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CBPF–CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Notas de Física

QUENCHED BOND-MIXED CUBIC FERROMAGNET IN PLANAR
SELF-DUAL LATTICE: CRITICAL BEHAVIOR

by

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Abstract

The critical behavior of the quenched bond-mixed ferromagnetic cubic model, on a planar self-dual hierarchical lattice, is investigated within a simple real space renormalization group. We obtain the complete phase diagram of the system, exhibiting three phases. This phase diagram is nearly exact for the hierarchical lattice, and believed to be of high precision for the square lattice. The correlation-length critical exponents and the universality classes are determined as well.

KEY-WORDS: Cubic model, Criticality, Renormalization group, Hierarchical lattice.

1 Introduction

The study of the quenched bond(site)-diluted and bond(site)-mixed models is motivated by both theoretical interest and various possible experimental applications on disordered magnetic systems. Several works have been dedicated to these models, very particularly, to the Heisenberg model(see [1] and references therein), the Ising model(see [2] and references therein) and the cubic model[3].

The cubic model or discrete N-vector model[3-7] has been conveniently applied to describe phase transitions in rare-earth compounds[7], molecular oxygen adsorbed on graphite[8], order-disorder transition in atomic oxygen on tungsten[8], among many other applications(see [3] and references therein). The dimensionless Hamiltonian associated with the cubic model is given by

$$\beta H = -NK \sum_{<ij>} \vec{S}_i.\vec{S}_j \tag{1}$$

where $\beta \equiv 1/k_B T$, (ij) runs over all the couples of first-neighboring sites, the spin $\vec{S}_i$ at any given site is an N-component vector which points along the edges of an N-dimensional hypercube, i.e., $\vec{S}_i = (\pm 1, 0, 0, ..., 0)$, $(0, \pm 1, 0, ..., 0)$, ..., $(0, 0, 0, ..., \pm 1)$. As we shall see later on, the model is closed, under the renormalization group(RG) transformation, if a quadrupolar interaction is included into the Hamiltonian, i.e.,

$$\beta H = -NK \sum_{<ij>} \vec{S}_i.\vec{S}_j - N^2 L \sum_{<ij>} (\vec{S}_i.\vec{S}_j)^2. \tag{2}$$

This Hamiltonian is an interesting one since it contains, as limiting cases, various important statistical models, namely, the Ising model(N=1), the Z(4) model(N=2), the 2N-state Potts model(NL=K), the N-state Potts model(K=0) and the grand-canonical statistics of the self-avoiding walk $(N \rightarrow 0)$[9].

The purpose of the present paper is to study, for the first time as far as we know, the criticality of the quenched bond-mixed ferromagnetic cubic model on the square lattice which we approximate here by a self-dual hierarchical lattice. The RG formalism is presented in Section 2, our results in Section 3 and our conclusions in Section 4.

2 Quenched Bond-mixed Model and RG Formalism

The quenched bond-mixed model is defined by Hamiltonian (2) where the following probability law is assumed for the coupling constants K and L :

$$P(K,L) = (1-p)\delta(K-K_1)\delta(L-L_1) + p\delta(K-K_2)\delta(L-L_2) \quad (3)$$

where $0 \leq p \leq 1$. By imposing the ground state to be ferromagnetic we obtain that $K_1 > 0$, $K_2 > 0$, $K_1 + NL_1 > 0$ and $K_2 + NL_2 > 0$.

Our treatment consists in constructing a real space RG which associates the probability distribution given by (3) with each bond of the self-dual Wheatstone-bridge(fig. 1), a graph which guarantees an excellent approximation for the square lattice[10]. Let us introduce for convenience the transmissivity variables [6,11,16−19]

$$t_1 = \frac{1 - \exp(-2NK)}{1 + 2(N-1)\exp[-N(K+NL)] + \exp(-2NK)} \quad \epsilon\,[0,1] \quad (4)$$

$$t_2 = \frac{1 - 2\exp[-N(K+NL)] + \exp(-2NK)}{1 + 2(N-1)\exp[-N(K+NL)] + \exp(-2NK)} \quad \epsilon\,[0,1] \quad (5)$$

hence

$$\exp[-N(K+NL)] = \frac{1 - t_2}{1 + Nt_1 + (N-1)t_2} \quad (6)$$

$$\exp(-2NK) = \frac{1 - Nt_1 + (N-1)t_2}{1 + Nt_1 + (N-1)t_2}. \quad (7)$$

The transmissivity (t_1, t_2) generalizes the scalar one introduced for the Ising[13], Potts[11] and Z(4)[12] models. The equivalent transmissivity

