ISSN 0029-3865

CBPF-NF-030/89

GEOMETRICAL PROPERTIES OF AN INTERNAL LOCAL OCTONIONIC SPACE IN A NON-RIEMANNIAN MANIFOLD

by

Sirley MARQUES-BONHAM

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

* This work was supported by the Foundation for Research of Rio de Janeiro (FAPERJ), Brasil.

Abstract

It is summarized here a geometrical treatment on a flat tangent space local to a generalized complex, quaternionic, and octonionic space-time.

I. Introduction

It was shown recently [6] that it is possible to develop a non-Riemannian manifold over each of the algebras stated at the Hurwitz theorem, namely: the real (**R**) algebra, the complex (**C**) algebra, the quaternionic (**Q**) algebra, and the octonionic (**O**) or Cayley algebra [1]. An important application of this theory is that we can geometrically construct a unified field theory including the gravitation, the electromagnetism, and the Yang-Mills fields, which are usually treated in gauge field theories on the (flat) Minkowski space-time. Therefore, the General Relativity is a theory developed on a Riemannian manifold over the real algebra [2], the Einstein-Schroedinger non-symmetric theory is developed over the complex algebra [3] [4], and a matrix (Yang-Mills) theory [5], for the case of dimension n = 2, can be reinterpreted as the quaternions algebra. The extension of the theory to include (split) octonions algebra was developed by S. Marques and C.G. Oliveira in ref. [6]. They also developed a tentative theory on the local tangent space [7], which was later modified by S. Marques [8] to develop a generalized Dirac equation on this extended manifold[1].

The present work exposes, in a compact form, the geometrical properties of a tangent space local to the non-Riemannian manifold connected to each of the algebras: **R**, **C**, **Q**, and **O**, reffered to above. With this objective, the main geometrical properties are organized on a table, so that we can compare the developing generalization of the manifold when we go from the **R**-algebra to the **O**-algebra. This form of presentation will avoid a long and repetitive reading. The repetition of symbols can be noticed throughout to keep a mnemonic similarity when we go from the real theory to the octonionic (Cayley) theory, even though it implicates calculations specific to each of them.

II. The Hurwitz Theorem and the Cayley Algebra

A composition algebra over the real numbers is defined as an algebra A with the identity element and with a non-degenerated quadratic form[2] Q defined over A such that Q permits

[1]It will not be described here each of these four theories, which are thoroughly studied in refs. [6], [7], and [8]

[2]A quadratic form is non-degenerated if the scalar product associated to it is non-degenerated, i.e., if $x \neq 0$, there exists y such that $(x, y) \neq 0$.

composition, i.e., for each element x, y of A,

$$Q(xy) = Q(x)Q(y) \quad , \qquad Q(\alpha x) = \alpha^2 Q(x) \quad , \tag{2.1}$$

where α is a real number. We can associate to this quadratic form a scalar product:

$$(x,y) = \frac{1}{2}[Q(x+y) - Q(x) - Q(y)] \quad , \tag{2.2}$$

which is a symmetrical non-degenerated bilinear form. The norm of a vector x is defined as:

$$N(x) = Q(x) \quad , \quad \text{i.e.,} \quad \mathrm{N(x) = (x,x)} \quad . \tag{2.3}$$

The Hurwitz theorem states that: "the only composition algebras on the real numbers, except isomorfisms, are the algebra of real numbers **R** (dim. $n = 1$), the algebra of complex numbers **C** (dim. $n = 2$), the algebra of quaternions **Q** (dim. $n = 4$), and the algebra of octonions **O** (dim. $n = 8$). From these the quaternions are non-commutatives and the octonions are non-commutatives and non-associatives". The composition algebra is said to be a division algebra if the quadratic form Q is anisotropic, i.e., if $Q(x) = 0$), implies $x = 0$. If not, the algebra is called "split".

A. The Algebra of Quaternions

The quaternion algebra **Q** has four generators $\{e_0, e_i, i = 1,2,3\}$, where e_0 is the identity element of the algebra. They satisfy the relation,

$$e_i e_j = \epsilon_{ijk} e_k - \delta_{ij} e_0 \quad . \tag{2.4}$$

Every element x of **Q** can be written in terms of the generators as:

$$x = x_0 e_0 + x_i e_i \quad , \qquad \overline{x} = x_0 e_0 - x_i e_i \quad , \quad i = 1,2,3 \quad , \tag{2.5}$$

where $\overline{x}$ is the quaternionic conjugate of x. Therefore, the squared norm of x is given by:

$$x\overline{x} = \overline{x}x = (x_0{}^2 + x_i{}^2)\mathbf{1} \quad , \tag{2.6}$$

where $\mathbf{1} \equiv e_0$.

