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ON THE SCALAR CASIMIR ENERGIES IN SPACETIMES WITH $M^d \times T^q$ STRUCTURE

by

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ABSTRACT

The Casimir energy density of a scalar field quantized in a $M^d \times T^q$ spacetime is calculated. The field is supposed to satisfy Dirichlet and periodic boundary conditions in the (d-1)- and q-dimensional submanifolds, respectively. On account of this non-trivial topology, the sign of the Casimir energy is shown to have the same peculiar and entangled dependence on the number of finite sides of the hyperparallelepipedal cavity and on the spacetime dimension, with only one exception which is discussed.

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The motivation to study the Casimir effect [1] in spacetimes with an arbitrary number of dimensions is frequently attributed to the present development of unifying forces programs as, for instance, those inspired on the Kaluza-Klein ideas [2] or superstring theories [3]. It is unquestionable that any higher-dimensional theory should explain *why* and *how* a certain number of dimensions are to be compactified, in order to have their predictions checkable in the four-dimensional phenomenological world. Presumably these dimensions are curled out in a volume whose scale is set by Planck length but the way for implementing a dynamical compactification scheme is not unique.

Although in the past, several authors [4,5] pointed out that the Casimir effect could be a plausible explanation of a spontaneous compactification of extra dimensions, this question is still far from being widely understood. Inasmuch as the nature of the Casimir effect is strongly influenced by the topological features of spacetime, e.g. compactness, and the vacuum renormalized energy density has been calculated only for volumes with its boundaries placed in the infinity [6] or for completely compact space [7], it is certainly needful to analyse the case of geometric boundary configurations separated by finite distances in a manifold with some compact dimensions. Our point is that, choosing these feasible configurations able to measure the Casimir effect, once the Casimir energy is computed in different manifolds, the analysis of its nature and magnitude and its confront with experimental measurements (in the case of electromagnetism) [8] may select possible topologies for the curled out dimensions.

In this letter, the vacuum renormalized energy density associates with a scalar field defined in a D-dimensional flat manifold, with a $M^d \times T^q$ structure (with d+q=D), bounded by a hyperparallelepipedal cavity made of perfectly reflecting plates, is calculated. The result is discussed and compared to other well known results.

Several techniques has been developed in the literature to compute the Casimir energy: the cut-off method, the Green function method, the dimensional regularization one and the zeta-function technique [9]. It is hopeful that all methods will necessarily lead to the same result, but a general proof of the equivalence of all those methods is lacking. For a simple case, an analytical proof of the equivalence between the cut-off and the zeta-function methods is given in [10]. Using the same technique applied in a recent paper [11], an expression for the Casimir energy associated to a scalar field, more convenient to carry on the aforementioned discussion is obtained. Non-flat structures were studied in [12].

A Hermitian massless scalar field $\Phi(x^0,x^i,y^j)$ can be defined in a D-dimensional $M^d \times T^q$ manifold; d is the number of non-compact and q of the compact dimensions. Let us assume that the compact manifold has the topology of a q-torus. This scalar field obeys the generalized Klein-Gordon equation;

$$\left[\left(\frac{\partial}{\partial x^0}\right)^2 - \sum_{i=1}^{d-1} \left(\frac{\partial}{\partial x^i}\right)^2 - \sum_{j=1}^{q} \left(\frac{\partial}{\partial y^j}\right)^2 \right] \Phi(x,y) = 0 \tag{1.1}$$

and in the (d-1)-dimensional sub-manifold it is confined in the interior of a rectangular cavity with edges L_1, L_2, ..., L_{d-1}, satisfying Dirichlet boundary conditions over the box surface $\partial\Omega$, i.e., $\Phi(x,y)|_{\partial\Omega}=0$. For the q-dimensional compact sub-manifold the field satisfies periodic b.c, i.e.,

$$\Phi(x^0,x^i,y^1,\dots,y^j,\dots, y^q) = \Phi(x^0,x^i,y^1,\dots,y^j+\bar{\ell}^j,\dots, y^q), \tag{1.2}$$

for $\forall j$, where the $\bar{\ell}^j$'s are the torus cincunference legths.

