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SPHERICALLY SYMMETRIC RELATIVISTIC MODEL FOR SPIRAL GALAXIES AND DENSE STARS

by

R. HOJMAN*, Ligia M.C.S. RODRIGUES and F.D. SASSE

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*Departamento de Física, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile and Centro de Física El Traúco, Fanor Velasco, 41D, Santiago, Chile.

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ABSTRACT

The behaviour of the pressure and the density as well as the gravitational field of a dense star are studied in some detail. For such a purpose and to take into account relativistic effects, we find a family of exact solutions of the Tolman-Oppenheimer-Volkov equation, which contains as a particular case solutions corresponding to a γ-law equation of state. The mentioned family can also be used to model the (luminous or dark) matter content of spiral galaxies, as it fits the observed data for their orbital velocities profiles.

Key-words: Relativistic astrophysics; Spiral galaxies; Gravitation and astrophysics; Dark matter and galaxies halos.

I. INTRODUCTION

General Relativity has little effect on the equilibrium configuration of stars with $p \ll \rho$ and $m(r) \ll r$, $m(r)$ being the (geometrical) mass up to a distance r from the center of the star (Weinberg 1972). Nonetheless, it has been shown that its influence can become important when studying the stability of stars and, in fact, the interior of cold catalysed stars (Misner, Thorne and Wheeler 1970) can be adequately modeled by a relativistic static spherically symmetric self-gravitating ideal fluid with energy density ρ and pressure p.

If such a picture is adopted, Einstein equations for the static spherically symmetric line element

$$ds^2 = e^{2\Phi(r)}dt^2 - e^{2v(r)}dr^2 - r^2(d\theta^2 + \mathrm{sen}^2\theta d\phi^2) \tag{1}$$

can be reduced to a system of two coupled ordinary non-linear differential equations. One of them is the relativistic version of Laplace equations, that is,

$$\frac{d\Phi(r)}{dr} = \frac{m(r) + 4\pi r^3}{r[r-2m(r)]} \tag{2}$$

where $\Phi(r)$ is the gravitational potential.

The other is the Tolman-Oppenheimer-Volkof (TOV) equation

$$\frac{dp}{dr} = - (p+\rho)\frac{m(r) + 4\pi r^3 p}{r[r-2m(r)]} \quad , \tag{3}$$

which is the generalization of the classical equation of hydrostatic support.

The energy density ρ is related to p through the equation of state

$$p = p(\rho) \tag{4}$$

and the geometrical mass is given by

$$m(r) = \int_0^r 4\pi r^2 \rho dr \ . \tag{5}$$

Relativistic models (Weinberg 1972) are usually constructed by specifying an equation of state and a value for the central pressure, p_c (or equivalently for the central density). Eqs. (2), (3) and (5) are integrated outwards from $r = 0$ to the surface of the star where the pressure vanishes, subjected to the conditions $p(r = 0) = p_c$, $\Phi(r = 0) = \Phi_0$ and $m(r = 0) = 0$. Furthermore, Φ is chosen so as to have $\Phi \to 0$ when $r \to \infty$ or by requiring Φ to match onto the Schwarzschild value at the surface of the star.

An alternative prescription (Berger et al 1983), which demands the choice of an arbitrary master function G(r) that reduces the problem to quadratures, can be used. In section II, following this method we construct a family of spherically symmetric solutions that is a good model for relativistic stars. Conditions to render it realistic are analysed.

Another astrophysical application of that family of solutions can be found. It is presently believed that in spiral galaxies the central luminous core is surrounded by massive non-luminous halos (Rubin 1983). One observational evidence is the non-decreasing of the rotational velocities of stars far from the center of the galaxy in spite of the fact that luminosity decreases exponentially with increasing radial distance. In section III, we will show that our spherically symmetric geometries reasonably fit the observational data for orbital velocities concerning the luminous or dark[1] region of spiral galaxies.

[1]For the question of nature and origin of dark matter in the universe, see E.W. Kolb and M.S. Turner, "The Early Universe", Addison-Wesley, 1990.

II. A FAMILY OF SOLUTIONS

Employing the method developped by Berger et al (1983), we will take the master function G(r) proportional to r^3 (Hojman and Rodrigues 1990) that is,

$$G(r) = ar^3 \tag{6}$$

As will become clear below, it is convenient to take the constant a in the form

$$a = \frac{\gamma}{4-5\gamma} \quad , \quad \gamma \neq 0, \ \gamma \neq \frac{4}{5} \tag{7}$$

We will obtain a family of geometries (1) that is adequate to describe both dense relativistic stars and the matter content of spiral galaxies[2].