A possible realization of the quaterionic algebra that will be used in this work, is through the Pauli matrices:

$$w_i = i^{-1}\sigma_i \quad , \qquad i = 1,2,3 \quad ,$$
$$w_0 = \sigma_0 = \mathbf{1} \quad . \tag{2.7}$$

where $\sigma_i, i = 1,2,3$, are the Pauli matrices and w_i, and w_0, are the (matrix) generators of a quaternion algebra. Therefore, the "numbers" $\{w_0, w_i,\ i+1,2,3\}$, satisfy a product of the kind (2.4) above.

B. The Cayley Algebra. Realization via Zorn matrices

The octonions algebra has eight dimensions and its base-vectors $\{e_0, e_i,\ i = 1,...,7\}$, satisfy the product law:

$$e_0 e_i = e_i e_0 = e_i \quad ,$$
$$e_i e_j = \epsilon_{ijk} e_k - \delta_{ij} e_0 \quad , \tag{2.8}$$

ϵ_{ijk} is now an object completely skew-symmetric with seven non-zero matrix elements: ϵ_{123}, ϵ_{516}, ϵ_{624}, ϵ_{435}, ϵ_{471}, ϵ_{572}, ϵ_{673}. This algebra is also called Cayley algebra [1]. It is neither associative, nor commutative, but belongs to the class of alternative algebras with the property that, for any three octonions x, y, z, their associator is given by:

$$\{x, y, z\} = (xy)y - x(yz) \quad . \tag{2.9}$$

It changes sign when any two of its arguments change position. Also, x, y, z are called Cayley numbers.

Any octonion x can be written in terms of the base vectors as:

$$x = x_0 e_0 + x_i e_i, \quad \bar{x} = x_0 e_0 - x_i e_i \quad , \quad i = 1,...,7 \quad , \tag{2.10}$$

where $\bar{x}$ is the octonionic conjugate of x. The squared norm of x is then defined as:

$$Q(x) = x\bar{x} = \bar{x}x = ({x_0}^2 + {x_i}^2)\mathbf{1} \quad , \tag{2.11}$$

where $\mathbf{1} = e_0$ is the identity element of the algebra.

The Cayley algebra with the base given in (2.8), belongs to the class of division algebras (real base), but it also can be presented as an split-algebra if we use a new base in the complex domain. This is given as:

$$u_0 = \tfrac{1}{2}(e_0 + ie_7) \ , \ u_i = \tfrac{1}{2}(e_i + e_{i+3}) \ ,$$
$$u_0^* = \tfrac{1}{2}(e_0 - ie_7) \ , \ u_i^* = \tfrac{1}{2}(e_i - ie_{1+3}) \ , \qquad (2.12)$$
$$i = 1,2,3.$$

From this definition follows the multiplication table:

$$\begin{array}{lll} u_i^* u_j^* = \epsilon_{ijk} u_k \ , & u_i u_j = \epsilon_{ijk} u_k^* \ , & \\ u_i u_j^* = -\delta_{ij} u_0 \ , & u_i^* u_j = \delta_{ij} u_0^* \ , & \\ u_i^* u_0^* = 0 \ , & u_i u_0 = 0 \ , & \\ u_0^* u_i^* = u_i^* \ , & u_0 u_i = u_i \ , & u_0^* u_0 = u_0 u_0^* = 0 \ , \\ u_0^{*2} = u_0^* \ , & u_0^2 = u_0 \ , & \\ u_i^* u_0 = u_i^* \ , & u_i u_0^* = u_i \ , & \\ u_0 u_i^* = 0 \ , & u_0^* u_i = 0 \ , & \end{array} \qquad (2.13)$$

It is of our interest a convenient realization for the elements of the base (u_0, u_i, u_0^*, u_i^*), through the Pauli matrices. This is possible by means of the following identification:

$$u_0 = \begin{pmatrix} 0 & 0 \\ 0 & w_0 \end{pmatrix} \ , \quad u_0^* = \begin{pmatrix} w_0 & 0 \\ 0 & 0 \end{pmatrix} \ ,$$
$$\qquad\qquad i = 1,2,3 \qquad (2.14)$$
$$u_i = \begin{pmatrix} 0 & 0 \\ w_i & 0 \end{pmatrix} \ , \quad u_i^* = \begin{pmatrix} 0 & -w_i \\ 0 & 0 \end{pmatrix} \ .$$

w_0 and w_i were given in (2.7). Therefore, for any octonion A, we have:

$$A = au_0^* + bu_0 + x_i u_i^* + y_i u_i = \begin{pmatrix} a & -\vec{x} \\ \vec{y} & b \end{pmatrix} \ . \qquad (2.15)$$

