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ON THE DIRAC EQUATION IN THREE DIMENSIONS

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Abstract

We argue that, in three dimensions, spinors should have four components as a consequence of the algebraic structure realized from the Clifford algebra related to the Dirac equation.

Key-words: Dirac equation; Dirac-Kähler equation;
Clifford algebras.

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1 Preliminaries

In this letter we analyze the kinematics of the Dirac equation in three dimensions. Our interest is concentrated on the spacetime with Minkowski metric $(+--)$, but our results are easily extended to other metrics and to Euclidean space as well.

In recent years, we have studied carefully the four-dimensional Dirac equation [1]. We have discovered that there is a spacetime $SU(4)$ symmetry closely related to the formulation of the Dirac equation in terms of the Clifford algebra of the Dirac matrices or of differential forms (the Dirac–Kähler equation). Actually, we believe that the former is a representation of the latter [2,3]. This work in three dimensions is a step forward in the foundation of such ideas.

In the framework of differential forms, an $SU(4)$ symmetry was also found by Becher and Joos [4] in their pioneer work on the subject. They, however, interpreted it quite differently. Our point is that the Lie-algebraic structure related to the Dirac–Kähler equation, or its all-important matrix Dirac counterpart, follows directly from the Clifford algebra structure endowed on spacetime.

We conjectured [1,2] that in three dimensions the related symmetry group would be $SU(2) \times SU(2)$. This we show in what follows and it has deep consequences.

We use the formulation of the Clifford algebra related to the Dirac equation in terms of differential forms as a first tool. This aims to indicate that the algebraic structure has its roots in spacetime, something which is somehow not properly seen when using matrices.

Once the $SU(2)\times SU(2)$ structure for differential forms (endowed with a Clifford product) is demonstrated, we pass to the representation in terms of matrices, profiting from the isomorphism stated by Graf [5]. We show that the usual work with Dirac matrices in three dimensions is not in agreement with the algebraic structure.

Further, we show that the action of discrete symmetry operators such as parity and time reversal precludes the use of a two-component formalism for spinors in three dimensions.

2 Differential forms in 2+1 dimensions

The space of differential forms in three dimensions has eight components,

$$1, dx^{\mu}, dx^{\mu}\wedge dx^{\nu}, dx^{0}\wedge dx^{1}\wedge dx^{3}\equiv \varepsilon,$$

where μ and ν run from 0 to 2 and $\wedge$ denotes the usual representation of the exterior product of differential forms. The duality $*$ operator defined by Hodge links the subspace of forms with degree p with that of those with degree $3-p$; thus, the set of four components $(1, dx^{\mu})$ maps into the remaining four, that is, $(dx^{\mu}\wedge dx^{\nu}, \varepsilon)$.

We assume further, following Kähler [6,4], that a Clifford product is defined such that

$$dx^{\mu}\vee dx^{\nu} = dx^{\mu}\wedge dx^{\nu} + g^{\mu\nu}. \tag{1}$$

With this operation, we look for the Clifford commutator between differential forms [4]:

$$\left[dx^{K}, dx^{L}\right]_{\vee} = dx^{K}\vee dx^{L} - dx^{L}\vee dx^{K}, \tag{2}$$

where dx^K and dx^L represent two out of the eight basic differential forms.

In order to be able to exhibit the Lie-algebraic structure relevant to three dimensions, we give explicitly the Clifford commutators for all forms:

$$
\begin{array}{ll}
[dx^0, dx^1]_\vee = 2dx^0 \wedge dx^1 & [dx^1, dx^1 \wedge dx^2]_\vee = -2dx^2 \\
[dx^0, dx^2]_\vee = 2dx^0 \wedge dx^2 & [dx^2, dx^0 \wedge dx^2]_\vee = 2dx^0 \\
[dx^0, dx^0 \wedge dx^1]_\vee = 2dx^1 & [dx^2, dx^1 \wedge dx^2]_\vee = 2dx^1 \\
[dx^0, dx^0 \wedge dx^2]_\vee = 2dx^2 & [dx^0 \wedge dx^1, dx^0 \wedge dx^2]_\vee = -2dx^1 \wedge dx^2 \\
[dx^1, dx^2]_\vee = 2dx^1 \wedge dx^2 & [dx^0 \wedge dx^1, dx^1 \wedge dx^2]_\vee = -2dx^0 \wedge dx^2 \\
[dx^1, dx^0 \wedge dx^1]_\vee = 2dx^0 & [dx^0 \wedge dx^2, dx^1 \wedge dx^2]_\vee = 2dx^0 \wedge dx^1,
\end{array}
$$

all others being zero. We notice that the volume form ε commutes with all the remaining forms. This indicates somehow the algebraic structure that is to be expected to arise.