We find

$$p = p_0 r^\Gamma + \frac{\Theta}{r^2} \tag{8a}$$

$$\rho = p_0 \Delta r^\Gamma + \frac{\Sigma}{r^2} \tag{8b}$$

$$e^{2\Phi} = g_0^2 r^{\frac{4(\gamma-1)}{\gamma}} \tag{8c}$$

and

[2]This solution was first obtained by Tolman, R.C., Phys. Rev., 55 (1939) 364.

$$e^{2\nu} = 1 - \sum - \frac{\gamma}{4-5\gamma} p_0 r^{\Gamma+2} \tag{8d}$$

where p_0 and g_0 are integration constants and

$$\Gamma = \frac{4(\gamma-1)(2-\gamma)}{\gamma(3\gamma-2)} \tag{9}$$

$$\text{Ⓗ} = \frac{4(\gamma-1)^2}{\gamma^2 + 4\gamma - 4} \tag{10}$$

$$\Delta = - \frac{\gamma+2}{3\gamma-2} \tag{11}$$

$$\sum = \frac{4(\gamma-1)}{\gamma^2 + 4\gamma - 4} \tag{12}$$

When $p_0 = 0$,

$$p = \frac{4(\gamma-1)^2}{\gamma^2 + 4\gamma - 4} \frac{1}{r^2} \tag{13}$$

$$\rho = \frac{4(\gamma-1)}{\gamma^2 + 4\gamma - 4} \frac{1}{r^2} \tag{14}$$

and therefore, the γ-law equation of state is obtained as a particular case for the general equation of state relating (8a) and (8b), that is,

$$\left[p - (\gamma-1)\rho\right]\left[(3\gamma-2)\rho + (\gamma+2)p\right]^{\Gamma/2} = p_0(\gamma^2+4\sigma-4)\left[\frac{4(\gamma-1)}{3\gamma-2}\right]^{\Gamma/2} \qquad (15)$$

The analysis concerning the range of values γ for which both the pressure and the density are positive is summarized in Table I.

An inspection of Table I indicates that realistic models can be constructed only if $1 < \gamma < 2$. Moreover, if the surface of the star is defined by the vanishing of the pressure for a certain $r = r_p = \left(-\frac{Ⓗ}{p_0}\right)^{1/\Gamma}$ and reminding that Ⓗ is strictly positive for those values of γ, it is seen that relativistic stars will be properly modeled only when negative values of p_0 are taken.

III. GEODESIC MOTION AND VELOCITY PROFILES OF SPIRAL GALAXIES

We will show that the family of solutions generated by $G(r)\alpha r^3$ also models the (luminous or dark) content of spiral galaxies. For this purpose we will study the geodesic motion of a star (considered as a test particle) in the background geometry given by eqs. (8).

The geodesic equations for the metric given by (1) and (8) to (12) when $\theta = \frac{\pi}{2}$ imply that

$$r^{\frac{\gamma}{4-5\gamma}}\,\dot{t} = E \qquad (16)$$

$$r^{\frac{\gamma}{4-5\gamma}}\,\dot{\phi} = \ell \qquad (17)$$

where E and ℓ are constants of motion and the dot stands for the derivative with respect to the arc length s.

As usual, instead of deriving the r-component of the equation of motion, it is easier to work with the normalization condition for the 4-velocity, uu = 1.

By combining the above equations one gets,

$$\dot{r}^2 = \left[\frac{\gamma^2}{\gamma^2+4\gamma-4} - p_0 \frac{\gamma}{4-5\gamma} r^{\Gamma+2}\right] r^{\frac{4(1-\gamma)}{\gamma}} \left[E^2 - r^{\frac{4(\gamma-1)}{\gamma}} - \ell^2 r^{\frac{2(\gamma-2)}{\gamma}}\right] \tag{18}$$

We will study the conditions for the existence of stable orbits through the analysis of the effective potential

$$V_{eff} = r^{\frac{4(\gamma-1)}{\gamma}} + \ell^2 r^{\frac{2(\gamma-2)}{\gamma}} \tag{19}$$

If

$$\left(\frac{\gamma^2}{\gamma^2+4\gamma-4} - p_0 \frac{\gamma}{4-5\gamma} r^{\Gamma+2}\right) > 0 \tag{20}$$

then V(r) has a minimum for

$$r^2 = \frac{2-\gamma}{2(\gamma-1)} \ell^2 \tag{21}$$

which implies that no circular orbits can exist out of the range $1 < \gamma < 2$, consistently with eq. (20) if $p_0 > 0$. Also, when $p_0 = 0$. V_{eff} always has the minimum (21) for $1 < \gamma < 2$.

For circular motion ($\dot{r} = 0 = \ddot{r}$), the orbital velocity defined by

$$v^2_{orb} = r^2 \frac{\dot{\phi}^2}{\dot{t}^2} \tag{22}$$

is given by

$$v^2 = r\ e^{2\Phi}\ \frac{d\phi}{dr} \tag{23}$$

From (8.c) we get

$$v^2_{orb} = \frac{2(\gamma-1)}{\gamma}\ g_0^2 r^{\frac{4(\gamma-1)}{\gamma}} \tag{24}$$

Notice that v_{orb} varies as r^s with $0 < s < 1$.