The conjugate of A is defined by:

$$\overline{A} = bu_0^* + au_0 - x_i u_i^* - y_i u_i = \begin{pmatrix} b & \vec{x} \\ -\vec{y} & a \end{pmatrix} \ . \qquad (2.16)$$

The norm of A is then given by:

$$A\overline{A} = \overline{A}A = (ab + \vec{x}.\vec{y}).\mathbf{1} \quad , \tag{2.17}$$

where $\mathbf{1}$, the identity element of the algebra, is now written as: $\mathbf{1} = 1.u_0^* + 1.u_0$. The "matrices" above are called "Zorn matrices" or "vectorial matrices" ref. [1]. The product of any two of these matrices is defined by:

$$AB = \begin{pmatrix} a & -\vec{x} \\ \vec{y} & b \end{pmatrix} \begin{pmatrix} c & -\vec{u} \\ \vec{v} & d \end{pmatrix} = \begin{pmatrix} ac - \vec{x}.\vec{v} & -(a\vec{u} + d\vec{x} + \vec{y} \times \vec{v}) \\ c\vec{y} + b\vec{v} + \vec{x} \times \vec{u} & bd - \vec{y}.\vec{u} \end{pmatrix} \quad , \tag{2.18}$$

which guarantees the non-associativity of the product. The multiplication table for the u's is reproduced in this Zorn-matrix notation. Some other properties for the octonions in this Zorn notation are:

$$\begin{aligned} &A\mathbf{1} = \mathbf{1}A = A \quad , \qquad \overline{AB} = \overline{B}\,\overline{A} \quad , \\ &A + \overline{A} = Tr(A)\mathbf{1} \quad , \quad Tr(AB) = Tr(BA) \quad , \end{aligned} \tag{2.19}$$

where the trace (Tr) operation is taken on the Zorn matrices. From (2.19), and using the definition of an associator given in (2.9), we have that:

$$Tr[(AB)C] = Tr[A(BC)] = Tr[ABC] \quad , \tag{2.20}$$

and that the Tr operation on a product of Zorn matrices follows the cyclic order of the factors.

In general, the complex (split) Cayley algebra contains seven Euclidean (vectorial) sub-algebras, as well as seven quaternion sub-algebras. This property follows from the multiplication rules of the complex (split) base given in (2.13) and from the definition (2.12). Finally, observe that we always have been using $A \equiv Z(A)$, the Zorn matrix of the octonion A, as the Zorn algebra is isomorphic to the algebra of the split octonions.

III. Internal Transformations for Quaternions and Octonions

When we have a complex unimodular matrix A,

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

then its inverse is defined as:

$$A^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

The conjugation operation of quaternions and octonions is equivalent to the inversion of a unimodular matrix.

A. Quaternionic Transformations

Let us take U as the quaternionic transformation matrix, or "Q-transformations",

$$U = m_0 w_0 + m_i w_i \quad , \tag{3.1}$$

which is also a transformation matrix of the $SU(2)$ group, i.e., it satisfies the condition

$$U^{-1} = U^{\dagger} \quad , \qquad \text{and} \qquad detU = 1 \quad , \tag{3.2}$$

where $U^{\dagger} = U^{T*}$ is the Hermitian conjugate of U. As U is a unimodular matrix, we also have that:

$$U^{-1} \equiv \overline{U} = m_0 w_0 - m_i w_i \quad , \tag{3.3}$$

where $\overline{U}$ is the quaternionic conjugate of U. Thus, the following expression is valid:

$$\overline{U}U = U\overline{U} = U^{-1}U = UU^{-1} = (m_0^2 + m_i^2)w_0 = 1 \quad , \tag{3.4}$$

where

$$m_0^2 + m_i^2 = detU = 1 \quad . \tag{3.5}$$

We must have for the general form for U :

$$U = e^{i\vec{\lambda}.\vec{\sigma}} = e^{-\vec{\lambda}.\vec{w}} \quad , \tag{3.6}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ are real parameters.