$(t_1^{(s)}, t_2^{(s)})((t_1^{(p)}, t_2^{(p)}))$ of a series(parallel) array of two bonds with transmissivities $(t_1^{(1)}, t_2^{(1)})$ and $(t_1^{(2)}, t_2^{(2)})$ is given by

$$t_r^{(s)} = t_r^{(1)} t_r^{(2)} \qquad (r = 1, 2) \qquad (8)$$

$$(t_r^{(p)})^D = (t_r^{(1)})^D (t_r^{(2)})^D \qquad (r = 1, 2) \qquad (9)$$

with

$$(t_1^{(q)})^D = \frac{1 - N t_1^{(q)} + (N-1) t_2^{(q)}}{1 + N t_1^{(q)} + (N-1) t_2^{(q)}} \qquad (q = 1, 2, p) \qquad (10)$$

$$(t_2^{(q)})^D = \frac{1 - t_2^{(q)}}{1 + N t_1^{(q)} + (N-1) t_2^{(q)}} \qquad (q = 1, 2, p) \qquad (11)$$

where D stands for dual(see [14] and references therein).

The series and parallel algorithms enable the calculation of the equivalent transmissivity of any(two-terminal) graph which is reducible in series/parallel operations. For those graphs(e.g., the Wheatstone-bridge array of fig. 1) which are not reducible, the Break-collapse method[15,16] can be used. We have applied this method to fig. 1 and have obtained the following equivalent transmissivity $(t_1^{(W)}, t_2^{(W)})$ where

$$t_1^{(W)} = \frac{N_1^{(W)}}{D^{(W)}} \qquad (12)$$

$$t_2^{(W)} = \frac{N_2^{(W)}}{D^{(W)}} \qquad (13)$$

with

$$
\begin{aligned}
N_1^{(W)} = {} & y_1 w_1 + x_1 z_1 + u_1 y_1 z_1 + u_1 w_1 x_1 + (N-1)^2 (u_1 x_2 y_1 z_1 w_2 + u_1 x_1 y_2 z_2 w_1) \\
& + (N-1)(x_2 y_1 z_2 w_1 + x_1 z_1 y_2 w_2 + u_1 y_1 z_1 w_2 + u_1 x_1 z_2 w_1 \\
& + u_1 w_1 x_1 y_2 + u_1 z_1 x_2 y_1 + u_2 w_2 z_1 x_1 + u_2 x_1 y_2 z_1 + u_2 w_1 z_2 y_1 + u_2 w_1 x_2 y_1) \\
& + (N-1)(N-2)(u_2 w_2 x_1 y_2 z_1 + x_2 y_1 z_2 w_1 u_2)
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
N_2^{(W)} =\ & y_2w_2 + x_2z_2 + u_2y_2z_2 + u_2w_2x_2 + N(N-1)x_1y_1z_1w_1u_2 \\
& + (N-2)(N-3)x_2y_2z_2w_2u_2 + N(N-2)(x_1y_1z_2w_2u_1 + x_2y_2z_1w_1u_1) \\
& + (N-2)(x_2y_2z_2w_2 + x_2y_2z_2u_2 + y_2z_2w_2u_2 + x_2z_2w_2u_2 + x_2y_2w_2u_2) \\
& + N(x_1y_1z_1w_1 + y_2z_1w_1u_1 + x_2z_1w_1u_1 + x_1y_1w_2u_1 + x_1y_1z_2u_1) \quad (15)
\end{aligned}
$$

$$
\begin{aligned}
D^{(W)} =\ & 1 + N(u_1z_1w_1 + u_1y_1x_1) + (N-1)(u_2y_2x_2 + u_2w_2z_2) + Nx_1z_1w_1y_1 \\
& + (N-1)x_2y_2z_2w_2 + N(N-1)u_2y_1z_1w_1x_1 \\
& + (N-1)(N-2)x_2z_2w_2u_2y_2 \\
& + N(N-1)(u_1x_1y_1z_2w_2 + u_1x_2y_2z_1w_1). \quad (16)
\end{aligned}
$$

