ISSN 0029-3865

CBPF-NF-037/89

# ON THE $\zeta$-FUNCTION REGULARIZATION OF CHIRAL JACOBIANS FOR SINGULAR DIRAC OPERATORS

by

C.E.I. CARNEIRO*, S.A. DIAS+ and M.T. THOMAS**

+Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*Universidade de São Paulo
Instituto de Física
Caixa Postal 20516
01498 - São Paulo, SP - Brasil

**Universidade Federal Fluminense
Instituto de Física
Outeiro São João Batista s/nº
24020 - Niterói, RJ - Brasil

ABSTRACT

We propose a definition of the chiral Jacobian which uses the invariance of the generating functional under chiral rotations. This definition takes into account the contributions of all terms which after the rotation, get a dependence on the chiral parameter $\alpha$. We show that when the Dirac operator has zero eigenvalues the presence of fermionic sources gives an additional dependence on $\alpha$. Our definition, by considering this $\alpha$-dependence, reconciles the $\zeta$-function method of calculating chiral Jacobians with Fujikawa's.

PACS NUMBERS: 10; 11; 11.30. Rd

KEYWORDS: Chiral Jacobians, $\zeta$-function regularization, singular Dirac operators

## 1. INTRODUCTION

The study of anomalies using the path integral formalism began when Fujikawa[1] discovered that the fermionic measure is not invariant under chiral rotations. He calculated the Jacobian associated with these transformations and showed how it was related to the anomaly. Since then these results have been rederived in many ways. In particular, Gamboa-Saraví, Muschietti, Schaposnik and Solomin[2],[3] (GSMSS) proposed a very elegant method based on the $\zeta$-function regularization[4] and Seeley's[5] expansion coefficients, which permits one to extend in a natural way the evaluation of chiral Jacobians to theories which contain non-Hermitian operators.

In the GSMSS approach the quadratic fermionic path integral is regularized as

$$\int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp(-\langle\, \bar{\psi}\, D\, \psi\, \rangle) \equiv \mathrm{Det}\, D \equiv \exp\left\{\frac{-d\zeta(s,D)}{ds}\right\}_{s=0}, \qquad (1.1)$$

where

$$\zeta(s,D) = \sum_{j} \lambda_j^{-s} \qquad (1.2)$$

is the $\zeta$-function associated with operator D, $\lambda_j$ are the eigenvalues of D and < > denotes integration. Throughout this paper we work in Euclidean space.

Performing the chiral transformation

$$\psi(x) \longrightarrow e^{\alpha(x)\gamma_5}\psi(x) \equiv \Omega_5(x)\psi(x) \quad , \tag{1.3a}$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}(x)e^{\alpha(x)\gamma_5} \equiv \bar{\psi}(x)\Omega_5(x) \quad , \tag{1.3b}$$

in the path integral (1.1) and using its invariance under changes in the integration variables, we obtain

$$\text{Det } D = J(\alpha)\ \text{Det}(\Omega_5 D\ \Omega_5) \tag{1.4}$$

Thus, the Jacobian $J(\alpha)$ can be defined as the ratio of two regularized determinants. This definition treats Hermitian and non-Hermitian operators on an equal footing.

However, in many cases the operator D has zero-eigenvalues. For instance, in QCD $D = \gamma_\mu(i\partial_\mu + A_\mu)$, where $\gamma_\mu$ are hermitian Dirac matrices. If $A_\mu$ belongs to a non-trivial topological sector, D has zero eigenvalues. ´t Hooft[6] calculated the zero-eigenmodes when $A_\mu$ was an instaton or anti-instanton configuration for fermions in the fundamental representation. Jackiw and Rebbi[7] studied other representations. If there are zero eigenmodes, equation (1.4) cannot be applied directly. Gamboa-Saraví, Muschietti and Solomin[8] extended their method to non-invertible operators defining

$$\text{Det}'(D) = J_{GSMSS}(\alpha)\ \text{Det}'(\Omega_5 D\ \Omega_5) \quad , \tag{1.5}$$

where

$$\text{Det}'D = \lim_{\epsilon\to 0^+} \frac{\text{Det}(D + \epsilon I)}{\epsilon^{n_0}} \quad , \tag{1.6}$$

and $n_o$ is the number of zero eigenvalues. Notice that this procedure is not equivalent to adding a mall mass $\epsilon$ to D from the beginning[9]. According to (1.5) and (1.6) the mass term is added after the chiral rotation. For QCD their method, when $\alpha(x)$ is infinitesimal, gives[8]

$$\ln J_{GSMSS}(\alpha) = 2 \int dx\, \alpha(x)\left( \frac{1}{48\pi^2}\, tr\, {}^*F_{\mu\nu}F_{\mu\nu}\right) +$$

$$+ 2 \sum_{i=1}^{n_o} \int dx \varphi_{oi}^{+}(x)\gamma_5 \alpha(x)\varphi_{oi}(x) \quad , \qquad (1.7)$$

where $\varphi_{oi}(x)$ , $i=1,\ldots,n_o$, are the zero eigenmodes of D and the trace is over the color matrices.