We define a Q-transformation of a quaternion K by:

$$K' = UKU^{-1} \quad . \tag{3.7}$$

If K is written in terms of components as:

$$K = \alpha_0 w_0 + \alpha_i w_i \quad , \tag{3.8}$$

K' is given in terms of components, using (3.7), as:

$$K' = (m_0\alpha_0 m_0 + m_0\alpha_k m_k + m_k\alpha_0 m_k - m_k\alpha_k m_0)w_0$$

$$+[m_0\alpha_p m_0 - m_0\alpha_0 m_p + m_p\alpha_0 m_0 + m_k\alpha_k m_p$$

$$+ \epsilon_{kip}(m_k\alpha_l m_0 - m_0\alpha_k m_l) - \epsilon_{ijk}\epsilon_{klp}m_i\alpha_j m_l]w_p \quad . \tag{3.9}$$

The symmetry group of this transformation is the $SU(2)$ group, which is homomorphic to the rotation group, O_3. This can be expressed through the relation:

$$Uw_iU^{-1} = R_{ij}w_j \quad , \qquad i,j, = 1,2,3, \tag{3.10}$$

where R_{ij} is the transformation matrix for the O_3 group. In terms of components this can be written as:

$$m_0^2\delta_{ip} + m_i m_p - 2\epsilon_{ijp}m_0 m_j + \epsilon_{ijk}\epsilon_{klp}m_l m_j = R_{ip} \quad . \tag{3.11}$$

For the local Q-transformations, which are used in the space-time manifold connected to an internal quaternionic space, we must have

$$U = U(x) = m_0(x)w_0 + m_i(x)w_i \quad , \tag{3.12}$$

and

$$U(x) = e^{-\vec{\lambda}(x).\vec{w}} \quad , \tag{3.13}$$

where now, $\vec{\lambda}(x) = (\lambda_1(x), \lambda_2(x), \lambda_3(x))$ are real functions. Also, the coefficents of K in (3.8) are functions of space-time coordinates.

B. Octonionic Transformations

When we consider octonions, we may define an octonionic transformation law, or O-transformations by means of the octonion **U**, in this case, split-O: split:

$$\mathbf{U} = p_0u_0^* + p_iu_i^* + q_0u_0 + q_iu_i, \quad , \qquad i = 1,2,3, \tag{3.14}$$

where $p_0,\ p_i,\ q_0,\ q_i,\ \ i = 1,2,3$, are real coefficients. The conjugate of **U** is then given by:

$$\overline{\mathbf{U}} = q_0u_0^* - p_iu_i^* + p_0u_0 - q_iu_i \quad . \tag{3.15}$$

Comparing with the situation for the quaternions, we can define:

$$\mathbf{U}^{-1} \equiv \overline{\mathbf{U}} \ . \tag{3.16}$$

Actually,

$$\overline{\mathbf{U}}\mathbf{U} = \mathbf{U}\overline{\mathbf{U}} = \mathbf{U}^{-1}\mathbf{U} = \mathbf{U}\mathbf{U}^{-1} = (p_0 q_0 + p_i q_i)\mathbf{1} \ , \tag{3.17}$$

which will be equal to **1** if

$$p_0 q_0 + p_i q_i = 1 \ . \tag{3.18}$$

In this case, we have for **U**,

$$\mathbf{U} = e^{-\vec{\delta}.\vec{u}^* - \vec{\gamma}.\vec{u}} \tag{3.19}$$

where $\vec{\delta} = (\delta_1, \delta_2, \delta_3)$ and $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ are real parameters.

For an extension of the case valid for quaternions, the O-transformations of an octonion **K** is defined by:

$$\mathbf{K}' = \frac{1}{2}[(\mathbf{UK})\mathbf{U}^{-1} + \mathbf{U}(\mathbf{KU}^{-1})] \ . \tag{3.20}$$

We can show easily that

$$(\mathbf{UK})\mathbf{U}^{-1} = \mathbf{U}(\mathbf{KU}^{-1}) \ , \tag{3.21}$$

which simplifies the relation (3.21) above to

$$\mathbf{K}' = \mathbf{UKU}^{-1} \ . \tag{3.22}$$

If **K** is written in terms of components by

$$\mathbf{K} = \rho_0 u_0^* + \rho_i u_i^* + \kappa_0 u_0 + \kappa_i u_i \ , \tag{3.23}$$

we have, from (3.22),

$$\begin{aligned} \mathbf{K}' = {} & (p_0\rho_0 q_0 + p_0\rho_k q_k + p_k\kappa_0 q_k - p_k\kappa_k q_0)u_0^* + (q_0\kappa_0 q_0 + q_0\kappa_k p_k + q_k\rho_0 p_k - q_k\rho_k p_0)u_0 \\ & +[p_0\rho_p p_0 - p_0\rho_0 p_p + p_p\kappa_0 p_0 + p_p\kappa_k p_k + \epsilon_{klp}(p_0\kappa_l q_k - q_0\kappa_k q_l) - \epsilon_{ijk}\epsilon_{klp}p_i\rho_j q_l]u_p^* \\ & + [q_0\kappa_p q_0 - q_0\kappa_0 q_p + q_k\rho_0 p_k + q_k\rho_k q_p + \epsilon klp(p_k\rho_l q_0 - p_0\rho_k p_l) - \epsilon_{ijk}\epsilon_{klp}q_i\kappa_j p_l]u_p \ . \end{aligned} \tag{3.24}$$