Let us now focus the bond-mixed problem. Distribution (3) can be equivalently rewritten as follows:

$$
P(t_1, t_2) = (1-p)\delta(t_1 - t_1^{(1)})\delta(t_2 - t_2^{(1)}) + p\delta(t_1 - t_1^{(2)})\delta(t_2 - t_2^{(2)}) \quad (17)
$$

where $(t_1^{(1)}, t_2^{(1)})$ are related to (K_1, L_1) through eqs. (4) and (5) (which also provide the relationship between $(t_1^{(2)}, t_2^{(2)})$ and (K_2, L_2)). If we associate now distribution (17) with each bond of the Wheatstone-bridge array of fig. 1, we obtain for the equivalent distribution $P_W(t_1, t_2)$ the following expression:

$$
P_W(t_1, t_2) = \sum_{i=1}^{14} F_i (1-p)^{n_i} p^{5-n_i} \delta(t_1 - t_1^{(i)})\delta(t_2 - t_2^{(i)}) \quad (18)
$$

where F_i and n_i are respectively the weights and exponents associated with the possible bond-configurations in the (self-dual) Wheatstone-bridge. The $\{F_i\}$ satisfy $\sum_{i=1}^{14} F_i = 2^5$, and the $\{(t_1^{(i)}, t_2^{(i)})\}$ are straightforwardly obtained

from eqs. (12-16). The present scaling operation does not preserve the original binary form since P_W has 14 terms. At this level, we shall introduce an approximation, namely to approach P_W by the following binary distribution:

$$P^{'}(t_1, t_2) = (1 - p^{'})\delta(t_1 - t_1^{(1)'})\delta(t_2 - t_2^{(1)'}) + p^{'}\delta(t_1 - t_1^{(2)'})\delta(t_2 - t_2^{(2)'}) \quad (19)$$

where $(p^{'}, t_1^{(1)'}, t_2^{(1)'}, t_1^{(2)'}, t_2^{(2)'})$ are functions of $(p, t_1^{(1)}, t_2^{(1)}, t_1^{(2)}, t_2^{(2)})$ to be determined. To do this we impose

$$< t_1 >_{P'} = < t_1 >_{P_W} \quad (20)$$

$$< t_2 >_{P'} = < t_2 >_{P_W} \quad (21)$$

$$< t_1 t_2 >_{P'} = < t_1 t_2 >_{P_W} \quad (22)$$

$$< (t_1)^2 t_2 >_{P'} = < (t_1)^2 t_2 >_{P_W} \quad (23)$$

$$< t_1 (t_2)^2 >_{P'} = < t_1 (t_2)^2 >_{P_W} \quad (24)$$

where $< ... >$ denotes the standard mean value (for example, $< (t_1)^n >_{P'} = (1 - p^{'})(t_1^{(1)})^n + p^{'}(t_1^{(2)})^n$). This type of approximation(in which the first relevant momenta are preserved) has been successfully used for various problems[2,4,19] which are recovered herein as particular cases.

The set of equations (20-24) provides $(p^{'}, t_1^{(1)'}, t_2^{(1)'}, t_1^{(2)'}, t_2^{(2)'})$ as explicit functions of $(p, t_1^{(1)}, t_2^{(1)}, t_1^{(2)}, t_2^{(2)})$. Its iteration yields the RG flow in the $(p, t_1^{(1)}, t_2^{(1)}, t_1^{(2)}, t_2^{(2)})$ space(or equivalently in the (p, K_1, L_1, K_2, L_2) space), and consequently the phase diagram. The thermal critical exponents are obtained through the calculation of the relevant eigenvalues λ_i ($\lambda_i > 1$) of the Jacobian $\partial(p^{'}, t_1^{(1)'}, t_2^{(1)'}, t_1^{(2)'}, t_2^{(2)'})/\partial(p, t_1^{(1)}, t_2^{(1)}, t_1^{(2)}, t_2^{(2)})$ associated with

the unstable fixed points. More precisely, the correlation-length critical exponent ν_i is given by $\nu_i = ln(b)/ln(\lambda_i)$, where b is the RG linear expansion factor(b=2 for fig. 1). The crossover critical exponents Φ_{ij} are defined whenever more than one relevant eigenvalue exist, and are given by $\Phi_{ij} = ln(\lambda_j)/ln(\lambda_i)$.

3   Results

Within the present RG we verify the existence of three phases, namely the paramagnetic(P), ferromagnetic(F) and intermediate(I) ones. These phases were already present in the pure[6] and diluted[4] cases and are characterized by

$$\langle \vec{S}_i \rangle \equiv \langle (S_i^1, S_i^2, ..., S_i^N) \rangle = 0$$

and

$$\langle (S_i^\alpha)^2 \rangle - 1/N = 0 \quad \forall\, \alpha \,\epsilon\, \{1, 2, ..., N\} \quad (P),$$

$$\langle \vec{S}_i \rangle = 0$$

and

$$\langle (S_i^\alpha)^2 \rangle - 1/N > 0 \quad \text{for } \alpha = \alpha_0 \,\epsilon\, \{1, 2, ..., N\}$$
$$< 0 \quad \text{for } \alpha \neq \alpha_0 \qquad (I),$$

$$\langle \vec{S}_i \rangle \neq 0$$

and

$$\langle (S_i^\alpha)^2 \rangle - 1/N > 0 \quad \text{for } \alpha = \alpha_0 \,\epsilon\, \{1, 2, ..., N\}$$
$$< 0 \quad \text{for } \alpha \neq \alpha_0 \qquad (F).$$

These phases are associated, in the $(p, t_1^{(1)}, t_2^{(1)}, t_1^{(2)}, t_2^{(2)})$ space, with the following fully stable fixed points:

(0,0,0,0,0) and (1,0,0,0,0)   (P)

(1,1,1,1,1) and (0,1,1,1,1)   (F)

(1,0,1,0,1) and (0,0,1,0,1)   (I) .