On the other hand, if one uses Fujikawa's method to calculate the QCD Jacobian one obtains only the first term of expression (1.7). Apparently only in the trivial topological sector both methods give the same result. One may think that Fujikawa's derivation does not hold when there are zero eigenmodes. We believe that this is not the case. In our notation his definition of the chiral Jacobian is

$$\ln J(\alpha) = -2\int dx\, \alpha(x) \lim_{M\to\infty} \sum_n \varphi_n^{+}(x)\gamma_5\, e^{(D/M)^2}\varphi_n(x) \; , \qquad (1.8)$$

where $\{ \varphi_n \}$ is a complete set of eigenfunctions of D. This set is complete only if one includes the zero eigenmodes. In his calculation Fujikawa changes the basis vectors $\varphi_n$ to plane waves and here the completness of the set $\{ \varphi_n \}$ is crucial. In addition, since the number of zero eigenmodes is finite in each topological sector, no ill defined expression results if they are included in the sum of equation (1.8).

The purpose of this paper is to show that it is possible to reconcile the $\zeta$-function regularization with Fujikawa's method by changing the definition of the chiral Jacobian. Instead of using expression (1.5) we define $J(\alpha)$ by imposing that the generating functional is invariant under chiral rotations. In case of non-singular Dirac operator, this requirement reduces to (1.5). However, the presence of external sources in the singular Dirac operator case, induces new non-trivial $\alpha$-dependent terms in the Jacobian.

The lay-out of this paper is as follows. In section 2 we calculate the chiral Jacobian in the presence of sources. We show how the sources get an $\alpha$-dependence which exactly cancells the second term in equation (1.7). Using our results for constant $\alpha$ we rederive the Atiyah-Singer theorem[1],[10]. In section 3 we obtain Fujikawa's result by adding to D a small mass from the beginning. The discussion of the results and our conclusions are presented in section 4. Finally, we determine the zero-eigenmodes of the chirally rotated operator in the Appendix.

## 2. CALCULATION OF THE CHIRAL JACOBIAN IN THE PRESENCE OF SOURCES AND THE ATIYAH-SINGER THEOREM

The fundamental object in the path integral formalism is the generating functional

$$Z\,[\bar{\eta},\eta] = \int \mathcal{D}\psi\ \mathcal{D}\bar{\psi}\ \exp\Big\{-<\bar{\psi}D\psi> + <\bar{\eta}\psi> + <\bar{\psi}\eta>\Big\}\ , \qquad (2.1)$$

where for simplicity we have written down only the fermionic fields. $Z[\bar{\eta},\eta]$ is invariant under changes of the integration variables. In particular, it is invariant under the chiral transformation (1.3)

$$Z[\bar{\eta},\eta] = Z[\bar{\eta},\eta;\alpha] = J(\alpha)\int \mathcal{D}\psi\ \mathcal{D}\bar{\psi}\ \exp\{-<\bar{\psi}\ D(\alpha)\psi> +$$

$$+ <\bar{\eta}(\alpha)\psi> + <\bar{\psi}\eta(\alpha)>\}\ , \qquad (2.2)$$

with

$$D(\alpha) \equiv e^{\alpha(x)\gamma_5}\,D\,e^{\alpha(x)\gamma_5}\ ;\ \bar{\eta}(\alpha) \equiv \bar{\eta}\,e^{\alpha(x)\gamma_5}\ ;\ \eta(\alpha) \equiv e^{\alpha(x)\gamma_5}\eta\ . \qquad (2.3)$$

If the operator D does not have zero eigenvalues it is invertible an so is $D(\alpha)$. In this case the sources can be extracted from the path integral and their dependence on $\alpha$ cancells. In order to prove this we perform the shift

$$\psi(x) \longrightarrow \psi(x) + \int S(x,\ y;\ \alpha)\ \eta(y;\ \alpha)dy \quad , \tag{2.4a}$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}(x) + \int \bar{\eta}(y;\ \alpha)\ S(y,\ x;\ \alpha)dy \quad , \tag{2.4b}$$

where

$$S(x,y;\alpha) \equiv e^{-\alpha(x)\gamma_5} \sum_n \frac{\varphi_n(x)\varphi_n^+(y)}{\lambda_n}\ e^{-\alpha(y)\gamma_5} \equiv e^{-\alpha(x)\gamma_5}\ S(x,y)\ e^{-\alpha(y)\gamma_5} \tag{2.4c}$$

is the inverse of $D(\alpha)$ defined in equation (2.3). This can be easily demonstrated using the completeness of the set $\{\ \varphi_n(x)\ \}$ of eigenfunctions of the non-rotated operator D. Equation (2.2) becomes

$$Z[\bar{\eta},\eta;\alpha] = J(\alpha)\ \exp\{\ <\ \bar{\eta}\ S\ \eta\ >\ \}\ \int \mathcal{D}\psi\ \mathcal{D}\bar{\psi}\ \exp\{-\ <\ \bar{\psi}\ D(\alpha)\ \psi\ >\ \}. \tag{2.5}$$