We are going to define a similar relation to (3.10), for O-transformations, which is:

$$\mathbf{U}(u_i^* + u_i)\mathbf{U}^{-1} = k_{ij}u_j^* - l_{ij}u_j \ , \quad i,j = 1,2,3 \ . \tag{3.25}$$

In terms of components this is:

$$
\begin{aligned}
p_0^2\delta_{ip} + p_i p_p - \epsilon_{ilp} q_j (p_0 + q_0) + \epsilon_{ijk}\epsilon_{klp} p_j q_l = k_{ip} \quad , \\
q_0^2\delta_{ip} + q_i q_p - \epsilon_{ilp} p_j (p_0 + q_0) + \epsilon_{ijk}\epsilon_{klp} q_j p_l = l_{ip} \quad ,
\end{aligned}
\tag{3.26}
$$

with the additional condition

$$p_0 q_i - p_i q_0 = 0 \quad . \tag{3.27}$$

From the relations (3.26), we can see clearly that in the limit $q_0 \to p_0$, $q_i \to p_i$, the O-transformations, through **U**, will be equivalent to the Q-transformations, through U. Therefore, the O-transformations are homomorphics to the rotation group O_3 and so, they must be $SU(2)$-like (in our case). Besides, we observe a certain symmetry with regard to the positioning of the components terms, in expressions (3.26).

In the case of local O-transformations, which we use in the octonionic space-time manifold, we must have:

$$\mathbf{K} = \mathbf{K}(x) = \rho_0(x) u_0^* + \rho_i(x) u_i^* + \kappa_0(x) u_0 + \kappa_i(x) u_i \quad , \tag{3.28}$$

$$\mathbf{U} = \mathbf{U}(x) = p_0(x) u_0^* + p_i(x) u_i + q_0(x) u_0 + p_i(x) u_i \quad , \tag{3.29}$$

and

$$\mathbf{U}(x) = e^{-\vec{\delta}(x).\vec{u}^* - \vec{\gamma}(x).\vec{u}} \quad , \tag{3.30}$$

where $\vec{\delta}(x)$ and $\vec{\gamma}(x)$ are now functions of the space-time coordinates.

IV. The Tangent Space

The definition of a tangent space local to the Riemannian or non-Riemannian space-time manifold starts with the correspondence principle. For the General Relativity it states that at each point of the curved (Riemannian) space-time, there exists a local tangent space, with the structure of a flat space-time, where the metric is given by the Minkowiski tensor, η_{ab}. The line element on the curved space-time is then, locally equivalent to the one on the flat spacetime. This principle can be extended to non-Riemannian space-time manifolds, as for example, the complex non-Riemannian manifold of the Einstein-Schroedinger theory (ref. [3]) used here. One way of doing this, is to define complex vierbeins This has the advantage of keeping the metric on the tangent space as the Minkowski tensor.

The quaternionic theory is the next one permited by the Hurwitz theorem. It stems from a more general matrix theory (ref. [5]), which includes the n-dimensional Yang-Mills field. This quaternionic theory corresponds to the bidimensional matrix theory, with realization via Pauli matrices. A reasoning similar to the complex theory was used when we define quaternionic vierbeins on the non-Riemannian manifold.

The octonionic theory is described through the vectorial Zorn matrices and also, with realization through the Pauli matrices. This was done due the obvious possibility to reobtain the quaternionic theory in a convenient limit, which makes it physically reasonable. It is defined there, octonionic vierbeins on the octonionic non-Riemannian manifold.

Table I displays the geometrical objects obtained in the development of a geometrical theory for the tangent space local to a non-Riemannian space-time manifold, for each of the algebras stated in the theorem of Hurwitz. That means a space-time manifold to which is attached an internal space, namely: the space of the real numbers (dimension n = 1), the space of the complex numbers (dimension n = 2), the space of the quaternions (dimension n = 4), or the space of the (split) octonions (dimension n = 8). This is possible through the definitions of corresponding generalized vierbeins refferenced above. The internal space has its origin on a spinorial space defined through the wave functions of a particle of mass m in the presence of gravitation, electromagnetism, and Yang-Mills fields.