If γ is nearly 1, the orbital velocity increases slightly as r grows which is precisely the behaviour reported for the velocity profile for the arms of certain spiral galaxies (Rubin 1983). Nevertheless, it has to be pointed out that the model is realistic only if $p_0 = 0$.

As the value of γ approaches 2, the model is more suitable for the central luminous core of those galaxies, as the orbital velocity is then very close to a linear function of the distance from the nucleus.

IV. DISCUSSION

The geometry studied here is a candidate to adequately represent the gravitational fields due to extremely dense objects. It was shown that it can fulfill the requirements to be a realistic model: the conditions of positive energy density and pressure restrict the parameter γ to $1 < \gamma < 2$ and negative values of the integration constant p_0 guarantee that the model naturally accomodates a radius for which the pressure vanishes and that can then be intepreted as the surface of the star.

For values of $\gamma \simeq 1$ and $\gamma \simeq 2$, this family of solutions also fits the observational data for velocity profiles of spiral galaxies. Let us take for example the SC galaxy NGC801 (Rubin, 1983). From the center of the galaxy up to a distance r of approximately 5 kpc, the rotational velocities v_{rot} of stars in the field of luminous matter - which dominates in this region - are almost linear functions of r. This function can be approximated by (23), with γ very close to 2. An inspection of Table I shows us that positive values of p and ℓ are guaranteed either for positive or negative values of p_0. In both cases, we see also that p_0 has to be very small so that r_2 (in case $p_0 > 0$) or r_3 ($p_0 < 0$) can fit radial distances values of the order of 5 kpc[3].

For $r > 5$ kpc, v_{rot} becomes almost constant and again a good approximation can be obtained for $\gamma \simeq 1$. As in spiral galaxies luminosity decreases exponentially with the increasing of the radial distance, in this region dark matter largely dominates and v_{orb} is kept from decreasing due to its gravitational attraction. Again, table I reveals that for both $p_0 > 0$ and $p_0 < 0$, for radial distances larger than a certain critical radius (r_2 or r_3), either ρ or p is negative. Therefore, as formerly proposed (Hojman et al 1989), only the case $p_0 = 0$ is suitable for modeling the dark matter dominated region of spiral galaxies.

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[3]For example, for $\gamma = 1.9$, $r_2 = (\frac{1}{2p_0})^{2/5}$.

TABLE I

Analysis concerning the range of values of γ for which both the pressure and the density are positive

	$p_0 > 0$		$p_0 < 0$	
γ interval	p	ρ	p	ρ
$(-\infty\ ,\ -2-2\sqrt{2})$	>0	<0	$>0\ ,\ r<r_3$ $<0\ ,\ r<r_3$	$>0\ ,\ r>r_4$ $<0\ ,\ r<r_4$
$(-2-2\sqrt{2}\ ,\ -2)$	$>0\ ,\ r>r_1$ $<0\ ,\ r<r_1$	$>0\ ,\ r<r_2$ $<0\ ,\ r>r_2$	<0	>0
$(-2\ ,\ 0)$	$>0\ ,\ r>r_1$ $<0\ ,\ r<r_1$	>0	<0	$>0\ ,\ r<r_4$ $<0\ ,\ r>r_4$
$(0\ ,\ 2/3)$	$>0\ ,\ r>r_1$ $<0\ ,\ r<r_1$	>0	<0	$>0\ ,\ r<r_4$ $<0\ ,\ r>r_4$
$(2/3\ ,\ -2+2\sqrt{2})$	$>0\ ,\ r>r_1$ $<0\ ,\ r<r_1$	$>0\ ,\ r<r_2$ $<0\ ,\ r>r_2$	<0	>0
$(-2+2\sqrt{2}\ ,\ 1)$	>0	<0	$>0\ ,\ r<r_3$ $<0\ ,\ r>r_3$	$>0\ ,\ r>r_4$ $<0\ ,\ r<r_4$
$(1\ ,2\ )$	>0	$>0\ ,\ r<r_2$ $<0\ ,\ r>r_2$	$>0\ ,\ r<r_3$ $<0\ ,\ r>r_3$	>0
$(2\ ,\ +\infty)$	>0	$>0\ ,\ r<r_2$ $<0\ ,\ r>r_2$	$>0\ ,\ r<r_2$ $<0\ ,\ r>r_3$	>0

$$r_1 \equiv \left(|\Theta|\, p_0\right)^{1/\Gamma+2} ; r_2 \equiv \left(\Sigma / |p_0|\Delta|\right)^{1/\Gamma+2} ; r_3 \equiv \left(\Theta / |p_0|\right)^{1/\Gamma+2} ; r_4 \equiv \left(\Sigma / |p_0|\Delta\right)^{1/\Gamma+2}$$

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