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A NEW CLASS OF QUANTUM POTENTIALS: THE CONDITIONALLY EXACTLY SOLUBLE ONES

by

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Abstract: We present a new class of quantum-mechanical potentials. These are in the midway between the exactly solvable potentials and the quasi-exactly ones. Their fundamental feature is that one can find the entire spectrum of a given potential, provided that some of its parameters be conveniently fixed.

Key-words: Schrödinger picture; Exact potentials; Analytic methods.

Since the appearing of the quantum mechanics, the searching for exactly soluble potentials (ES) has been a constant [1]. This relies in the importance of such solutions in many branches of Physics, and that these solutions can be used as basis to perform perturbative calculations in non-exact potentials. Until recently it was thought that there would exist only two classes of potentials in quantum mechanics. Namely, the exactly soluble and the non-exactly soluble ones. However about a decade ago [2], another class was discovered and, since then, has been widely discussed in its various aspects [3-7]. This class, the so-called quasi-exactly soluble potentials (QES), is characterized by the fact that it is only possible to have exactly a finite number of their energy eigenstates, others can only be obtained through numerical calculations, like in the non-exact soluble potentials. The importance in the study of these new potentials, apart from the intrinsic academic interest, rests on the possibility of using their solutions to test the quality of numerical methods and in the possible existence of real physical systems which they could represent.

Here we intend to report the finding of one more new class of potentials, what we call the conditionally exactly soluble potentials (CES), because its principal feature is that of having exact solutions only when certain conditions are provided, namely, that some of the parameters of the potential be fixed to a very specific value.

This new class of potentials are positioned amid the ES and the QES potentials. All of their energy levels can be exactly obtained like happens to the ES potentials, but their parameters cannot be arbitrarily chosen as in the QES potentials. The first feature put them in conditions to be used in perturbative calculations, having the advantage that, for the potentials belonging approximately to their form, they take into account the anarmonicity of the potential, unlike the traditional perturbative method where it is commonly used the harmonic oscillator or any other exact potential with more appropriate form. The second feature put the CES potentials in a more suitable situation in order to testify about the quality of a given numerical approach. Furthermore we will see that one could expect some fashion of reality in at least one of these potentials.

The way to get such potentials we use is that of looking for a mapping between them and a driven harmonic oscillator. This is done by performing non-linear coordinate transformations, and then requiring that a certain term vanishes.

We start with the Schroedinger equation for a given potential V(r), do the variable transformation r = f(u) and redefine the wave function like

$$\psi(r,t) = \sqrt{f'(u(r))}\ \chi(r,t),$$

where the prime denotes differentiation with respect to the variable u. This new wave function obeys the transformed Schroedinger equation

$$\left\{ - \frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial u^2} + V_T(u) \right\} \chi(u) = E_T\, \chi(u), \tag{1}$$

where μ is the mass, E_T is some constant resulting from the transformation which takes the role of the "energy" in the new equation. Furthermore it is straightforward to show that

$$V_T(u) - E_T = (f'(u))^2 \Big[V(f(u)) - E \Big] + \Delta V(u), \tag{2a}$$

with

$$\Delta V(u) = \frac{\hbar^2}{\mu} \left[- \frac{1}{4} \frac{f'''(u)}{f'(u)} + \frac{3}{8} \left(\frac{f''(u)}{f'(u)} \right)^2 \right]. \tag{2b}$$

This type of approach has been used extensively in the path-integral method of quantization [8]. However for transformations where $f(u) = u^\alpha$, with α being real, $\Delta V(u)$ will always produces a u^{-2} term, that should be removed in order to get a driven harmonic oscillator potential in the new variable. This is the origin, for the case here considered, of the very particular fixing of one of the potential parameters.

Let us now see the first two representatives of this new class of potentials. First of all we see that an extension of our old acquainted Coulomb potential belongs to it. This potential is

$$V(r) = \frac{A}{r} + \frac{B}{r^{1/2}} + \frac{G_0}{r^2} . \qquad (3)$$

The transformation is the same one that links the Coulomb potential with a three-dimensional harmonic oscillator $f(u) = u^2$ [8]. In this case the expression (2) reads

$$V_T(u) - E_T = -4\ E\ u^2 + 4\ B\ u + 4\ A + \left(4\ G_0 + \frac{3\ \hbar^2}{8\ \mu}\right)u^{-2}, \qquad (4)$$

where we can identify E_T with 4 A, the frequency of the oscillator $\omega = \sqrt{-8E/\mu}$, and the driving force is 4B. In order to get an exact solution one can follow one of two possible ways. The first one is the elimination of the driving force (B = 0), leading to the solution of the usual Coulomb potential. The second way is that of imposing the elimination of the centrifugal-barrier-type term, this is obtained by imposing that $G_0 = - 3\hbar^2/32\mu$. This last case lead us to the first CES presented in this work. It is interesting to observe that for great distances it behaves like a s-wave Coulomb potential.