The transformation laws that rule each of these theories are analyzed in the next section. They will permit obtaining covariant derivatives which automatically define the connections, namely: space-time connections and internal connections. The corresponding curvatures are obtained in the standard way through the difference: $A^{\mu}_{|\alpha\beta} - A^{\mu}_{|\beta\alpha}$, where A^{μ} is a space-time vector with internal indices (**R**, **C**, **Q**, and **O** indices).

With the geometrical objects defined on the extended space-time manifold, it is then possible to determine a Minimal Action Principle, which will permit the obtaining of field equations (or dynamical equations) for the theory.

V. An Analysis of the Transformation Law in a Tangent Space Associated to the Complex, Quaternionic, and Octonionic spaces

A. The Complex-Theory

The transformation law on a tangent space local to the Riemannian space-time manifold follows the group $SL(2C)$. Therefore, for the case of a complex non-Riemannian tangent space treated here, there is also an internal (complex) part for the transformation law. Taking the vierbein $e^a_\mu(x)$, it will transform on the tangent space as,

$$e'^a_\mu(x) = L^a{}_b(x) e^b_\mu(x) \quad , \tag{5.1}$$

where $L^a{}_b(x)$ are the local Lorentzian rotational matrices, which have the property

$$L^T \eta L = \eta \quad . \tag{5.2}$$

The vierbein $e^a_\mu(x)$ is a complex object, $e^a_\mu = e^a_{\mu_R} + e^a_{\mu_I}$. Thus,

$$\overline{e}^a_\mu = e^a_{\mu_R} - e^a_{\mu_I} \tag{5.3}$$

is the conjugate of e^a_μ. Therefore, e^a_μ is an object with indices on the "internal" C-space. The transformation law for an object of this internal space, K, is

$$K' = U(1) K \quad , \tag{5.4}$$

where $U(1)$ stands for a unitary 1×1 (local) transformation matrix, $U(1) = e^{i\phi(x)}$, and

$$\overline{K}' = \overline{U}(1) \overline{K} \quad , \tag{5.5}$$

where $\overline{U}(1) = U^{-1}(1) = e^{-i\phi(x)}$. A more general transformation law for the complex vierbeins should be then:

$$\begin{aligned} e'^a_\mu(x) &= L^a{}_b(x)(U(1) e^b_\mu(x)) \quad , \\ \overline{e}'^a_\mu(x) \equiv e'^{*a}_\mu(x) &= L^a{}_b(x)(U^{-1} e^{*b}_\mu(x)) \quad . \end{aligned} \tag{5.6}$$

The covariant derivative for these vierbeins on the tangent space are now given by:

$$e^{a}_{\mu||\nu} = e^{a}_{\mu,\nu} + \Lambda_{\nu}{}^{a}{}_{b} e^{b}_{\mu} + C_{\nu} e^{a}_{\mu} \quad ,$$

$$e^{*a}_{\mu||\nu} = e^{*a}_{\mu,\nu} + \Lambda_{\nu}{}^{a}{}_{b} e^{*b}_{\mu} - C_{\nu} e^{*a}_{\mu} \quad , \tag{5.7}$$

where $\Lambda_{\nu}{}^{a}{}_{b}$ is the tangent connection related to the Minkowskian space and C_{ν} is the "internal connection". Their transformation laws are, respectively,

$$\Lambda'_{\nu} = L\Lambda_{\nu}L^{-1} - L_{,\nu}L^{-1} \qquad (\text{space} - \text{time transformations}) \quad ,$$

$$C'_{\nu} = U(1)C_{\nu}U^{-1}(1) - U_{,\nu}(1)U^{-1}(1) \qquad (\text{internal transformations}) \quad . \tag{5.8}$$

When we consider the particular case of the infinitesimal $U(1) = 1 + i\phi$, the internal connection C_{ν} transforms in first order, as

$$C'_{\nu} = C_{\nu} + i\phi \quad , \tag{5.9}$$

which is the same as a gauge transformation law for an electromagnetic potential. An internal curvature can be obtained:

$$P_{\mu\nu} = C_{\mu,\nu} - C_{\nu,\mu} \quad , \tag{5.10}$$

which can also be considered to correspond to the Maxwell electromagnetic tensor.

Field equations for the complex theory on the non-Riemannian complex manifold are obtained in ref. [8]. The corresponding ones for the complex vierbeins are presented in Table II.