3 The algebraic structure

To illustrate the algebraic structure with an example, let us consider the three forms dx^0, idx^1 and $dx^0 \wedge dx^1$. Notice that all have a unit norm. The Clifford commutators are

$$
\begin{aligned}
\left[dx^0, idx^1\right]_\vee &= 2idx^0 \wedge dx^1 \\
\left[idx^1, dx^0 \wedge dx^1\right]_\vee &= 2idx^0 \\
\left[dx^0 \wedge dx^1, dx^0\right]_\vee &= 2i(idx^1).
\end{aligned}
$$

Defining

$$X_1 = \frac{1}{2}dx^0, \quad X_2 = \frac{1}{2}idx^1, \quad X_3 = \frac{1}{2}dx^0 \wedge dx^1, \tag{3}$$

the above commutators can be synthesized as

$$[X_k, X_\ell]_\vee = i\epsilon_{k\ell m} X_m, \tag{4}$$

with $\epsilon_{k\ell m}$ the usual totally antisymmetric symbol. The dual Hodge forms are $dx^1 \wedge dx^2$, $dx^0 \wedge dx^2$ and dx^2 for the original set. Consider now the commutators among the set $idx^1 \wedge dx^2$, $-dx^0 \wedge dx^2$ and idx^2. One finds

$$\left[idx^1 \wedge dx^2, -dx^0 \wedge dx^2\right]_\vee = 2idx^0 \wedge dx^1$$
$$\left[idx^2, idx^1 \wedge dx^2\right]_\vee = 2i(idx^1)$$
$$\left[-dx^0 \wedge dx^2, idx^2\right]_\vee = 2idx^0 \wedge dx^1,$$

so that by defining

$$Y_1 = \frac{1}{2}idx^1 \wedge dx^2, \quad Y_2 = -\frac{1}{2}dx^0 \wedge dx^2, \quad Y_3 = \frac{1}{2}idx^2, \tag{5}$$

the above commutators can be summarized in the relation

$$[Y_k, Y_\ell]_\vee = i\epsilon_{k\ell m}X_m. \tag{6}$$

The stage is set. Following the usual manipulations, define now

$$W_k^+ = \frac{1}{2}(X_k + Y_k) \tag{7}$$
$$W_k^- = \frac{1}{2}(X_k - Y_k). \tag{8}$$

Application of the rules of the game results in

$$\left[W_k^+, W_\ell^+\right]_\vee = i\epsilon_{k\ell m}W_m^+ \tag{9}$$
$$\left[W_k^+, W_\ell^-\right]_\vee = 0 \tag{10}$$
$$\left[W_k^-, W_\ell^-\right]_\vee = i\epsilon_{k\ell m}W_m^-. \tag{11}$$

This is precisely the structure of an $SU(2)\times SU(2)$ algebra for the differential forms. That is, conversely, any differential form can be represented as a linear combination of $SU(2) \times SU(2)$ generators, the unit matrix and the volume form. For reasons of convenience, we shall write the product as $SU(2)_+ \times SU(2)_-$.

Moreover, the following properties are valid:

$$*W_k^{\pm} = \pm i W_k^{\mp} \quad (k = 1,2) \tag{12}$$

$$*W_3^{\pm} = \mp i W_3^{\mp} \tag{13}$$

and, subsequently,

$$**W_k^{\pm} = W_k^{\pm} \quad (k = 1,2,3). \tag{14}$$

A glance at the commutator table shows that the original set is not a preferred one. We can choose, for instance,

$$X_1 = \frac{1}{2} i dx^1, \quad X_2 = \frac{1}{2} dx^0 \wedge dx^1, \quad X_3 = \frac{1}{2} dx^0 \tag{15}$$

$$Y_1 = \frac{1}{2} dx^0 \wedge dx^2, \quad Y_2 = \frac{1}{2} i dx^2, \quad Y_3 = \frac{1}{2} i dx^1 \wedge dx^2. \tag{16}$$

This choice, as we shall show later, corresponds to what is called the Dirac–Pauli picture (or representation) in four dimensions. Also, the choice

$$X_1 = \frac{1}{2} i dx^1, \quad X_2 = \frac{1}{2} i dx^2, \quad X_3 = \frac{1}{2} i dx^1 \wedge dx^2 \tag{17}$$

$$Y_1 = -\frac{1}{2} dx^0 \wedge dx^2, \quad Y_2 = \frac{1}{2} dx^0 \wedge dx^1, \quad Y_3 = \frac{1}{2} dx^0 \tag{18}$$

will be seen to correspond to the Kramers–Weyl picture.