Using the solution of the driven harmonic oscillator, and returning to the original physical variable, it is not difficult to obtain the eigenfunction

$$\psi_n(r) = \left(2^{(n-1)}n!\right)^{-1/2}\left(\frac{r\ \beta_n^2}{\pi}\right)^{1/4} H_n\left[\beta_n\left(r^{1/2} - \frac{B}{2E_n}\right)\right] \cdot$$

$$\cdot \exp\left[-\frac{\beta_n^2}{2}\left(r^{1/2} - \frac{B}{2E_n}\right)^2\right], \tag{5}$$

where we defined

$$\beta_n \equiv \left(-\frac{8\ \mu\ E_n}{\hbar^2}\right)^{1/4},$$

and the energy obeys a polynomial equation, coming from that of the driving harmonic oscillator, of the third degree

$$\hbar^2(n + 1/2)^2E_n^3 + 2\ \mu\ A^2E_n^2 + \mu\ A\ B^2E_n + \mu\ B^4/8 = 0. \tag{6}$$

From the three solutions of the above equation we shall discard two. This can be done by a simple physical criteria. As can be seen from the form of the potential, we see that when B = 0, the energy spectrum must be reduced to that one of the Coulomb type. Using this criteria one gets the solution

$$E_n = \left[R + \left(Q^3 + R^2\right)^{1/2}\right]^{1/3} + \left[R - \left(Q^3 + R^2\right)^{1/2}\right]^{1/3} - \frac{a_1}{3}, \tag{7a}$$

where

$$Q \equiv (3\ a_2 - a_1^2)/9\ ,\ R \equiv (9\ a_1a_2 - 27\ a_3 - 2\ a_1^3)/54, \tag{7b}$$

and

$$a_1 = \frac{2\ \mu\ A^2}{\hbar^2(n + 1/2)^2}\ ,\ a_2 = \frac{\mu\ A\ B^2}{\hbar^2(n + 1/2)^2}\ ,\ a_3 = \frac{\mu\ B^4}{8\ \hbar^2(n + 1/2)^2}\ . \tag{7c}$$

The second example, as far as we know, does not have in some limit any other well known ES potential, as happened in the previous example. The potential appears like

$$V(r) = A\ r^{2/3} + \frac{B}{r^{2/3}} + \frac{g_0}{r^2}\ , \tag{8}$$

where g_0 is chosen to be equal to $-5\hbar^2/72\mu$, in a completely analogous fashion to the previous case. It is remarkable to observe that a non-exact version of this potential was considered recently [9], in connection with an effective quark-antiquark potential model for heavy and light mesons. In fact, it is easy to see that for high radial quantum numbers the above CES will have its spectrum more and more close to the non-exact case appearing in [9]. The transformation function in this case is $f(u) = u^{3/2}$ and the Eq.(2) looks like

$$V_T(u) - E_T = (9A/4)\ u^2 - (9E/4)\ u + 9B/4\ , \tag{9}$$

and now we identify E_T with 9B/4, $\omega = \sqrt{9A/2\mu}$, and the driving force is represented by - 9E/4. This time the eigenfunctions are

$$\psi_n(r) = \left(2^{(n+1)} n!/3\right)^{-1/2} \left(\frac{r^{2/3}\beta^2}{\pi}\right)^{1/4} H_n\left[\beta\left(r^{2/3} - \frac{E_n}{2A}\right)\right] .$$

$$.\exp\left[-\frac{\beta^2}{2}\left(r^{2/3} - \frac{E_n}{2A}\right)^2\right], \tag{10}$$

with $\beta \equiv (9\mu A/2\hbar^2)^{1/4}$. The equation for the spectrum is simply

$$E_n^2 = \frac{16A}{9}\left[(n + 1/2)\sqrt{9A/2\mu} + 9B/4\right], \tag{11a}$$

with the solutions

$$E_n = \pm \frac{4\sqrt{A}}{3}\left[(n + 1/2)\sqrt{9A/2\mu} + 9B/4\right]^{1/2}. \tag{11b}$$

It is not difficult to convince oneself that, for the cases where the parameter B is positive, the positive solution is quite good, but if B is negative one can see that there will exist prohibited quantum numbers. This can be seen by imposing that E_n must be a real quantity so,

$$n \geq \mathrm{Int}\left[\left(\frac{9B^2}{8A}\right)^{1/2} - \frac{1}{2}\right], \tag{12}$$

where Int[.] stands for the first integer after the value of its argument. This, however, does not says that there are not such energy levels but that, perhaps, it should be searched in the negative solutions in (11b). It is only a matter of experimentation to verify that, for instance, with n = 1 the negative solution looks like a ground state eigenfunction, because it does not have any node.

The problem is that this potential, in contrast with the former, does not have some type of limit as a guide to decide about what solution to use. This problem is under investigation and we intend to report on it in a further publication.

As should be expected, these two cases can be mapped one into another through a suitable transformation, in the first one, for example, it would be $f(u) = u^{4/3}$.

Some extensions can be thought, like looking for supersymmetric partners, one can find another CES representatives belonging to Poschl-Teller, Rosen-Morse, and others classes of ES potentials.

We can also study the possibility of applying the second potential presented above to the case of quark-antiquark potential models. These and other problems related to the CES are presently under investigation, and we expect to report on them in the near future.

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