B. The Quaternionic-Theory

The transformation law for the vectors on the tangent space is defined, as usual, through the Lorentzian rotational matrices. Therefore, a more general transformation law for a 2-dimentional matrix tangent vectors, with realization via Pauli matrices, shall be in the case of the vierbein $E^{a}_{\mu}(x)$:

$$E'^{a}_{\mu}(x) = L^{a}{}_{b}(x)(U(2)E^{a}_{\mu}(x)U^{\dagger}(2)) \quad . \tag{5.11}$$

The internal transformation matrix $U(2)$ is unitary and unimodular, and is then reinterpreted through the "quaternionic transformation matrix", U, defined above.

The internal covariant derivative, in this situation, is now called quaternionic covariant derivative or Q-derivative. For example, the Q-derivative of a quaternion $K(x)$ is defined as:

$$K_{\|\mu} = K_{,\mu} + [\Gamma_\mu, K] \quad , \tag{5.12}$$

where

$$\Gamma_\mu = \vec{B}_\mu . \vec{w} \tag{5.13}$$

is called the "quaternionic connection", or Q-connection, and it transforms as

$$\Gamma'_\mu = U\Gamma_\mu U^{-1} - U_{,\mu}U^{-1} \tag{5.14}$$

under the Q-transformations.

The operation of (total) covariant differentiation for the vierbein E^a_μ on the quaternionic tangent space, is defined as:

$$E^a_{\underset{+}{\mu}|\nu} = E^a_{\mu,\nu} - \Omega^\rho_{\mu\nu} E^a_\rho + \Lambda_\nu{}^a{}_b E^b_\mu + [\Gamma_\nu, E^a_\mu] \quad , \tag{5.15}$$

where $\Omega^\rho_{\mu\nu}$ and $\Lambda_\nu{}^a{}_b$ are the curved and tangent space connections, respectively.

A quaternionic curvature can be obtained in the standard way, performing the difference:

$$K_{\|\mu\nu} - K_{\|\nu\mu} = P_{\mu\nu}K - KP_{\mu\nu} \quad , \tag{5.16}$$

where

$$P_{\mu\nu} = \Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} - [\Gamma_\mu, \Gamma_\nu] \tag{5.17}$$

is the internal quaternionic curvature.

C. The Octonionic-Theory

The general transformation law for the octonionic vierbein $H^a_\mu(x)$ is defined as:

$$H'^a_\mu(x) = L^a{}_b(x)(\mathbf{U}H^a_\mu(x)\mathbf{U}^{-1}) \quad . \tag{5.18}$$

The "internal transformation" in this expression, corresponding to the part $(\mathbf{U}H^a_\mu(x)\mathbf{U}^{-1})$, is the "octonionic transformation", or O-transformation, defined in section III (above).

The octonionic covariant derivative, or **O**-derivative, of an octonion **K**, for example, is defined by:

$$\mathbf{K}_{\|\mu} = \mathbf{K}_{,\mu} + [\mathbf{\Gamma}_\mu, \mathbf{K}] \quad , \tag{5.19}$$

where $\mathbf{\Gamma}_\mu$ is the "octonionic connection", or **O**-connection, such that $\mathbf{K}_{\|\mu}$ transforms like an octonion under **O**-transformations[3]:

$$\mathbf{K}' = \mathbf{U}\mathbf{K}\mathbf{U}^{-1} \quad ,$$

$$\mathbf{K}'_{\|\mu} = \mathbf{U}\mathbf{K}_{\|\mu}\mathbf{U}^{-1} \quad , \tag{5.20}$$

and

$$\mathbf{\Gamma}'_\mu = \mathbf{U}\mathbf{\Gamma}_\mu\mathbf{U}^{-1} - \mathbf{U}_{,\mu}\mathbf{U}^{-1}. \tag{5.21}$$

The octonions $\mathbf{U}(x)$, define local-octonionic transformations that are homomorphic to the rotational group O_3.

It is imposed in this theory, that $\mathbf{\Gamma}_\mu$ is a trace-free Zorn matrix of the Yang-Mills type (Pauli matrices representation):

$$\mathbf{\Gamma}_\mu = -\vec{L}_\mu.\vec{u}^* - \vec{K}_\mu.\vec{u}$$

$$\equiv \begin{pmatrix} 0_2 & \vec{L}_\mu.\vec{w} \\ -\vec{K}_\mu.\vec{w} & 0_2 \end{pmatrix} \quad , 0_2 \equiv 2\times 2 \text{ zero matrix.} \tag{5.22}$$

The reason for this choice is that we can get an exact doubling of the quaternionic theory with realization via Pauli matrices (Yang-Mills theory), in the limit $\vec{K}_\mu \to \vec{L}_\mu$ in this split-**O** theory.