If we equate $Z[\bar{\eta},\ \eta;\ \alpha]$ with $Z[\bar{\eta},\ \eta]$ the source term cancells and we recover the definition (1.4) of the chiral Jacobian proposed by GSMSS. This result is not surprising. We might have performed the shift (2.4) with $\alpha = 0$, before the chiral rotation. In this way we would have extracted the sources from the beginning, preventing them from getting any $\alpha$-dependence. The final result cannot depend on the order we perform these transformations. This simple argument suggets that there may be difficulties when D does not have an inverse and we cannot extract the sources before the chiral rotation. We are going to show that when zero eigenmodes are present a residual coupling between the sources and the fermionic fields remains. Thanks to this coupling the sources will get an $\alpha$-dependence which has to be taken into account in the definition of $J(\alpha)$.

Let us calculate expression (2.2) when D has $n_o$ zero eigenmodes. As we did before we perform the shift (2.4a,b,c), where now the sum in (2.4c) is over all eigenfunctions with non-zero eigenvalues. Since the completeness relation is now

$$\sum_{i=1}^{n_o} \varphi_{oi}(x)\, \varphi^{+}_{oi}(y) + \sum_{j} \varphi_j(x)\, \varphi^{+}_j(y) = 1\, \delta(x-y) \quad , \tag{2.6}$$

where 1 is the identify of the inner-space, $\varphi_{oi}(x)$ are the zero eigenvalues of D, and $S(x, y; \alpha)$ satisfies

$$D(x;\alpha)\, S(x,y;\alpha) = 1\, \delta(x-y) - e^{\alpha(x)\gamma_5} A_o(x,y)\, e^{-\alpha(y)\gamma_5} \quad , \tag{2.7}$$

where

$$A_o(x,y) = \sum_{i=1}^{n_o} \varphi_{oi}(x)\, \varphi^{+}_{oi}(y) \ , \tag{2.8}$$

is the projection operator on the sub-space of zero eigenvectors of D.

The generating functional (2.2) becomes

$$Z[\bar{\eta},\eta;\alpha] = J(\alpha)\, e^{\langle \bar{\eta}\, S\, \eta \rangle} \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp\Big\{ - \langle \bar{\psi}\, D(\alpha)\psi \rangle + $$

$$+ \langle \bar{\psi}\, e^{\alpha\gamma_5} A_o\, \eta \rangle + \langle \bar{\eta}\, A_o\, e^{\alpha\gamma_5}\, \psi \rangle \Big\} \quad , \tag{2.9}$$

which is the same result that we would have obtained if we had performed the shift, with $\alpha = 0$, before the chiral rotation.

We shall assume that D and hence $D(\alpha)$ have a complete orthonormal set of eigenfunctions $\{ \varphi_n(\alpha) \}$ and the fermionic variables can be expanded as[11]

$$\psi(x) = \sum_{i=1}^{n_o} a_{oi}(\alpha)\, \varphi_{oi}(x;\alpha) + \sum_j a_j(\alpha)\, \varphi_j(x;\,\alpha)\ , \qquad (2.10a)$$

$$\bar\psi(x) = \sum_{i=1}^{n_o} \bar a_{oi}(\alpha)\, \varphi^+_{oi}(x;\,\alpha) + \sum_j \bar a_j(\alpha)\, \varphi^+_j(x;\,\alpha)\ , \qquad (2.10b)$$

where $\varphi_n(x;\,\alpha)$ are the eigenfunctions of $D(\alpha)$ with eigenvalue $\lambda_j(\alpha)$, and $\varphi_{oi}(x;\,\alpha)$ are the zero eigenfunctions.

It is convenient to expand $\eta(x)$ and $\bar\eta(x)$ in terms of the non-rotated eigenfunctions

$$\eta(x) = \sum_{i=1}^{n_o} b_{oi}\, \varphi_{oi}(x) + \sum_j b_j\, \varphi_j(x)\ , \qquad (2.11a)$$

$$\bar\eta(x) = \sum_{i=1}^{n_o} \bar b_{oi}\, \varphi^+_{oi}(x) + \sum_j \bar b_j\, \varphi^+_j(x)\ . \qquad (2.11b)$$

As usual the path integral measure is defined as

$$\mathcal{D}\psi\, \mathcal{D}\bar\psi \equiv \prod_{i=1}^{n_o} da_{oi}(\alpha)\, d\bar a_{oi}(\alpha) \prod_j da_j(\alpha)\, d\bar a_j(\alpha)\ . \qquad (2.12)$$