In any case, the procedure described above for the construction of the $SU(2) \times SU(2)$ algebra applies. Interesting properties are the following ones:

$$X_1 \vee X_2 \vee X_3 = \frac{1}{8} i \mathbb{1}, \qquad Y_1 \vee Y_2 \vee Y_3 = -\frac{1}{8} \varepsilon. \tag{19}$$

For the $SU(2)$ generators, one has

$$W_1^+ \vee W_2^+ \vee W_3^+ = \frac{1}{16}(i\mathbb{1} - \varepsilon) \tag{20}$$

$$W_1^- \vee W_2^- \vee W_3^- = \frac{1}{16}(i\mathbb{1} + \varepsilon). \tag{21}$$

The resemblance of the right-hand side with the costumary chirality projection operators in four dimensions is noticeable, provided the volume element is related to the matrix γ_5.

In what follows we shall refer to the pictures associated above as the Dirac–Pauli and the Kramers–Weyl pictures.

4 The Dirac–Kähler equation

At this point we refer briefly to the formalism which deals with spin-$\frac{1}{2}$ particles in terms of differential forms. Further references can be found in our previous work [1,2,3] and in the work of Becher and Joos [4].

The differential operators of exterior differentiation, d, and its adjoint (with respect to the "usual" scalar product), δ, combine to form the Dirac–Kähler operator, $i(d-\delta)$.[1] This operator leaves invariant the minimal left ideals of the Clifford algebra [3,4] and any ideal can be associated to a Dirac spinor. In fact, the result of the Dirac–Kähler operator on a minimal left ideal is exactly the same as the result of the usual Dirac operator with gamma matrices on a suitable spinor for Euclidean and Minkowski spaces of dimension two and four.

The fundamental concept involved is the isomorphism established by Graf [5] between differential forms endowed with a Clifford product and gamma matrices:

$$dx^\mu \vee \longleftrightarrow \gamma^\mu . \tag{22}$$

We take for granted this isomorphism for the present case of three dimensions

[1] We have slightly changed our conventions with respect to previous work, following now those of Curtis and Miller [7].

and will exploit it, using the algebraic structure shown above for differential forms, to construct representations of the differential forms in terms of matrices.

Since the work with differential forms for spin-$\frac{1}{2}$ particles is less familiar to theoretical physicists, we shall only state here that one can construct the set of four coupled linear differential equations of the first degree from the application of the Dirac–Kähler operator to a minimal left ideal, Ψ,

$$i(d-\delta)\Psi = m\Psi. \tag{23}$$

The four equations couple by pairs, as should be expected from the algebraic structure described above. We leave for the section on the representation by matrices and spinors a more explicit discussion of the meaning of this coupling.

5 Matrix representation and the Dirac matrices

In terms of matrices, the algebraic structure found with differential forms is well known. Since we work in the lowest dimensional representation, we have, indeed, 4×4 matrices:

$$W_k^+ = \begin{pmatrix} \sigma_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad W_k^- = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_k \end{pmatrix}, \tag{24}$$

where the σ_k are the Pauli matrices.

The matrices representing the Dirac matrices and their products will turn out to be block-diagonal matrices, i.e., the direct sum of 2×2 Dirac matrices. To wit:

Dirac–Pauli

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \gamma^1 = -i \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \gamma^2 = -i \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}; \tag{25}$$

Kramers–Weyl

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = -i \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \gamma^2 = -i \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{26}$$

The reason for the proposed names of these two pictures seems now evident: γ^0 is diagonal and has the sign of the energy in the rest frame for the Dirac–Pauli picture, while γ^0 and $i\gamma^1\gamma^2$ exchange places when going to the Kramers–Weyl picture.

A comparison between these matrices and those quoted in the literature for the four-dimensional representation [8] shows a substantial difference: there is always a relative minus sign for one of the blocks in the $SU(2)_-$ sector. This is not accidental, it is a direct consequence of the exact spacetime algebraic structure proper to three dimensions.

It is worth remembering that from the Clifford commutators for differential forms, the volume 3-form was found to commute with the rest. In terms of matrices, its appearance is transparent. By calculation from the expressions above or from clever symmetry arguments, the only possible block matrix commuting with the others, and still not proportional to the unit matrix is

$$i\gamma^0\gamma^1\gamma^2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The particular form of this outstanding matrix agrees with the arguments in the classical work by Brauer and Weyl [9]. (In fact, Brauer and Weyl even introduced the concept analogous to Hodge duality for Dirac matrices, but did not completely finalize the arguments leading to the actual form of the algebra.)