$\{\phi_{nm}, \phi^*_{nm}\}$ stands for a short notation of the basis in the space of solutions of the Klein-Gordon equation where the above b.c. are imposed, and are given (N is a normalization factor) by:

$$\phi_{\{nm\}}(x,y) = N \exp(-i\omega_{\{nm\}}x^0) \exp(i\vec{k}.\vec{x}) \prod_{j=1}^{q} \left[\exp(im_j\pi y^j/\ell_j)\right] \tag{1.3}$$

where $\{nm\}$ denotes $\{n_1, n_2, \dots, n_{d-1}, m_1, m_2, \dots, m_q\}$, $\ell_j = \bar{\ell}^j/2$,

$$\omega_{\{nm\}} = \left[k^2 + \left(\frac{m_1\pi}{\ell_1}\right)^2 + \left(\frac{m_2\pi}{\ell_2}\right)^2 + \dots + \left(\frac{m_q\pi}{\ell_q}\right)^2 \right]^{1/2} \tag{1.4}$$

and

$$k^2 = \left[\left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2 + \dots + \left(\frac{n_{d-1}\pi}{L_{d-1}}\right)^2 \right].$$

The scalar field can be expanded over this complete orthonormal set of mode solutions as follows:

$$\Phi(x) = \sum_{\substack{\{n\}=1 \\ \{m\}=-\infty}}^{\infty} \left[a_{\{nm\}} \, \phi_{\{nm\}} + a^{\dagger}_{\{nm\}} \, \phi^{*}_{\{nm\}} \right] \tag{1.5}$$

Inserting (1.5) in the free scalar field Hamiltonian and using orthonormality of the $\{\phi_{nm}, \phi^{*}_{nm}\}$ basis, it is straightforward to find

$$H = \sum_{\substack{\{n\}=1 \\ \{m\}=-\infty}}^{\infty} \omega_{\{nm\}} \left[a^{\dagger}_{\{nm\}} \, a_{\{nm\}} + 1/2 \right] \tag{1.6}$$

The vacuum (no particle state) expected value of the Hamiltonian operator (1.6) is

$$E_D^{(d,q)}(L_1,\ldots,L_{d-1}, \ell_1, \ldots, \ell_q) = \frac{1}{2} \sum_{\substack{n_1 n_2 \ldots n_{d-1} = 1 \\ m_1 m_2 \ldots m_q = -\infty}}^{\infty} \left[\left(\frac{n_1 \pi}{L_1}\right)^2 + \right.$$

$$\left. + \left(\frac{n_2 \pi}{L_2}\right)^2 + \ldots + \left(\frac{n_{d-1} \pi}{L_{d-1}}\right)^2 + \left(\frac{m_1 \pi}{\ell_1}\right)^2 + \ldots + \left(\frac{m_q \pi}{\ell_q}\right)^2 \right]^{1/2} \tag{1.7}$$

In the limit

$$L_1, L_2, \ldots, L_p << L_{p+1}, L_{p+2}, \ldots, L_{d-1}$$

and, for simplicity (without loss of generality for our purpose) assuming $L_1 \simeq L_2 \simeq \ldots \simeq L_p = L$, and $\ell_1 \simeq \ell_2 \simeq \ldots \simeq \ell_q = \ell$, we define the energy density (energy per unit hyperarea), $\varepsilon_D^{(p,q)}(L,\ell)$, as a function of the number p of finite length edges and of the number q of compact space dimensions:

$$\varepsilon_D^{(p,q)}(L,\ell) \equiv \frac{E_D^{(p,q)}}{\prod_{i=p+1}^{D-q-1} (L_i)} = \frac{1}{(2\sqrt{\pi})^{D-p-q-1}} \quad \frac{1}{\Gamma\left(\frac{D-p-q-1}{2}\right)} \times$$

$$\times \sum_{\substack{n_1 n_2 \ldots n_p = 1 \\ m_1 m_2 \ldots m_q = -\infty}}^{\infty} \int_0^{\infty} dr\, r^{D-p-q-2} \left[r^2 + \left(\frac{n_1 \pi}{L}\right)^2 + \ldots + \right.$$

$$\left. + \ldots + \left(\frac{n_p \pi}{L}\right)^2 + \left(\frac{m_1 \pi}{\ell}\right)^2 + \ldots + \left(\frac{m_q \pi}{\ell}\right)^2 \right]^{1/2} \qquad (1.8)$$