Substituting equations (2.10a,b), (2.11a,b) and (2.12) into (2.9) we obtain

$$Z[\bar{\eta},\eta;\alpha] = J(\alpha)\ e^{\langle\bar{\eta}\ S\ \eta\rangle} \int \prod_{i=1}^{n_o} da_{oi}(\alpha)\ d\bar{a}_{oi}(\alpha) \prod_j da_j(\alpha)\ d\bar{a}_j(\alpha)\ *$$

$$\exp\Big\{-\sum_j \Big(\lambda_j(\alpha)\bar{a}_j(\alpha)a_j(\alpha) - \bar{a}_j(\alpha)\xi_j(\alpha) - \bar{\xi}_j(\alpha)a_j(\alpha)\Big)\ +$$

$$+\ \sum_{i=1}^{n_o}\Big[\ \bar{a}_{oi}(\alpha)\xi_{oi}(\alpha) + \bar{\xi}_{oi}(\alpha)a_{oi}(\alpha)\Big]\Big\}\ ; \qquad (2.13a)$$

with

$$\xi_j(\alpha) = \sum_{k=1}^{n_o} b_{ok} \int \varphi_j^+(x;\ \alpha)\ e^{\alpha(x)\gamma_5}\ \varphi_{ok}(x)\ dx \quad , \qquad (2.13b)$$

$$\bar{\xi}_j(\alpha) = \sum_{k=1}^{n_o} \bar{b}_{ok} \int \varphi_{ok}^+(x)\ e^{\alpha(x)\gamma_5}\ \varphi_j(x;\ \alpha)\ dx \quad , \qquad (2.13c)$$

$$\xi_{oj}(\alpha) = \sum_{k=1}^{n_o} b_{ok} \int \varphi_{oj}^+(x;\ \alpha)\ e^{\alpha(x)\gamma_5}\ \varphi_{ok}(x)\ dx \quad , \qquad (2.13d)$$

$$\bar{\xi}_{oj}(\alpha) = \sum_{k=1}^{n_o} \bar{b}_{ok} \int \varphi_{ok}^+(x)\ e^{\alpha(x)\gamma_5}\ \varphi_j(x;\ \alpha)\ dx \quad , \qquad (2.13e)$$

and, as before, the suppression of $\alpha$ means non-rotated quantities. The integrals over $a_{oi}$, $\bar{a}_{oi}$ are completely decoupled from the integrals over $a_j$, $\bar{a}_j$.

The integrals over $a_{oi}$ and $\bar{a}_{oi}$ are trivially done. Making the change of variables: $a_j' = a_j - \xi_j/\lambda_j(\alpha)$, $\bar{a}_j' = \bar{a}_j - \bar{\xi}_j/\lambda_j(\alpha)$, whose Jacobians are equal to the identity, the integrals over $a_j$ and $\bar{a}_j$ recover the gaussian form[12] and we finally get,

$$Z[\bar{\eta},\eta;\alpha] = J(\alpha)e^{\langle\bar{\eta}\ S\ \eta\rangle}\mathrm{Det}'(D(\alpha))\exp\left\{\sum_j \frac{\bar{\zeta}_j(\alpha)\ \zeta_j(\alpha)}{\lambda_j(\alpha)}\right\} \prod_{i=1}^{n_o} \zeta_{oi}(\alpha)\bar{\zeta}_{oi}(\alpha), \tag{2.14}$$

where $\mathrm{Det}'(D(\alpha))$ is equal to the product of all non-zero eigenvalues of $D(\alpha)$. However, $\mathrm{Det}'(D(\alpha))$ only makes sense if some regularization procedure is adopted. We shall define it as GSMSS.

In order to compare the generating functional before and after the chiral rotation it is necessary to express the rotated eigenfunctions in terms of the non-rotated ones. This is not difficult if we restrict ourselvés to infinitesimal chiral rotations and work to first order in $\alpha(x)$. This is sufficient to derive the anomalous Ward-Takahashi identities.

Although

$$e^{\alpha(x)\gamma_5}\ D\ e^{\alpha(x)\gamma_5}\ e^{-\alpha(x)\gamma_5}\ \varphi_{oi}(x) = D(\alpha)e^{-\alpha(x)\gamma_5}\ \varphi_{oi}(x) = 0\ , \tag{2.15}$$

we cannot identify $\exp(-\alpha(x)\gamma_5)\ \varphi_{oi}(x)$ with $\varphi_{oi}(x;\ \alpha)$. The reason is that the $\exp(-\alpha(x)\gamma_5)\ \varphi_{oi}(x)$ are not orthonormal. We have to take linear combinations of them,

$$\varphi_{oi}(x;\ \alpha) = \sum_{j=1}^{n_o} B_{ij}(\alpha)\ e^{-\alpha(x)\gamma_5}\ \varphi_{oi}(x)\ . \tag{2.16}$$