6 The Dirac equation in three dimensions

As anticipated from the results for the Dirac–Kähler equation, the explicit construction of the representation for the Dirac matrices confirms that the four components of a spinor couple in pairs in the differential equation.

This would seem to imply that physically the world in three dimensions is made of two blocks labelled by the eigenvalues of $i\gamma^0\gamma^1\gamma^2$, which could perhaps be qualified as right handed ($SU(2)_+$) or left handed ($SU(2)_-$), according to the way coordinate axes are oriented. Particles and antiparticles are partners with the same handedness, as we shall see below.

However, as demonstrated for two and four dimensions of spacetime [1,2,3], the discrete transformations of space inversion and time reversal are crucial to the complete understanding of the underlying algebraic structure. In our case, they forbid one to isolate a single handedness and force the description of spin-$\frac{1}{2}$ particles in three dimensions by four-component spinors.

7 Discrete transformations: C, P, T and CPT

7.1 Charge conjugation

Let us begin by considering charge conjugation. Applying the standard procedures from textbooks [10], the matrix that implements the charge conjugation operator, C, should satisfy

$$C^{-1}\gamma^{\mu}C = -\gamma^{\mu t}, \tag{27}$$

where the superscript t denotes transposition. For three dimensions, it turns out that the Hodge duality properties of differential forms translates into the existence of two matrices in the matrix formalism as the representatives of differential forms related by Hodge duality. The matrices also depend on the picture. For the pictures considered above, we have

$\psi^c = \gamma^0\gamma^1\psi^*$ or $-i\gamma^2\psi^*$ in the first example, Eqs. (3) and (5),
$\psi^c = i\gamma^1\psi^*$ or $-\gamma^0\gamma^2\psi^*$ for the Dirac–Pauli and Kramers–Weyl pictures.

The difference between the two alternatives is a relative sign for the lower pair of components.

Since the matrices at work belong always to the class of block diagonal 4×4 matrices, the charge-conjugate spinor features only a reshuffling within the blocks corresponding to each of the eigenvalues of the matrix $i\gamma^0\gamma^1\gamma^2$.

7.2 Space inversion

In three dimensions, space inversion is different when compared to the neighbouring cases of even total dimensions $(2, 4)$. It is a simple drawing exercise to check that the simultaneous inversion of both space axes corresponds to a rotation through an angle π about the time axes. Accordingly, spinors are related by a simple rotation operator. There is again an equivalent action by the dual operator. For all pictures, one has

$$\psi'(\boldsymbol{x}' = -\boldsymbol{x}, t) = i\gamma^1\gamma^2\psi(\boldsymbol{x}, t),$$

or

$$\psi'(\boldsymbol{x}' = -\boldsymbol{x}, t) = \gamma^0\psi(\boldsymbol{x}, t).$$

Thus, the blocks with different handedness are not exchanged.

If it is desirable to exchange handedness (or chiralities), this can be performed by inversion of a single space axis. The matrix representing this is no longer of the block-diagonal class, but the ambiguity concerning the relative sign for the pair of lower components in the spinor persists. Calling $P_{(k)}$ the matrix for inversion of the k space axis, the candidates, for the pictures considered, are

$$\text{The first example:} \quad P_{(1)} = \begin{pmatrix} 0 & \pm\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \qquad P_{(2)} = \begin{pmatrix} 0 & \pm I \\ I & 0 \end{pmatrix}$$

$$\text{Dirac–Pauli:} \quad P_{(1)} = \begin{pmatrix} 0 & \pm\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad P_{(2)} = \begin{pmatrix} 0 & \pm I \\ I & 0 \end{pmatrix}$$

$$\text{Kramers–Weyl:} \quad P_{(1)} = -i\begin{pmatrix} 0 & \pm\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad P_{(2)} = \begin{pmatrix} 0 & \pm\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}.$$

This clearly shows the inadequacy of the representation by two-component spinors.

