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The Langevin and Fokker-Planck Equations in the Framework of a Generalized Statistical Mechanics

by

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Abstract. We introduce generalized Langevin and Fokker-Planck equations that have as stationary solution distribution introduced by Tsallis in the framework of a generalized statistical mechanics. We show that the Langevin and Fokker-Planck equations preserve the classic form provided we define a suitable generalized potential. As a by-product we obtain one of the famous Einstein relations for the Brownian motion.

Key-words: Langevin equation; Fokker-Planck equation; Generalized statistical mechanics; Tsallis entropy.

Based on multifractal concepts Tsallis [1] generalized the classic information theoretic definition of entropy [2]:

$$S = -k \sum_{i=1}^{W} p_i \ln p_i \tag{1}$$

where k is Boltzmann's constant and the p_i's are the probabilities of W microscopic states of the system under study ($\sum_{i=1}^{W} p_i = 1$). The generalized entropy reads:

$$S_q = \frac{k}{q-1} \sum_{i=1}^{W} p_i (1 - p_i^{q-1}) \tag{2}$$

with q being any real number. The limit $q \to 1$ recovers Shannon's form. For general values of q, S_q possesses the usual properties of positivity, equiprobability, irreversibility and concavity and generalizes the standard additivity: if p_i^A and p_i^B are the probabilities associated with two *independent* systems A and B then

$$S_q(p_{ij}^{A \cup B}) = S_q(p_i^A) + S_q(p_j^B) + (1-q) S_q(p_i^A) S_q(p_j^B) \tag{3}$$

S_q is additive only in the $q = 1$ case and we could think of $1 - q$ as a parameter that measures the lack of additivity (or extensivity) of the system. Actually the lack of additivity is a property of many more systems than one a priori would suspect. Problems ranging from stellar structure [3](a) to statistical inference in economics [4] share this characteristic. By means of suitable variational conditions on the generalized entropy, Tsallis [1] put the basis of a generalized statistical mechanics. Subsequently much work has been done by several authors accomplishing successful generalizations of equilibrium thermodynamics [5], the Boltzmann H-theorem [6] [7](a,b), the Schröedinger equation [8], the Ehrenfest theorem [3](b), the von Neumann equation [3](c), the variational method in statistical mechanics [9] and consistent testing in econometrics [4].

In the present work we pose the question of *whether it exists a subjacent stochastic dynamics in phase space that assymptotically, i.e. for long times, produces the equilibrium generalized canonical distribution introduced by Tsallis* [1]:

$$p_i = \frac{[1-\beta(1-q)V_i]^{\frac{1}{1-q}}}{Z_q} \tag{4}$$

with

$$Z_q = \sum_{i=1}^{W} [1-\beta(1-q)V_i]^{\frac{1}{1-q}} \tag{5}$$

where $\beta = 1/T$ is the inverse temperature, V the potential, Z_q the partition function and q a real parameter that characterizes a particular statistics.

As is well known the Langevin and Fokker-Planck equations that describe the Brownian motion play that role in classical statistical mechanics [10]. The simplest form of the Langevin equation for a particle moving in a potential V and subject to an additional random force $b(t)$ reads:

$$\frac{dX_i}{dt} = -\frac{1}{\eta}\frac{\partial V}{\partial X_i} + b_i(t) \tag{6}$$

where η is the viscosity and $b_i(t)$ is a gaussian distributed random variable with zero mean and variance

$$\langle b_i(t_1) b_j(t_2)\rangle = 2A\delta_{i,j}\delta(t_1 - t_2) \tag{7}$$

The associated Fokker-Planck equation that describes the temporal evolution of the probability distribution of X_i, $P(X_i, t)$, has the form:

$$\frac{\partial P}{\partial t} = \sum_i \left\{ \frac{\partial}{\partial X_i}\frac{1}{\eta}\left(\frac{\partial V}{\partial X_i}P\right) + A\frac{\partial^2 P}{\partial X_i^2}\right\} \tag{8}$$

It can be demonstrated that, under very general conditions on the potential V, this

Fokker-Planck equation has a stationary solution that corresponds to the canonical Boltzmann-Gibbs distribution $P \propto e^{-\beta V}$ [11].