In the Appendix we use the Gram-Schmidt orthonormalization procedure to determine $B_{ij}(\alpha)$ to first order in $\alpha$.

$$B_{ii}(\alpha) = 1 + \int dx\, \varphi^{+}_{oi}(x)\, \alpha(x)\, \gamma_5\, \varphi_{oi}(x) \qquad (2.17a)$$

$$B_{ij}(\alpha) = 2 \int dx\, \varphi^{+}_{oj}(x)\, \alpha(x)\, \gamma_5\, \varphi_{oi}(x) \ , \quad j < i \qquad (2.17b)$$

and $B_{ij}(\alpha) = 0$ in the other cases. Notice that non-diagonal elements are at least of order $\alpha$.

Let's show that $\xi_j(\alpha)$ and $\bar{\xi}_j(\alpha)$ are order $\alpha$. Since $\alpha$ is infinitesimal

$$\varphi^{+}_{j}(x;\, \alpha) = \varphi^{+}_{j}(x) + \mathcal{O}(\alpha) \ , \qquad (2.18)$$

and we can rewrite the integrals in expression (2.13b) as

$$\int \varphi^{+}_{j}(x)\, \varphi_{oi}(x)\, dx + \mathcal{O}(\alpha) = \mathcal{O}(\alpha) \ , \qquad (2.19)$$

where we used the orthogonality of the eigenfunctions of D. An analogous demonstration shows that $\bar{\xi}_j(\alpha)$ is also $\mathcal{O}(\alpha)$. In addition

$$\lambda_j(\alpha) = \lambda_j + \mathcal{O}(\alpha) \ , \qquad (2.20)$$

hence

$$\sum_j \frac{\bar{\xi}_j(\alpha)\, \xi_j(\alpha)}{\lambda_j(\alpha)} = \mathcal{O}(\alpha^2) \qquad (2.21)$$

can be safely neglected.

The integrals which appear in (2.13d,e) can be easily evaluated using equation (2.16) and the orthogonality of the $\varphi_{oi}(x)$.

$$\xi_{oi}(\alpha) = \sum_{j=1}^{i} b_{oj}\, B^{*}_{ij}(\alpha) \ , \ \bar{\xi}_{oi}(\alpha) = \sum_{j=1}^{i} \bar{b}_{oj}\, B_{ij}(\alpha) \ , \qquad (2.22)$$

where the sums run from 1 to i because $B_{ij} = 0$ for $j > i$.

Since,

$$\bar{b}_{oi}^{\,2} = b_{oi}^{\,2} = 0,$$

then

$$\prod_i \xi_{oi}(\alpha)\, \bar{\xi}_{oi}(\alpha) = \prod_{i=1}^{n_o} b_{oi}\, \bar{b}_{oi}\, B_{ii}(\alpha)\, B^{*}_{ii}(\alpha) \ . \qquad (2.23)$$

Finally substituting (2.23) into (2.14) and using (2.17a) and (2.21) we obtain

$$Z[\bar{\eta},\eta;\alpha] = J(\alpha)\, e^{\langle\bar{\eta}\ S\ \eta\rangle} \mathrm{Det}'(D(\alpha))[1 + 2 \sum_{i=1}^{n_o} \langle\varphi^{+}_{oi}\alpha\gamma_5\varphi_{oi}\rangle]\prod_i b_{oi}\bar{b}_{oi} \ . \qquad (2.24)$$

$J(\alpha)$ is obtained by equating $Z[\bar{\eta},\eta;\alpha]$ with $Z[\bar{\eta},\eta]$. The expression of $Z[\bar{\eta},\eta]$ is obtained from (2.24) by putting $\alpha=0$. To first order in $\alpha$,

$$J(\alpha) = \frac{\mathrm{Det}'(D)}{\mathrm{Det}'(D(\alpha))} \left[1 - 2 \sum_{i=1}^{n_o} \langle\, \varphi^{+}_{oi}\alpha\gamma_5\varphi_{oi}\rangle\right] . \qquad (2.25)$$

To the same order equation (1.7) gives

$$J_{GSMSS}(\alpha) \equiv \frac{\mathrm{Det}'(D)}{\mathrm{Det}'(D(\alpha))} = 1 + 2 \left\langle \frac{\alpha}{48\pi^2}\mathrm{tr}\,{}^{*}F_{\mu\nu}F_{\mu\nu}\right\rangle + 2 \sum_{i=1}^{n_o} \langle\varphi^{+}_{oi}\alpha\gamma_5\varphi_{oi}\rangle \qquad (2.26)$$

Inserting (2.26) into (2.25) the contribution from the zero eigenmodes is cancelled and we obtain Fujikawa's result.