Notice that, in the (I)-phase, the system has chosen one of the axes, but not a sense within that axis.

The model we are considering here contains, in many different ways, the Ising, the Potts and the bond percolation models as particular cases. In all of them, the square lattice exact critical points are recovered within the present RG. Furthermore, various slopes are almost exactly reproduced(see table I). We have consequently good confidence that the RG critical surfaces we obtain here can be considered as a high precision approximation of the corresponding ones in the square lattice. This satisfactory fact is clearly related to the self-duality of the Wheatstone-bridge array we have used, thus preserving the self-duality of the square lattice. The various critical exponents we have obtained (see table I) are exact for the hierarchical lattice but clearly not for the square lattice.

The subspace (p, t_1, t_1, t_2, t_2) is closed under RG; it corresponds to the particular case NL=K (2N-state Potts model with dimensionless coupling constant 2NK). For the case N=1 (Ising model) the present approach reproduces the solution obtained in [2]. The phase diagram associated with N=2 is presented in figure 2. The points marked A in this figure are fully unstable fixed ones and correspond to the bond percolation limit. The axes CBC and C'CC' correspond to the pure 2N-state Potts model. The point B is unstable out of the critical surface; within this surface it is fully stable if $N \leq N^* \approx 2.6$, and semi-stable if $N > N^*$; indeed, at $N = N^*$,

new fixed points appear through a bifurcation[20].

In fig. 3 we show typical cuts of the phase diagram in the 5-dimensional full space. In figs. 3a and 3b we present cuts for N=2. These diagrams share with the pure case(p=1) the presence of three phases. However, when the value of p decreases, the intermediate phase region shrinks. Fig 3c corresponds to the N=3 diluted case, and we verify that for $p < p_c = 0.5$ (critical percolation probability) the ferromagnetic phase disappears. The N=1 case is shown in fig 3d; let us stress that the P-I critical line must be considered as a mathematical artifact[6,15].

The behavior for arbitrary values of N has been analysed and the corresponding diagrams present the same characteristics mentioned above.

4 Conclusion

In this paper we have studied, within a real-space renormalization group method, the criticality of the quenched bond-mixed discrete ferromagnetic cubic model (N-vector model) in a square lattice, herein approached by a self-dual hierarchical lattice. The 5 parameters of the present problem ensure considerable freedom for renormalization, thus leading to results whose precision is quite higher than that obtained for the diluted model[4].

The present treatment provides an efficient method with few mathematical requirements. In fact, the present procedure is practically as simple as a Mean Field approximation, providing nevertheless quite superior results. As a final remark, it is worthy to stress that the criticality of the present model can be understood in terms of competitions between

the Ising, N-state and 2N-state Potts, cubic and percolation models.

Acknowledgements

Caption for figures and tables

Figure 1: Iteration associated with the Wheatstone-bridge hierarquical lattice(the full and open circles respectively denote the internal and terminal sites of the graph).

Figure 2: N=2 phase diagram and RG flow in the (p, t_1, t_1, t_2, t_2) subspace (2N-state Potts model). $\bigcirc, \bullet, \blacksquare$ denote, respectively, the fully stable, fully unstable and semi-stable fixed points. The paramagnetic(P) and ferromagnetic(F) phases are indicated. The critical surface is invariant under the transformation $(p, t_1, t_1, t_2, t_2) \rightarrow (1 - p, t_2, t_2, t_1, t_1)$. The line ABA lies the plane p=1/2; the line CBC correspond to $t_1 = t_2 = 1/(\sqrt{2N} + 1)$.

Figure 3:Typical cuts of the full 5-dimensional space $(p, 1/K_2, L_2/K_2, K_1/K_2, L_1/K_1)$: (a) $N = 2, K_1 = 2L_1, K_1/K_2 = 1$ (b) $N = 2, K_1 = 2L_1, K_1/K_2 = 0.8$ (c) $N = 3, K_1 = L_1 = 0$. (d) $N = 1, K_1 = L_1, K_1/K_2 = 0.8$. (P),(F) and (I) respectively refer to the paramagnetic, ferromagnetic and intermediate phases.

