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FROM GEODESICS TO METRIC

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ABSTRACT

The inverse problem of General Relativity is solved; Einstein equations for a static spherically - symmetric perfect fluid are integrated from the given (measurable) orbital velocity of circular geodesics, which thus fixes both the geometry generated by and the equation of state of the fluid. The features of the matter content of a region are therefore determined from the (observed) rotational velocities in its surroundings.

Key-words: Inverse problem in General Relativity; Spiral galaxies; Spherical symmetric relativistic model.

Relativistic static spherically-symmetric ideal fluids describe adequately certain final stages of stellar evolution. In fact, when nuclear fuel has been exhausted, radiation pressure can no longer counteract gravitational forces and the star undergoes gravitational collapse resulting in a highly dense distribution of matter which renders a relativistic description unavoidable [1]. On the other hand, under the depicted conditions viscosity and heat conduction can be ignored and the stellar fluid can be modeled by

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu - pg_{\mu\nu} \qquad (1)$$

where p is the pressure, ρ is the matter density, $g_{\mu\nu}$ is the metric and u^μ is the 4-velocity of the fluid particles.

If Einstein equations are written for the most general static spherically symmetric line element

$$ds^2 = e^{2\Phi(r)}dt^2 - e^{2\nu(r)}dr^2 - r^2d\Omega^2 \qquad (2)$$

one finds after some algebra that the problem problem can be reduced to (the so called Tolman - Oppenheimer - Volkov equation for stellar interiors)

$$\frac{dp}{dr} = - \frac{(p+\rho)(2m+pr^3)}{2r(r-2m)} \qquad (3)$$

where

$$m(r) = 4\pi\int_0^r r^2\rho(r)dr \tag{4}$$

is the (geometric) mass up to a distance r from the center of forces.

Eq. (3) is the relativistic generalization of the Newtonian equation of hydrostatic support [2].

The relativistic analogue of Laplace equation,

$$\frac{d\Phi}{dr} = \frac{2m(r) + pr^3}{2r[r-2m(r)]} \tag{5}$$

is also deduced from Eqs. (2) and it serves to find one of the metric coefficients (roughly speaking, Φ is the gravitational potential). The other one is given by

$$e^{2\nu(r)} = \left[1 - \frac{2m(r)}{r}\right]^{-1} \tag{6}$$

The system of differential equations must be supplemented with an equation of state $p = p(\rho)$.

The usual method of integration consists in arbitrarily singling out an equation of state; for some lucky choices, Eq. (3) can be analytically solved. Once p(r) has been found, $\rho(r)$ is determined through the equation of state, and the metric coefficients from Eqs. (5) and (6).

An alternative method, which presents some mathematical advantages, in which the unknown functions are determined in dependently, rather than sequentially, has been deviced [3,4]. As we will prove, that approach is full of physical significance in the sense that the geometry as well as the thermodynamics are determined from a single measurable quantity.

Let us briefly recall the above mentioned scheme. Given the auxiliary function

$$G(r) = - \frac{r - 2m(r)}{p + \frac{1}{r^2}} \tag{7}$$

it can be proven [4] that

$$p(r) = - \frac{1}{r^2} + e^{H(r)}\left[p_0 - 4 \int \frac{e^{-H(r)}}{r^3 - G} \, dr\right] \tag{8.a}$$

$$\rho(r) = \frac{1}{r^2}\left[Gp' + G'p + \frac{1}{r^3}(r^3 + rG' - 2G)\right] \tag{8.b}$$

$$e^{2\Phi(r)} = \frac{1}{r} \exp\left(- \int \frac{r^2}{G} \, dr\right) \tag{8.c}$$

$$e^{2\nu(\nu)} = \left(1 - \frac{2m(r)}{r}\right)^{-1} \tag{8.d}$$

$$\text{where } H(r) = \int \frac{(G+r^3)(G'+r^2)}{G(r^3-G)}\, dr \qquad (9)$$

and p_0 is an integration constant.

Notice that once G is (arbitrarily) chosen, the metric coefficients and the the equation of state given by expressions (8) are univocally determined.

In order to get some physical insight regarding the meaning of G let us study the geodesic motion of a test particle in the metric given by (2).

As t and ϕ are cyclic variables, two constants of the motion can be readily found:

$$e^{2\Phi}\dot{t} = \varepsilon = \text{constant} \qquad (10)$$

and

$$r^2\dot{\phi} = \ell = \text{constant} \qquad (11)$$

where the dot stands for differentiation with respect to the arc - length.