If $\alpha$ is constant it is easy to consider finite rotations which can be composed from infinitesimal ones by iteration

$$J(\alpha) = \lim_{N\to\infty} \left[1 - 2Q\,\frac{\alpha}{N}\right]^N = e^{-2Q\alpha} \quad , \tag{2.27}$$

where

$$Q = \langle -\frac{1}{48\pi^2}\,\mathrm{tr}\,{}^*F_{\mu\nu}F_{\mu\nu} \rangle \tag{2.28}$$

is the topological charge. For QCD $D(\alpha) = D = \gamma_\mu(i\partial_\mu + A_\mu)$ when $\alpha$ is constant, but not necessarily infinitesimal, and hence $\{\varphi_{io}(\alpha), \varphi_j(\alpha)\} = \{\varphi_{io}, \varphi_j\}$. Since $\gamma_5$ anti-commutes with D we can always choose our zero-eigenfunctions to have definite chirality,

$$\gamma_5\varphi_{oi} = +\varphi_{oi} \quad \text{or} \quad \gamma_5\varphi_{oi} = -\varphi_{oi} \; . \tag{2.29}$$

In this case expressions (2.13 b,c,d,e) give

$$\xi_j(\alpha) = \bar{\xi}_j(\alpha) = 0 \tag{2.30a}$$

and

$$\xi_{oi}(\alpha) = b_{oi}\, e^{(n_+ - n_-)\alpha} \quad ; \quad \bar{\xi}_{oi} = \bar{b}_{oi}\, e^{(n_+ - n_-)\alpha} \tag{2.30b}$$

Substituting (2.27), (2.30 a,b) into (2.14) and equating $Z[\bar{\eta},\eta;\alpha]$ with $Z[\bar{\eta},\eta]$ we obtain the Atiyah-Singer theorem

$$n_- - n_+ = Q. \tag{2.31}$$

In our derivation the sources play an important role. GSMSS and Fujikawa obtain (2.31) in a different way analising only the chiral Jacobian.

## 3. MASS REGULARIZATION

We can understand better how the zero eigenmodes get an $\alpha$-dependence if we rederive our results in a different way. In order to avoid non-invertible operators we can add a small mass to the Dirac operator from the beginning. This regularization, in principle, might be problematic since it does not respect the symmetries of the theory. For QCD, $D = \gamma_\mu(i\partial_\mu + A_\mu)$ and it is invariant under global chiral rotations. This invariance is lost if we substitute D for $D+\epsilon \equiv D_\epsilon$ . In spite of this, we are going to show that we recover Fujikawa's result when, at the end of the calculation, we put $\epsilon = 0$.

$$Z_\epsilon[\bar\eta,\eta] = \int \mathcal{D}\psi\ \mathcal{D}\bar\psi\ \exp\left\{- < \bar\psi\ D_\epsilon\psi > + < \bar\eta\ \psi > + < \bar\psi\ \eta >\right\} \quad . \tag{3.1}$$

After the chiral rotation $Z_\epsilon[\bar\eta,\eta]$ becomes

$$Z_\epsilon[\bar\eta,\eta;\alpha] = J_\epsilon(\alpha)\ \int \mathcal{D}\psi\ \mathcal{D}\bar\psi\ \exp\left\{- < \bar\psi\ D_\epsilon(\alpha)\psi > + < \bar\eta(\alpha)\psi > + < \bar\psi\eta(\alpha)>\right\}, \tag{3.2a}$$

with

$$D_\epsilon(\alpha) = e^{\alpha(x)\gamma_5}(D + \epsilon)e^{\alpha(x)\gamma_5} . \tag{3.2b}$$

The operators $D_\epsilon$ and $D_\epsilon(\alpha)$ can be inverted

$$D_\epsilon^{-1}(x,y;\alpha) = S_\epsilon(x,y;\alpha) = e^{-\alpha(x)\gamma_5} S_\epsilon(x,y) e^{-\alpha(x)\gamma_5} , \tag{3.3a}$$

where

$$S_\epsilon(x,y) = \frac{1}{\epsilon} \sum_{i=1}^{n_o} \varphi_{oi}(x)\, \varphi_{oi}^+(y) + \sum_j \frac{\varphi_j(x)\varphi_j^+(y)}{\lambda_j} . \tag{3.3b}$$

Shifting the integration variables as in equations (2.4a,b) with $S_\epsilon(x,y;\alpha)$ replacing $S(x,y;\alpha)$ we obtain

$$Z_\epsilon[\bar\eta,\eta;\alpha] = J_\epsilon(\alpha)\, e^{\langle\bar\eta\, S_\epsilon\, \eta\rangle} \mathrm{Det}'(D_\epsilon(\alpha))\epsilon^{n_o} , \tag{3.4}$$

where $\epsilon^{n_o}$ is the product of the shifted eigenvalues and we use the GSMSS prescription to regularize $\mathrm{Det}'(D_\epsilon(\alpha))$.