Table I: RG values for some fixed points(exact results), correlation-length and crossover exponents and limiting slopes. Whenever available, exponents and slopes are compared with the square lattice exact results. ν_t and ν_p are the correlation-length critical exponents; Φ_{tp} is the crossover critical exponent.
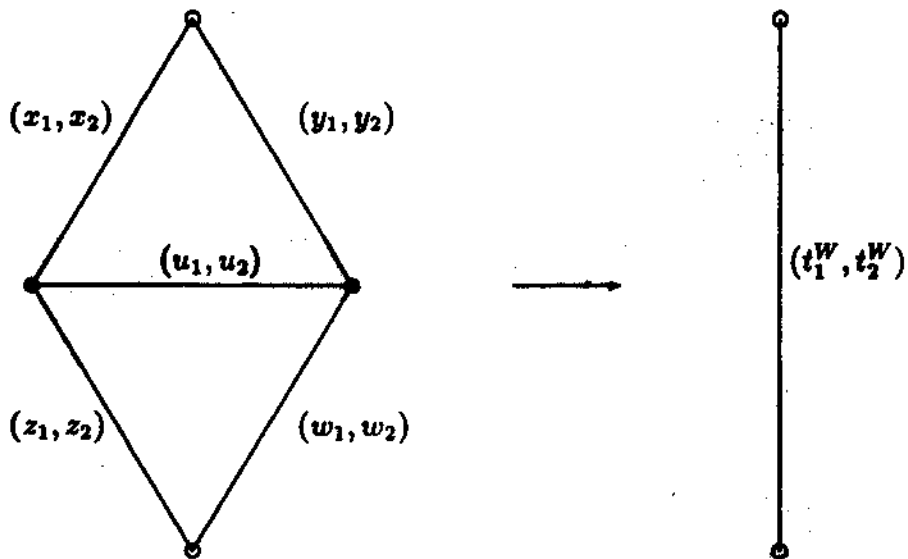