Rather than writing the r component of the geodesic, it is simpler to consider the normalization condition uu = 1 for the 4-velocity, with $\theta = \frac{\pi^2}{2}$:

$$e^{-2\Phi}\varepsilon^2 - e^{2\nu}\dot{r} - \frac{\ell^2}{r^2} = 1 \qquad (12)$$

(use has been made 10 and 11).

Circular orbits are found by imposing $\dot{r} = 0 = \ddot{r}$:

$$e^{-\Phi}\varepsilon^2 = 1 + \frac{\ell^2}{r^2} \tag{13}$$

$$e^{-2\Phi}\Phi'\varepsilon^2 = \frac{\ell^2}{r^3} \tag{14}$$

Any metric must be well behaved at infinity if it describes a (localized) physical objects. By virtue of Birkhoff theorem, a junction of the solution considered and Schwarzschild exterior solution must be performed for some r_0.

The orbital velocity of the test particle as measured by an observer at infinite is given by

$$v^2 = re^{2\Phi}\Phi' \tag{15}$$

Then,

$$e^{2\Phi} = \int 2\,\frac{v^2}{r}\,dr \tag{16}$$

From eqs. (8.c) and Eq. (15), the relation between G and v can be found

$$G = -\frac{r^3}{1 + 2r\Phi'} \tag{17}$$

It is worthwhile to reconsider in the present frame some physically relevant solutions.

Exterior Schwarzchild solution can be recovered by putting $p_0 = 0$ and $v^2 = M/r$ (M a constant) in expressions above or equivalent $G = - r^2(r - 2M)$ in expressions (8) (see ref. 3).

Observational data regarding the orbital velocity of many spiral galaxies have been reported during the last decade suggesting the presence of great amounts of non - luminous matter surrounding their central luminous core [5]. Exact relativistic models taking into account the above observations can be constructed [6,7]. The measured profile of the orbital velocity v_{orb} of spiral galaxies [5] shows that $v_{orb} \sim r^a$, with $0 < a < 1$. In that case, from (Eq. 17) we get

$$G = - \frac{1}{1 + 2a} r^3 \tag{18}$$

and the corresponding solution is given by [8,9]

$$p = p_0 r^\Gamma + \frac{\theta}{r^2} \tag{19.a}$$

$$\rho = p_0 \Delta r^\Gamma + \frac{\Sigma}{r^2} \tag{19.b}$$

$$e^{2\Phi} = r^{\frac{2a}{a}} \tag{19.c}$$

$$e^{-2\nu} = 1 - \sum + \frac{1}{1 + 2a} p_0 r^{\Gamma+2} \tag{19.d}$$

where

$$\Gamma = \frac{2a(1-a)}{2 - a} \tag{20}$$

$$\theta = \frac{a^2}{1 + 2a - a^2} \tag{21}$$

$$\Delta = \frac{3 - a}{1 + a} \tag{22}$$

$$\sum = \frac{1(2-a)}{1 + 2a - a^2} \tag{23}$$

To summarize, we have exhibited a scheme that allows the complete determination of the geometry and the thermodynamics of a static spherically symmetric distribution of an ideal fluid, from the measured orbital velocity profile. In other words, we have succeeded in solving explicitly the inverse problem of General Relativity by finding the metric from the geodesics.

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REFERENCES

[1] S. Weinberg, "Gravitation and Cosmology", Wiley (1972), ch. 11.

[2] R. Adler, M. Brazin and M. Schiffer, "Introduction to General Relativity", Mc.Graw Hill (1975), ch. 14.

[3] R. Hojman and J. Santamarina, J. Math. Phys. 25, 1973 (1984).

[4] S. Berger, R. Hojman and J. Santamarina, J. Math. Phys. 28, 2949 (1987). A generalization of this method for charged spheres was recently presented by A. Patriño and H. Rago, GRG 21, 637 (1989).

[5] V. Rubin, Science, 220, 1339 (1983).

[6] R. Hojman, L. Peña and N. Zamorano, "Exact Solutions for Galactic Dynamics in a dark matter background", "Proceedings of the 12th Symposium on General Relativity, (1988) (in press).

[7] R. Hojman, Ligia M.C. Rodrigues and F.D. Sasse, "Spherically symmetric relativistic models for spiral galaxies and dense stars", CBPF, Notas de Física 035/90, (submitted for publication).

[8] This solution was first obtained by R.C. Tolman, Phys. Rev. 55, 364 (1939).

[9] B. Kuchowicz, Acta Phys. Polon. 33, 541 (1968).