Now we will look for a suitable generalization of eqs.(6) and (8) that will generate the generalized canonical distribution (4). Let us introduce a "generalized potential":

$$\overline{V} = \frac{1}{\beta(q-1)} \ln\left[1 + \beta(q-1)V\right] \tag{9}$$

We can write the Langevin and Fokker-Planck equations in terms of $\overline{V}$ *preserving the form of the equations*:

$$\begin{aligned} \frac{dX_i}{dt} &= -\frac{1}{\eta}\frac{\partial \overline{V}}{\partial X_i} + b_i(t) && (10) \\ \frac{\partial P}{\partial t} &= \sum_i \left\{ \frac{\partial}{\partial X_i}\frac{1}{\eta}\left(\frac{\partial \overline{V}}{\partial X_i} P\right) + A\frac{\partial^2 P}{\partial X_i^2} \right\} && (11) \end{aligned}$$

It is straitforward to show that the generalized canonical distribution (4) is a stationary solution ($\frac{\delta P}{\delta t} = 0$) of the generalized Fokker-Planck equation:

$$0 = \sum_i \frac{\partial}{\partial X_i}\left[\left(\frac{1}{\eta}\frac{\partial \overline{V}}{\partial X_i} + A\frac{\partial}{\partial X_i}\right) P\right] \tag{12}$$

If $P \propto [1-\beta(1-q)V]^{\frac{1}{1-q}}$ then:

$$\begin{aligned} \frac{\partial P}{\partial X_i} &= -\frac{\beta P}{[1-\beta(1-q)V]}\frac{\partial V}{\partial X_i} \\ &= \frac{P}{(1-q)}\frac{\partial}{\partial X_i}\left\{\ln\left[1-\beta(1-q)V\right]\right\} && (13) \\ &= -\beta P\frac{\partial \overline{V}}{\partial X_i} \end{aligned}$$

Now

$$0 = \sum_i \frac{\partial}{\partial X_i}\left[\left(\frac{1}{\eta} - A\beta\right)\frac{\partial \overline{V}}{\partial X_i} P\right] \tag{14}$$

This last equality is satisfied if $\eta A\beta = 1$. This is one of the celebrated Einstein relations for the Brownian motion that is *preserved invariant in the present generalization*. It is interesting to note that the generalized canonical distribution can be written as a standard canonical distribution in terms of $\overline{V}$:

$$
\begin{aligned}
P \propto [1 - \beta(1-q)V]^{\frac{1}{1-q}} &= e^{-\beta \frac{1}{\beta(q-1)} \ln[1-\beta(1-q)V]} \\
&= e^{-\beta \overline{V}}
\end{aligned}
\tag{15}
$$

and the particular functional form of the generalized potential $\overline{V}$ is the one that recovers additivity in the present formalism.

Summarizing we have shown that the generalized canonical distribution (4) can be obtained as the assymptotic equilibrium distribution of suitable Langevin and Fokker-Planck equations by means of a properly defined generalized potential. The Langevin and Fokker-Planck equations preserve the standard form and as a consequence one can obtain a generalized Einstein relation that is also preserved invariant.

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References

[1] C.Tsallis, J.Stat.Phys. **52**, 479 (1988).

[2] C.E.Shannon, Bell Sys. Tech. Journ. **27**, 379-243, 623-656 (1948).

[3] A.R.Plastino and A.Plastino,

(a) Phys.Lett.**A174**, 384 (1993)

(b) Phys.Lett.**A177**, 177 (1993)

(c) preprint (1993).

[4] C.Tsallis, preprint (1993).

[5] E.M.F.Curado and C.Tsallis. J.Phys.**A24**,L69, (1991), (errata: J.Phys.**A24**, 3187 (1991) and **A25**, 1019 (1992)).

[6] A.M.Mariz, Phys. Lett. **A165**, 409 (1992).

[7] J.D.Ramshaw,

(a) Phys. Lett. **A175**, 169 (1993)

(b) Phys. Lett. **A175**, 171 (1993)

[8] R.N.Silver and C.Tsallis. preprint (1993).

[9] A.Plastino and C.Tsallis, preprint (1993).

[10] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*. North-Holland, 1981.

[11] G.Parisi, *Statistical Field Theory*, Addison-Wesley Pub. Co., 1988.