This vacuum energy density is clearly divergent. A finite result can be obtained by using the zeta-function regularization procedure. The Casimir density energy $\bar{\varepsilon}_D$ is thus given by:

$$\bar{\varepsilon}_D^{(p,q)}(L,\ell) = \frac{L^{p+q-D}}{2^{D-q+1}} \rho^q \sum_{k=0}^{p} (-1)^{k+1} C_p^k (\sqrt{\pi})^{k-D} \Gamma\left(\frac{D-k}{2}\right) \times$$

$$\times A(1,\ldots,1;\rho^2,\ldots,\rho^2;D-k) \qquad (1.9)$$

where $\rho \equiv \ell/L$, and there are p-k terms (1,...,1) and q terms $(\rho^2,\ldots,\rho^2)$ as argument of the Epstein function $A(a_1,a_2,\ldots,a_k;2s)$ [13], defined as:

$$A(a_1,a_2,\ldots,a_k;2s) = \sum_{n_1 n_2 \ldots n_k=-\infty}^{\infty}{}' \left[a_1 n_1^2 + a_2 n_2^2 + \ldots + a_k n_k^2 \right]^{-s} \quad (1.10)$$

($a_i>0$ and the prime means that the term $n_1=n_2=\ldots=n_k=0$ has to be excluded).

The expression (1.9) is not appropriate to analize the behaviour of the Casimir energy density and its limit for the very physical interesting limit $\rho \ll 1$. A more convenient expression can be obtained by inserting in Eq. (1.9) the integral representation for the Gamma function,

$$w^{-s}\,\Gamma(s) = \int_0^\infty dt\; e^{-wt}\; t^{s-1}\;,\; \mathrm{Re}\; w > 0$$

yielding

$$\bar{\varepsilon}_D^{(p,q)}(L,\ell) = \frac{L^{p+q-D}}{2^{D-q+1}}\;\frac{\rho^q}{\pi^{D/2}}\int_0^\infty dt\; t^{D/2-1}\;\sum_{k=0}^{p}(-1)^{k+1}\,C_p^k \;\times$$

$$\times \left(\sqrt{\frac{\pi}{t}}\right)^k \left[\left(\sum_{n=-\infty}^{\infty} e^{-n^2 t}\right)^{p-k} \left(\sum_{m=-\infty}^{\infty} e^{-m^2 t}\right)^{q} - 1 \right]$$

Performing the sum over k we obtain

$$\bar{\varepsilon}_D^{(p,q)}(L,\ell) = \frac{L^{p+q-D}}{2^{D-q+1}}\;\frac{\rho^q}{\pi^{D/2}}\int_0^\infty dt\; t^{D/2-1}\;\left[\left(1 - \sqrt{\frac{\pi}{t}}\right)^p + \right.$$

$$- \left(\sum_{n=-\infty}^{\infty} e^{-n^2 t} - \sqrt{\frac{\pi}{t}} \right)^p \left(\sum_{m=-\infty}^{\infty} e^{-m^2 \rho^2 t} \right)^q \Bigg] \tag{1.11}$$

which generalizes the result of Ref. [11]. Indeed, for q=0 (no compact dimensions) we retrieve the expression

$$\bar{\varepsilon}_D^{(p,0)}(L) = \frac{L^{p-D}}{2^{D+1}} \pi^{-D/2} \int_0^\infty dt \, (\sqrt{t})^{D-2} \left[\left(1 - \sqrt{\frac{\pi}{t}} \right)^p + \right.$$

$$\left. - \left(\vartheta_3(0,e^{-t}) - \sqrt{\frac{\pi}{t}} \right)^p \right] \tag{1.11'}$$

where $\vartheta_3(0,e^{-t})$ is the Elliptic theta function defined as [14]

$$\vartheta_3(z,q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos (2nz),$$

obtained in [11], where the behaviour of $\bar{\varepsilon}_D^{(p)}$ was carefully studied.

Doing p=0 in Eq. (1.11) we recover the result obtained in [8].

To analize the sign and the magnitude of the Casimir density energy it is useful to discuss both the limits $\rho \ll 1$ and $\rho \gg 1$. To this purpose, it is convenient to rewrite Eq. (1.11) as:

$$\bar{\varepsilon}_D^{(p,q)}(L,\ell) = \left(\frac{\rho}{\sqrt{\pi}} \right)^q \bar{\varepsilon}_D^{(p,0)}(L) - \frac{L^{p+q-D}}{2^{D-q+1}} \frac{\rho^q}{\pi^{D/2}} \int_0^\infty dt \, t^{D/2-1} \times$$

$$\times \left(\vartheta_3(0,e^{-t}) - \sqrt{\frac{\pi}{t}} \right)^p \left(\vartheta_3^q(0,e^{-\rho^2 t}) - 1 \right) \equiv \bar{\varepsilon}_1 + \bar{\varepsilon}_2 \tag{1.12}$$