$$e^{\langle\bar\eta\, S_\epsilon\, \eta\rangle} = \frac{1}{\epsilon^{n_o}} \prod_{i=1}^{n_o} \bar b_{oi}\, b_{oi} \exp \langle\bar\eta \sum_j \frac{\varphi_j\varphi_j^+}{\lambda_j + \epsilon}\eta\rangle + O\left(\epsilon^{-n_o+1}\right) . \tag{3.5}$$

As in equation (2.11a,b) $\bar b_{oi}$ , $b_{oi}$ are the components of the expansion of $\bar\eta$ , $\eta$ in terms of the eigenfunctions of $D_\epsilon$ . Substituting (3.5) into (3.4) and neglecting $O\left(\epsilon^{-n_o+1}\right)$ terms, which will not contribute in the limit $\epsilon \to 0$, we obtain

$$Z_\epsilon[\bar\eta,\eta;\alpha] = J_\epsilon(\alpha) \prod_{i=1}^{n_o} \bar b_{oi}\, b_{oi} \exp\left[\langle\bar\eta \sum_j \frac{\varphi_j\varphi_j^+}{\lambda_j + \epsilon}\eta\rangle\right]\mathrm{Det}'(D_\epsilon(\alpha)) . \tag{3.6}$$

Equating $Z_\epsilon[\bar{\eta},\eta;\alpha]$ with $Z_\epsilon[\bar{\eta},\eta]$ and taking the limit $\epsilon\to 0^+$ we get

$$J(\alpha) = \lim_{\epsilon\to 0^+} \frac{\mathrm{Det}'(D_\epsilon)}{\mathrm{Det}'(D_\epsilon(\alpha))} \tag{3.7}$$

This expression was calculated by GSMSS in reference 8 and it is equal to Fujikawa's Jacobian.

## 4. DISCUSSION OF THE RESULTS AND CONCLUSIONS

Comparing the results of sections 2 and 3 it is easy to understand the $\alpha$-cancellation mechanism. After the chiral rotation we can absorb the $\exp(\alpha\,\gamma_5)$ term in the sources and in the operator D

$$\bar{\psi}\,\eta \longrightarrow \bar{\psi}\,e^{\alpha\gamma_5}\eta \equiv \bar{\psi}\eta(\alpha) \quad ; \quad \bar{\eta}\,\psi \longrightarrow \bar{\eta}\,e^{\alpha\gamma_5}\psi \equiv \bar{\eta}(\alpha)\psi \;;$$

$$\bar{\psi}D\psi \longrightarrow \bar{\psi}\,e^{\alpha\gamma_5}\,D\,\;e^{\alpha\gamma_5}\,\psi \equiv \bar{\psi}\,D(\alpha)\psi\;. \tag{4.1}$$

On the other hand, when the Dirac operator D has an inverse, or when we add to it a small mass

$$D^{-1} \longrightarrow \;e^{-\alpha\gamma_5}\,D^{-1}\,e^{-\alpha\gamma_5} \equiv D^{-1}(\alpha)\;. \tag{4.2}$$

Thus, if the sources couple to $D^{-1}$ the $\alpha$-dependence cancells. This is precisely what happens after we perform the fermionic integration

$$\int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp\left\{ - \langle \bar{\psi}\, D\, \psi \rangle + \langle \bar{\eta}\, \psi \rangle + \langle \bar{\psi}\, \eta \rangle \right\} = \exp\left\{ \langle \bar{\eta}\, D^{-1} \eta \rangle \right\} \mathrm{Det}(D). \tag{4.3}$$

However, when zero eigenvalues exist the components $\bar{b}_{oj}, b_{oj}$ of the sources $\bar{\eta}, \eta$ along the zero eigenfunctions $\varphi_{oj}$ decouple completely from D (see equation 2.13a). In this case a residual $\alpha$-dependence remains.

We have shown that, independently on how one treats the Dirac operator, it is possible to reconcile the $\zeta$-function calculation of the chiral Jacobian with Fujikawa's. This was done by defining the chiral Jacobian in such a way that the generating functional with sources is invariant under chiral rotations. This definition takes into account all sources of $\alpha$-contributions for the path integral and it seems to work in all cases we studied.

ACKNOWLEDGMENTS

The authors are grateful to R.Jackiw and F.Schaposnik for useful discussions. We would like to thank CNPq and FAPERJ for partial financial support. One of us (S.A.D) is also grateful to L.C.Botelho for pointing out some interesting aspects related to singular Dirac operators.