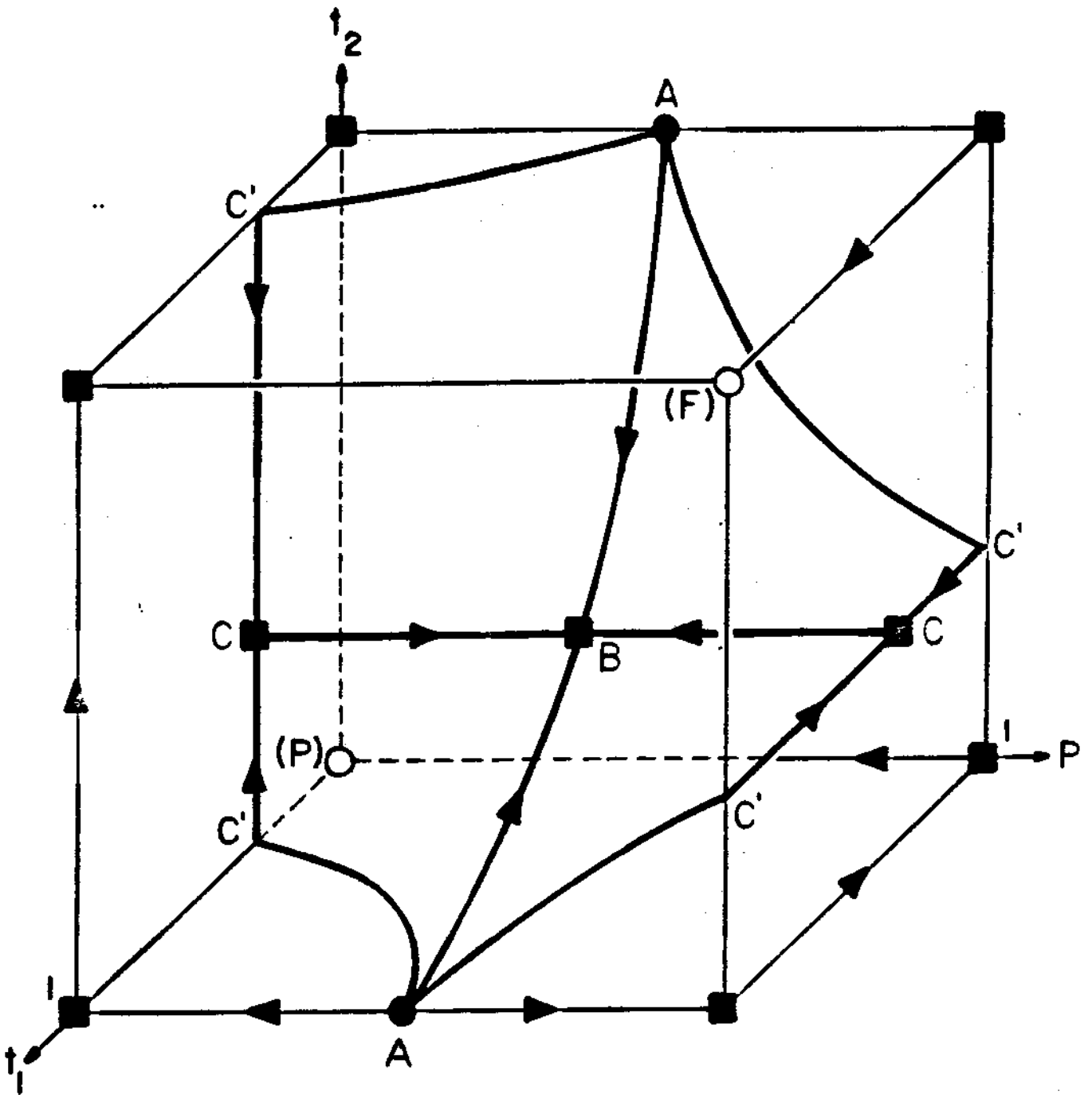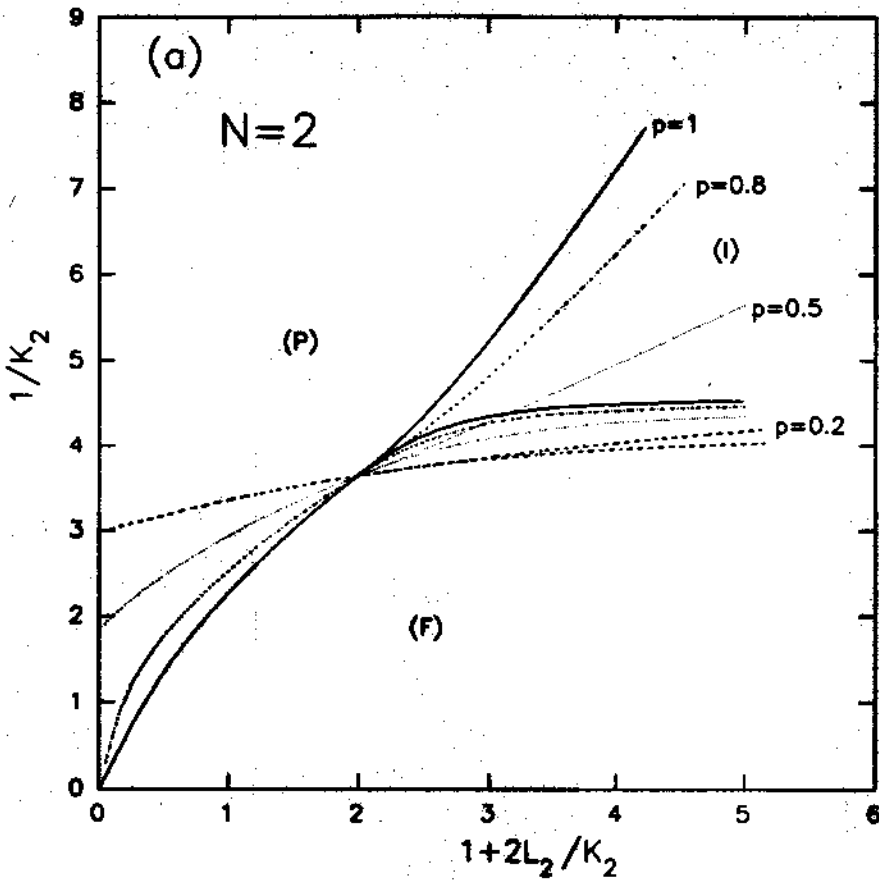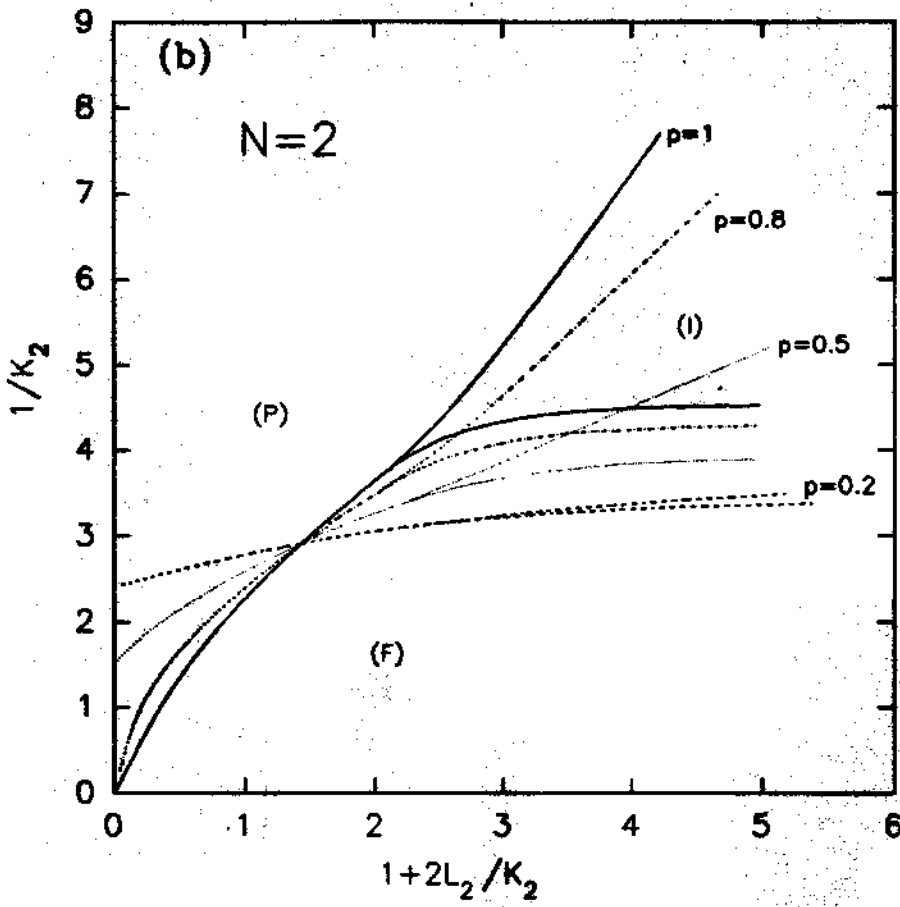
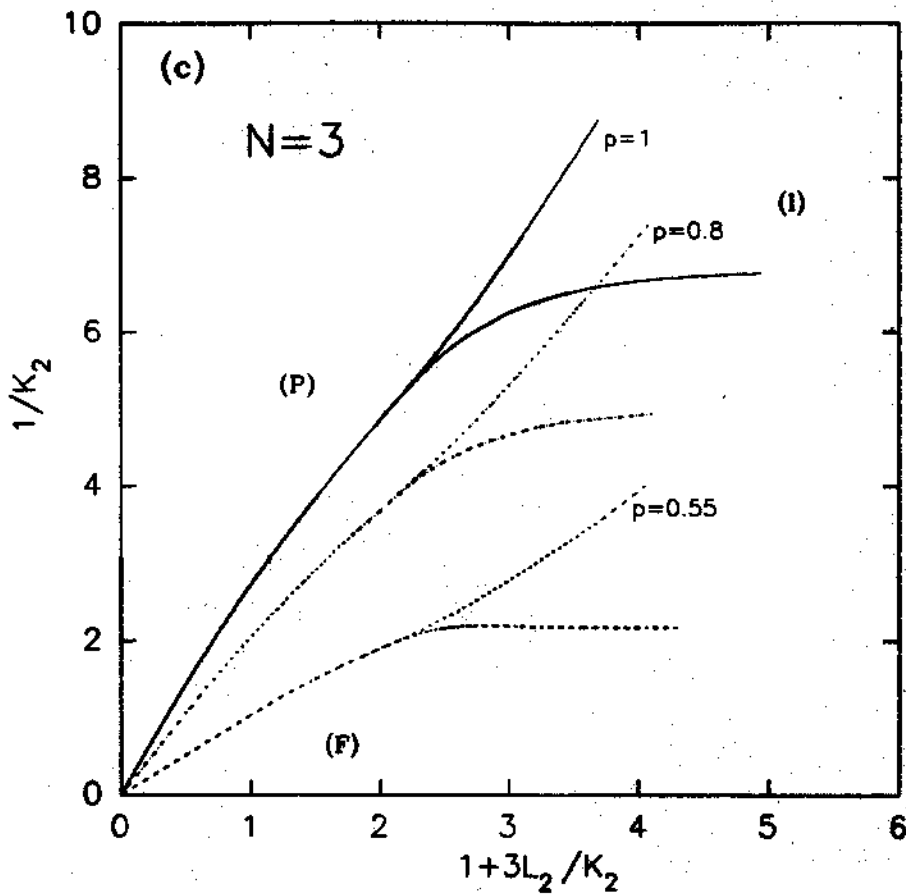figure 1

