the set of objects Y_k (again, $2^{D/2-1} - 1$ in number) satisfy a commutator algebra, with

$$[Y_k, Y_l] = c_{klm} X_m. \tag{7}$$

In fact, one can easily show that, for any pair (Y_k, Y_l), the corresponding c_{klm} and X_m are those for the objects without the index D.

Besides, one sees that

$$[X_k, Y_l] = c_{klm} Y_m, \tag{8}$$

as a corollary of the way the objects are made.

By the usual procedures, one can build from X_k and Y_k two sets of cardinality $2(2^{(D-1)/2} - 1)$, that is,

$$W_k^{\pm} = X_k \pm Y_k, \tag{9}$$

which generate two separate $su(2^{(D-1)/2})$ algebras:

$$\left[W_k^{\pm}, W_l^{\pm}\right] = c_{klm} W_m^{\pm} \tag{10}$$

$$\left[W_k^{+}, W_l^{-}\right] = 0. \tag{11}$$

Example. For $D = 3$, the explicit example for Dirac matrices has been built previously [2]. Let us take now the case of Minkowski spacetime with $D = 5$ and $g^{00} = 1$, $g^{kk} = -1$, $k = 1, \ldots, 4$.

Generators of the Clifford algebra: $\gamma^0, \gamma^1, \ldots, \gamma^4$.

Dirac ring:

$$
\begin{array}{rl}
I & \\
\gamma^0,\ i\gamma^k & (k = 1, \ldots, 4) \\
\gamma^0\gamma^k,\ i\gamma^k\gamma^l & (k < l,\ k, l = 1, \ldots, 4) \\
i\gamma^0\gamma^l\gamma^k,\ \gamma^l\gamma^k\gamma^m & (k < l < m,\ k, l, m = 1, \ldots, 4) \\
i\gamma^0\gamma^k\gamma^l\gamma^m,\ \gamma^1\gamma^2\gamma^3\gamma^4 & (k < l < m,\ k, l, m = 1, \ldots, 4) \\
\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 &
\end{array}
$$

Lie algebra generators: take the subset of the Dirac ring not having the index 4:

$$
X_1 = \frac{1}{2}\gamma^0,\ X_2 = \frac{1}{2}i\gamma^1, \ldots, X_4 = \frac{1}{2}i\gamma^3,\ X_5 = \frac{1}{2}\gamma^0\gamma^1, \ldots, X_7 = \frac{1}{2}\gamma^0\gamma^3,
$$
$$
X_8 = \frac{1}{2}i\gamma^1\gamma^2, \ldots, X_{10} = i\frac{1}{2}\gamma^2\gamma^3,\ X_{11} = \frac{1}{2}i\gamma^0\gamma^1\gamma^2, \ldots, X_{13} = \frac{1}{2}i\gamma^0\gamma^2\gamma^3,
$$
$$
X_{14} = \gamma^1\gamma^2\gamma^3,\ X_{15} = \frac{1}{2}i\gamma^0\gamma^1\gamma^2\gamma^3.
$$

Consider now the "complementary" subset:

$$Y_1 = \frac{1}{2}\gamma^1\gamma^2\gamma^3\gamma^4,\ Y_2 = \frac{1}{2}i\gamma^0\gamma^2\gamma^3\gamma^4, \ldots, Y_4 = \frac{1}{2}i\gamma^0\gamma^1\gamma^2\gamma^4,\ Y_5 = \frac{1}{2}\gamma^2\gamma^3\gamma^4,$$
$$Y_6 = -\gamma^1\gamma^3\gamma^4,\ Y_7 = \frac{1}{2}\gamma^1\gamma^2\gamma^4,\ Y_8 = i\frac{1}{2}\gamma^0\gamma^3\gamma^4,\ Y_9 = \frac{1}{2}i\gamma^0\gamma^2\gamma^4,$$
$$Y_{10} = -\frac{1}{2}\gamma^0\gamma^1\gamma^4,\ Y_{11} = -\frac{1}{2}i\gamma^3\gamma^4,\ Y_{12} = \frac{1}{2}i\gamma^2\gamma^4,\ Y_{13} = -\frac{1}{2}i\gamma^1\gamma^4,$$
$$Y_{14} = -\frac{1}{2}\gamma^0\gamma^4,\ Y_{15} = -\frac{1}{2}i\gamma^4.$$

It is easy to check that both sets, $\{X_k\}, \{Y_l\}$, $k, l = 1, \ldots, 15$, satisfy the following commutator algebra:

$$[X_k, X_l] = ic_{klm}X_m$$
$$[Y_k, Y_l] = ic_{klm}X_m$$
$$[Y_k, X_l] = ic_{klm}Y_m.$$

The way one recovers an uncoupled set of quantities is by defining

$$W_k^{\pm} = X_k \pm Y_k.$$

Simple algebra shows that

$$\begin{aligned} \left[W_k^{\pm}, W_l^{\pm}\right] &= ic_{klm}W_m^{\pm} \\ \left[W_k^{+}, W_l^{-}\right] &= 0, \qquad\qquad \text{for all } k, l. \end{aligned}$$

Notice that the algebras are $su(4)$, as the commutation relations are those of the Dirac ring with 15 generators, and there is a set of three generators commuting among themselves.