In the limit $\rho \ll 1$ the term $\bar{\varepsilon}_1$ goes trivially to zero since Eq. (1.11′) is independent of ρ. However, the integral contained in $\bar{\varepsilon}_2$ has a ρ dependence and needs a careful analysis. After some manipulations $\bar{\varepsilon}_2$ can still be written as

$$\bar{\varepsilon}_2 = \frac{-L^{p+q-D}}{2^{D-q+1}} \; \frac{\rho^{q-D}}{\pi^{D/2}} \int_0^\infty d\tau \; (\sqrt{\tau})^{D-2} \left(\vartheta_3^q(0,e^{-\tau}) - 1 \right) \times$$

$$\times \left(\vartheta_3(0,e^{-\tau/\rho^2}) - \rho \sqrt{\frac{\pi}{\tau}} \right)^p \qquad (1.13)$$

where $\tau = \rho^2 t$.

The absolute value of the integrand is bounded by

$$(\sqrt{\tau})^{D-2} \left(\vartheta_3^q(0,e^{-\tau}) - 1 \right)$$

which is integrable in $(0,\infty)$. So, using the dominated convergence theorem, it follows that

$$\lim_{\rho\to 0} \int_0^\infty d\tau \; (\sqrt{\tau})^{D-2} \left(\vartheta_3^q(0,e^{-\tau}) - 1 \right) \left(\vartheta_3(0,e^{-\tau/\rho^2}) - \rho \sqrt{\frac{\pi}{\tau}} \right)^p =$$

$$= \int_0^\infty d\tau \; (\sqrt{\tau})^{D-2} \left(\vartheta_3^q(0,e^{-\tau}) - 1 \right) = \Gamma(D/2) \; A(1,\ldots,1;D)$$

where now the $(1,\ldots,1)$ argument of the Epstein function consists of q terms. Then,

$$\bar{\varepsilon}_2 = \frac{-L^{p+q-D}}{2^{D-q+1}} \frac{\rho^{q-D}}{\pi^{D/2}} \left(\Gamma(D/2)\, A(1,\ldots,1;D) + O(\rho) \right) \qquad (1.14)$$

which, from Eq. (1.12), is the dominant contribution to the Casimir energy density in the limit $\rho \to 0$ and, therefore, $\bar{\varepsilon}_D$ becomes negative and very large, *irrespective of* q ($q \neq 0$). Note that even for finite L, the dominant contribution is just the Casimir energy density corresponding to all space without boundaries, as found in [8].

In the limit $\rho \gg 1$ of Eq. (1.12) it is easy to see that $\bar{\varepsilon}_2$ goes to zero faster than $\bar{\varepsilon}_1$. Thus, the sign of the Casimir energy will be the same of $\bar{\varepsilon}_1$ which was analized in [11], concerning non-compact spacetimes. There, it was shown that:

i) when p is odd the force is always attractive for all spacetime dimensions (D);

ii) for p even and not very large, there does exist a critical spacetime dimensionality (D_c) for which the force is repulsive if $D < D_c$ and it is attractive if $D \geq D_c$. For p large enough (≥ 30) the force is always attractive.

For the compact case, we first consider even values of p, for which the term $\bar{\varepsilon}_2$ in Eq. (1.12) is always negative. Thus, if $D \geq D_c$ or $p \geq 30$ the Casimir force will be always attractive irrespective of the value of ρ. Otherwise, the force is attractive in the limit $\rho \ll 1$ and repulsive if $\rho \gg 1$, showing that there will be a particular value of ρ for which the system will be in equilibrium.

When p is odd, the analysis of the nature of the Casimir force is done remembering that $\bar{\varepsilon} < 0$ for both limits $\rho \ll 1$ and $\rho \gg 1$, and then showing that Eq. (1.11) has no zero. Indeed, this is a straightforward result if, in all terms of this Eq., use is made of the inequality:

$$\sum_{n=-\infty}^{\infty} e^{-n^2 t} < \sqrt{\frac{\pi}{t}} + 1.$$

Thus, one gets the conclusion that, for p odd, the Casimir force is always attractive, irrespective of the values of ρ, D and q.

So, it was shown in this note that the aforementioned peculiar and entangled dependence between the force's nature, the geometry (p) and spacetime dimensionality (D), still remains in the case of compact (toroidal) spacetimes with only one exception, namely: for p even and $D < D_c$, the Casimir force is repulsive for non-compact flat spacetimes (regardless of the magnitude of L), while, for compact (toroidal) spacetimes, the nature of the force still depends on the ratio $\rho \equiv \ell/L$, being attractive for $\rho \ll 1$ or repulsive for $\rho \gg 1$. In particular, there will be a value of L for which the system will be in stable equilibrium.

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