7.3 Time reversal

This transformation exchanges handedness by necessity. The novel feature in three dimensions is the link between the components of the complexconjugate spinor and a spinor transformed from the original one. The results for the time-reversed spinors in the pictures considered are

Dirac–Pauli

$$\psi'(x, t' = -t) = \begin{pmatrix} 0 & \sigma_1 \\ \pm\sigma_1 & 0 \end{pmatrix} \psi(x, -t) = \begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix} \psi^*(x, -t); \quad (28)$$

Kramers–Weyl

$$\psi'(x, t' = -t) = \begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix} \psi(x, -t) = \begin{pmatrix} 0 & \sigma_1 \\ \pm\sigma_1 & 0 \end{pmatrix} \psi^*(x, -t). \quad (29)$$

The operation for the first example is the same as in eq. (28). Again, we see that the matrices concerned are outside the realm of diagonal-block matrices.

It is interesting to remark that, when a relative sign appears between the non-diagonal blocks, $T^2 = -1$. For matrices with blocks of the same sign, $T^2 = 1$.

7.4 CPT

As follows from the considerations above, there are two classes of results for the combined operations. When space inversion is meant as a simultaneous reversal of both space axes, the pair of upper components takes the place of the pair of lower components. In general, there is, in addition, an exchange inside each pair, with relative phases being introduced. There is one particular exception, which occurs for the Kramers–Weyl picture, with

$$C = \gamma^2, \qquad P = i\gamma^1\gamma^2, \qquad T = \begin{pmatrix} 0 & \sigma_1 \\ \pm\sigma_1 & 0 \end{pmatrix}, \quad (30)$$

we have

$$(CPT)\psi = \begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix} \psi. \quad (31)$$

When space inversion means the reversal of a single space axis, CPT always results in a block-diagonal matrix acting on the original spinor. This block matrix turns out to be the gamma matrix with the spatial index that is not inverted, or its "dual". For $P_{(1)}$ it is γ^2 or $i\gamma^0\gamma^1$, and for $P_{(2)}$ the result is similar. The geometrical meaning of this result is not yet fully appreciated.

8 Conclusions

The main results of this work were listed at the beginning. Now that they are more explicit, let us add a few comments.

We believe that we have unveiled a most important algebraic structure of three-dimensional spacetime. These new results are based on the assumed validity, for any number of dimensions, of the Graf isomorphism between the differential forms with Clifford product and the matrices associated to the Dirac equation. In other words, the algebraic structure associated with spacetime follows from the related Clifford algebra structure.

We have also shown that this algebraic structure makes the discrete operations of space inversion, time reversal and charge conjugation in three dimensions rather peculiar. The understanding of this demands further investigation.

In the light of this structure, the need for a description through four-component spinors of spin-$\frac{1}{2}$ particles follows. These four-component spinors are certainly different from the ones currently quoted in the literature, and this may induce changes in several physical results obtained for three-dimensional systems.

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References

[1] C.A. Linhares and J.A. Mignaco, Phys. Lett. **153B** (1985) 85; "$SU(4)$ properties of the Dirac equation", preprint CBPF-NF-057-88.

[2] C.A. Linhares and J.A. Mignaco, "New symmetries for the Dirac equation", in *J.J. Giambiagi Festschrift*, H. Falomir, R. E. Gamboa-Saraví, P. Leal Ferreira and F. A. Schaposnik, eds., World Scientific, Singapore, 1990.

[3] C.A. Linhares and J.A. Mignaco, "$SU(4)$ properties of the Dirac–Kähler equation", preprint CBPF-NF-008-91, submitted for publication.

[4] P. Becher and H. Joos, Zeits. für Phys. **15** (1982) 343.

[5] W. Graf, Ann. Inst. Henri Poincaré A **29** (1978) 85.

[6] E. Kähler, Abh. Dt. Akad. Wiss. Berlin, Kl. für Math., Phys. u. Tech., Jahrg. 1960 **4** (1960); **1** (1961); Rendiconti di Matematica (Roma), Ser. V, **21** (1962) 425.

[7] W.D. Curtis and F.R. Miller, *Differential Manifolds and Theoretical Physics*, Academic Press and Harcourt Brace Jovanovich Publ., 1985.

[8] T. Appelquist, M.J. Bowick, E. Cohler and L.C.R. Wijewardhana, Phys. Rev. Lett **55** (1985) 1715; M. Gomes, R.S. Mendes, R.F. Ribeiro and A.J. da Silva, Phys. Rev. D **43** (1991) 3516.

[9] R. Brauer and H. Weyl, Am. J. Math. **57** (1935) 425; See also H. Boerner, *Representations of Groups*, North Holland, Amsterdam, 1963, pages 265ff.

[10] S.S. Schweber, *An Introduction to Quantum Field Theory*, Harper and Row, New York, 1961, p. 106.