5 Discussion and conclusions

The main results of this work are apparent in the statement of the theorems. The procedures are algebraic and direct.

The meaning of the unitary groups involved needs further clarification from the physical point of view. They embody the symmetry under Lorentz transformations and rotations of the Dirac equation (and the changes in the so-called "representation" of the Dirac matrices, which we prefer to call "picture", which we prefer to call "picture").

The algebraic structure allows for a clear definition of the picture of the matrices representing the Clifford algebra: a picture is associated with a choice for the matrices forming the Cartan subalgebra.

By the construction, it is natural that the algebras for the $(D-1)$ odd-dimensional Dirac ring follows from the even D-dimensional case, by freezing one generator of the Clifford algebra for the latter.

The case of even dimensions is not a surprise, it is only clarified the origin of the need for $2^{D/2}$ spinors. One could go further and relate the spinors with the minimal left ideals of the Lie algebra involved; in fact, the members of the Cartan subalgebra are used to build the idempotents that project on a given minimal left ideal. We shall leave for a later publication the discussion of this aspect.

The odd-dimensional case is far more controversial. Most physicists in the community of high-energy physics and field theory take for granted the representation provided by a finite-group reasoning [6], in which the generators for the Clifford algebra are the $D-1$ generators of the even next lower dimension and its hermitian product (in order to guarantee the property of anticommutation for D generator). For those physicists, our development would represent a curious but rather unnecessary complication. The points we wish to make are the following:

1. In dealing with the Dirac equation, the symmetry operations (Lorentz invariance, rotation invariance, etc.) are always implemented by automorphisms which come from the exponentiation of the relevant members of the Dirac ring. The automorphisms generated by the finite

group we think are completely unusual for this.

2. The usual representations coming from the finite-group considerations are not irreducible under inversion of any given coordinate axis. In the present algebraic form, this inversion exchanges the blocks with eigenvalue ± 1 of the product $\Gamma^{D+1} = a\Gamma^1 \cdot \Gamma^2 \cdot \ldots \cdot \Gamma^D$.

3. The consequences of the last point translate into different physics coming from both representations. This has, consequently, possible experimental confirmation.

4. Last, but meaningful from the mathematical point of view, only for $D = 3$ is the representation coming from the finite group faithful. For higher odd dimensions this is not so [10,11].

There is also a sensible difference between the physics resulting from the representations coming from finite groups and those from our algebraic considerations. As we have shown, for the case of electrodynamics in $D = 3$, for instance, there is no induced Chern–Simons term in the vacuum polarization at the lowest (one-loop) order of perturbation theory [2]. This comes about since, in the algebraic procedure, the trace of an odd number of Dirac matrices is always zero.

We have to make also a reference to duality for differential forms. In the case of odd dimensions, the generators $\{X_k\}$ and $\{Y_l\}$ introduced above are Hodge dual to each other [12], when dealing with differential forms endowed with a Clifford product. Notice that in their analysis with Dirac matrices, Brauer and Weyl [13] introduced an operation that closely resembles the Hodge duality for differential forms. In terms of these, the combinations $W_k^{\pm}$ turn out to be selfdual or anti-selfdual.

This is something which is related with the representations induced by the considerations from finite groups. To be specific, let us refer to the $D = 3$ Minkowski space with metric diag$(+ - -)$. It is currently used for it, from finite groups, the representation

$$\gamma^0 = \sigma_3,\ \gamma^1 = i\sigma_1,\ \gamma^2 = i\sigma_2, \tag{12}$$

in terms of Pauli matrices. Notice that

$$\gamma^0\gamma^1 = i\gamma^2. \tag{13}$$

Through the isomorphism between matrices and forms [5], this would mean, in the latter formalism,

$$dx^0 \vee dx^1 = dx^0 \wedge dx^1 = idx^2, \tag{14}$$

making equivalent the component of an antisymmetric tensor with the one of a vector. In this sense, one notes that this forces Hodge duality in an unsuitable way.

A last comment refers to the concept of chirality. It is commonly stated that it is interesting only for spacetimes of even dimension. From the Graf isomorphism, chirality in odd dimensions refers to the eigenvalue of the diagonal matrix Γ^{D+1}, which is always of block form, $a \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, with I the identity matrix in $2^{(D-1)/2}$ dimensions. One goes from one block to the other by inverting a single matrix or a coordinate) in the product. By the way, this handedness (or chirality) for a reference system translates into a sense of gyration in space. This is precisely what is referred to, when chirality is defined, for instance, for massless fermions in $D = 4$. One talks about "left-handed" or "right-handed" neutrinos, and the sense of rotation is the one in the space of dimension two orthogonal to the momentum.

Acknowledgments

The authors warmly acknowledge Prof. A. O. Caride for his interest and care in explaining them the relevant aspects of the theory of representation for finite groups. One of them (J.A.M.) acknowledges discussions with Prof. E. Remiddi, from Bologna, and the hospitality of the theoretical group at Turin and CERN, where part of this work was realized.

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