## APPENDIX : CALCULATION OF THE COEFFICIENTS $B_{ij}(\alpha)$ TO FIRST ORDER IN $\alpha$

Defining

$$\psi_{oi}(\alpha) = e^{-\alpha\gamma_5}\,\varphi_{oi} \quad , \qquad (A.1)$$

we can rewrite equation (2.16) as

$$\varphi_{oi}(\alpha) = \sum_{k=1}^{n_o} B_{ik}(\alpha)\,\psi_{ok}(\alpha) \; . \qquad (A.2)$$

To determine the $B_{ij}(\alpha)$ we use the Gram-Schmidt orthogonalization procedure:

$$H_1(\alpha) = \psi_{o1}(\alpha) \qquad (A.3a)$$

$$\vdots$$

$$H_i(\alpha) = \psi_{oi}(\alpha) - \sum_{k=1}^{i-1} \frac{\langle H_k^+(\alpha)\;\psi_{oi}(\alpha)\rangle}{\langle H_k^+(\alpha)\;H_k(\alpha)\rangle} H_k(\alpha) \; . \qquad (A.3b)$$

Notice that

$$\psi_{oi}(\alpha) = \varphi_{oi} + O(\alpha) \;\; , \quad H_i(\alpha) = \varphi_{oi} + O(\alpha) \; . \qquad (A.4)$$

On the other hand, in equation (A.3b) $k < i$ , thus

$$\langle H_k^+(\alpha)\;\psi_{oi}(\alpha)\rangle = O(\alpha) \qquad (A.5)$$

and (A.3b) becomes

$$H_i(\alpha) = \psi_{oi}(\alpha) - \sum_{k=1}^{i-1} < H_k^+(\alpha)\ \psi_{oi}(\alpha) > H_k(\alpha) + O(\alpha^2) \ . \qquad (A.6)$$

Iterating this equation we obtain

$$H_i(\alpha) = \psi_{oi}(\alpha) - \sum_{k=1}^{i-1} < \psi_{ok}^+(\alpha)\ \psi_{oi}(\alpha) > \psi_{ok}(\alpha) + O(\alpha^2) \ . \qquad (A.7)$$

$$< H_i^+(\alpha)\ H_i(\alpha) > = < \psi_{oi}^+(\alpha)\ \psi_{oi}(\alpha) > + O(\alpha^2) \ . \qquad (A.8)$$

Finally, normalizing the $H_i(\alpha)$ we obtain the $\varphi_{oi}(\alpha)$,

$$\varphi_{oi}(\alpha) = \frac{H_i(\alpha)}{\sqrt{<H_i(\alpha)H_i(\alpha)>}} = \frac{\psi_{oi}(\alpha)}{\sqrt{<\psi_{oi}^+(\alpha)\psi_{oi}(\alpha)>}} + $$

$$- \sum_{k=1}^{i-1} < \psi_{ok}^+(\alpha)\psi_{oi}(\alpha) > \psi_{ok} + O(\alpha^2) \ . \qquad (A.9)$$

From equation (A.9) we read the $B_{ij}(\alpha)$ :

$$B_{ii}(\alpha) = \frac{1}{\sqrt{<\psi_{oi}(\alpha)\psi_{oi}(\alpha)>}} = 1 + < \varphi_{oi}^+ \alpha\gamma_5 \varphi_{oi} > + O(\alpha^2) \ , \qquad (A.10a)$$

$$B_{ij}(\alpha) = - < \psi_{ok}^+(\alpha)\ \psi_{oi}(\alpha) > = 2 < \varphi_{ok}^+ \alpha\gamma_5 \varphi_{oi} > + O(\alpha^2) \ , \ i > j \ , \qquad (A.10b)$$

$$B_{ij}(\alpha) = 0 \quad , \quad i < j \ . \qquad (A.10c)$$

## REFERENCES

1. K.Fujikawa, Phys. Rev. Lett. 42, 1195(1979); Phys. Rev. D 21, 2848(1980); 22, 1499(1980);

2. R.E.Gamboa-Saraví, M.A.Muschietti and J.E.Solomin, Comm. Math. Phys. 89, 363(1983);

3. R.E.Gamboa-Saraví, M.A.Muschietti, F.A.Schaposnik and J.E.Solomin, Ann. Phys. 157, 360(1984);

4. R.T.Seeley, in Proceedings of the Symposium on Pure Mathematics (American Mathematical Society, Providence), (1967), vol. 10, p.288;

5. S.W.Hawking, Comm. Math. Phys. 55, 133(1977);

6. G.´t Hooft, Phys. Rev. Lett. 37, 8 (1976);

7. R.Jackiw and C.Rebbi, Phys. Rev. D16, 1052(1977);

8. R.E.Gamboa-Saraví, M.A.Muschietti and J.E.Solomin, Comm. Math. Phys. 93, 407(1984);

9. R.Rajaraman; Solitons and Instantons - An Introduction to Solitons and Instantons in Quantum Field Theory; North-Holland (1982);

10. M.Atiyah, I.Singer; Ann. Math. 87, 484(1968);

11. The chiral rotated operator $D(\alpha)$ has the same number of zero modes as the operator D. The number of eigenstates in each chirality is also preserved;

12. C.Itzykson, J-B. Zuber; Quantum Field Theory; Mc Graw-Hill (1980).