figure 2

figure 3

figure 3

figure 3

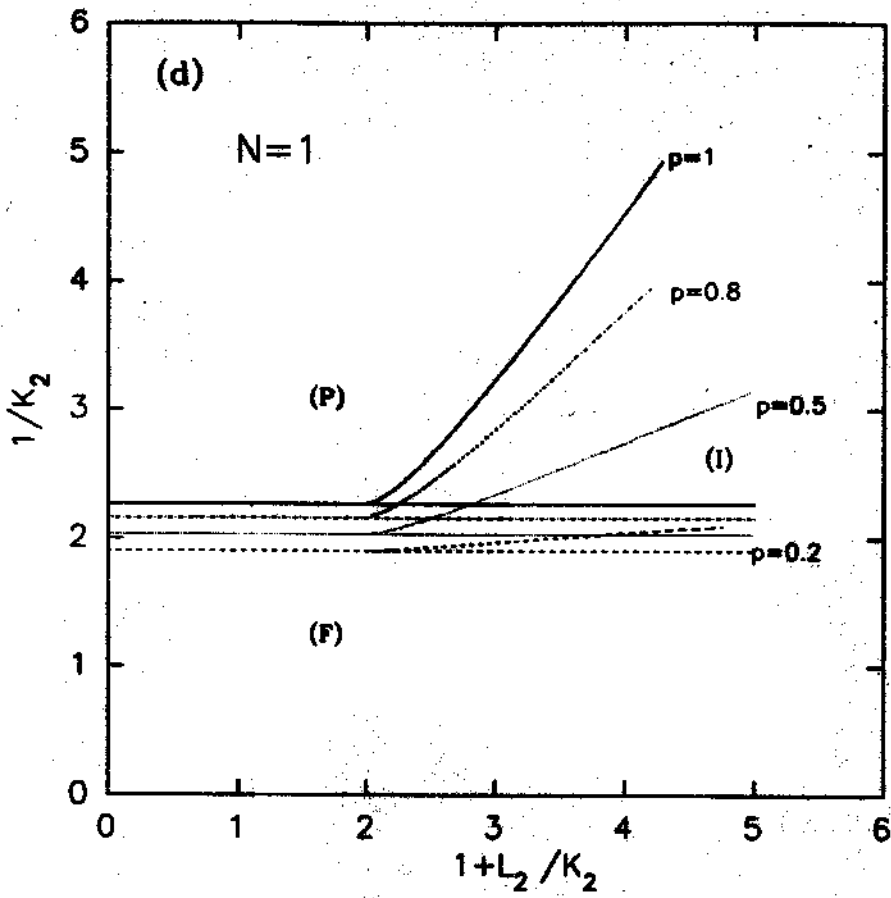figure 3

Table I

	critical points $(p, t_1^{(1)}, t_2^{(1)}, t_1^{(2)}, t_2^{(2)})$	critical exponents present RG	square lattice(exact)	slopes present RG	square lattice(exact)
N=1	(1,0.414,0.414,0.414,0.414)	$\nu_t \approx 1.15$	1 [22]	$-dt_1/dp \approx 0.45$	$6\sqrt{2}-8 \approx 0.48$ [21]
		$\Phi_{tp} \approx 1.56$	?		
	(1,0.414,0.414,0.414,1)	$\nu_t = 1.15$	1 [22]	$-dt_1/dp=0.45$	$6\sqrt{2}-8 \approx 0.48$ [21]
	(1,0.414,0.414,0.414,0.036)	$\nu_t = 1.15$	1 [22]	$-dt_1/dp \approx 0.45$	$6\sqrt{2}-8 \approx 0.48$ [21]
N=2	(1,0.414,0.414,0.333,0.333)	$\nu_t = 0.95$	2/3 [23]	$-dt_1/dp=0.44$	$4/9 \approx 0.44$ [27]
		$\Phi_{tp} = 2.72$	?		
	(1,0.414,0.414,0.414,0.172)	$\nu_t = 1.15$	1 [22]	$-dt_1/dp=0.55$	1/2
				$-dt_2/dt_2=0.5$	1/2 [25]
N=3	(1,0.34,0.34,0.34,0.21)	$\nu_t = 0.92$	1st order ?		
∀ N	(0.5,0,0,1,1)	$\nu_t = 1.43$	4/3 [24]	$-dt_1/dp=3.04$	$4\ln 2 \approx 2.77$ [26]
		$\nu_p = 1.43$	4/3 [24]		
		$\Phi_{tp} = 1$	1 [24]		

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