The total covariant derivative for the octonionic vierbein $H^a_\mu(x)$ can be written as:

$$H^a_{\underset{+}{\mu|\nu}} = H^a_{\mu,\nu} - \Omega^\rho_{\mu\nu}H^a_\rho + \Lambda_\nu{}^a{}_b H^b_\mu + [\mathbf{\Gamma}_\nu, H^a_\mu] \quad , \tag{5.23}$$

Again, an internal octonionic curvature can be obtained in the standard way, when we perform the difference:

$$\mathbf{K}_{\|\mu\nu} - \mathbf{K}_{\|\nu\mu} = \mathbf{P}_{\mu\nu}\mathbf{K} - \mathbf{K}\mathbf{P}_{\mu\nu} + \{\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_\nu, \mathbf{K}\} \quad , \tag{5.24}$$

---

[3]Note that expressions (5.20) and (5.21) have no parenteses, according to (3.22)

where:

$$\mathbf{P}_{\mu\nu} = \mathbf{\Gamma}_{\mu,\nu} - \mathbf{\Gamma}_{\nu,\mu} - [\mathbf{\Gamma}_{\mu}, \mathbf{\Gamma}_{\nu}] \tag{5.25}$$

is the internal octonionic curvature. In (4.25), the expression $\{\mathbf{\Gamma}_{\mu}, \mathbf{\Gamma}_{\nu}, \mathbf{K}\}$ is the associator of the fields $\mathbf{\Gamma}_{\mu}$, $\mathbf{\Gamma}_{\nu}$, and $\mathbf{K}$.

The expressions corresponding to the field equations obtained through a Minimal Variational Principle, for the quaternionic and the octonionic vierbeins, are obtained in ref. [8], and presented in Table II.

VI. Some Final Comments

It is important to mention, at this point, that we could have developed an octonionic geometrical theory on the real-base defined in (2.8). However, it would not be possible to recover the quaternionic (Yang-Mills) theory in a so obvious limit, as was done for the split-O theory. Also, a real-O theory would not be of Yang-Mills type, since the next group permited is the $SU(3)$, which would permit an internal (matrix) space with dimension $= 3^2 = 9$. The dimension of the real-O internal space is dimension $= 3^2 - 1 = 8$. Therefore, this realO theory would correspond to the introduction of a different field (say, internal-octonionic space) on the space-time manifold, which we could not currently interpret.

	The Riemannian theory (R-algebra)	The non-Riemannian theory (C-algebra)	The quartenionic theory (Q-algebra)	The octonionic theory (split-O algebra)
Line element	$ds^2 = dx^a dx^b \eta_{ab}$ $dx^a = h^a_\mu dx^\mu$	$ds^2 = dx^{*a} dx^b \eta_{ab}$ $dx^a = e^a_\mu dx^\mu, \quad dx^{*a} = e^{*a}_\mu dx^\mu$	$ds^2 = \frac{1}{2} Tr(dx^{\dagger a} dx^b \eta_{ab})$ $dx^a = E^a_\mu dx^\mu, \quad dx^{\dagger a} = E^{\dagger a}_\mu dx^\mu$	$ds^2 = \frac{1}{4} Tr(dx^{\dagger a} dx^b \eta_{ab})$ $dx^a = H^a_\mu dx^\mu, \quad dx^{\dagger a} = H^{\dagger a}_\mu dx^\mu$
Metric relation	$g^{\mu\nu} = h^\mu_a h^\nu_b \eta^{ab}$ $g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}$ $g_{\mu\nu} = g_{\nu\mu}$	$g^{\mu\nu} = e^{*\mu}_a e^\nu_b \eta^{ab}$ $g_{\mu\nu} = e^{*a}_\nu e^b_\mu \eta_{ab}$ $g^*_{\mu\nu} = g_{\nu\mu}$ $g_{\mu\nu} = g_{\underline{\mu\nu}} + i g_{\nu\mu}$	$G^{\mu\nu} = E^{\dagger\mu}_a E^\nu_b \eta_{ab}$ $G_{\mu\nu} = E^{\dagger a}_\nu E^b_\mu \eta_{ab}$ $G^\dagger_{\mu\nu} = G_{\nu\mu}$ $G_{\mu\nu} = q_{\mu\nu 0} w_0 + \vec{q}_{\mu\nu} . \vec{w}$	$G^{\mu\nu} = H^{\dagger\mu}_a H^\nu_b \eta_{ab}$ $G_{\mu\nu} = H^{\dagger a}_\nu H^b_\mu \eta_{ab}$ $G^\dagger_{\mu\nu}(s,r) = G_{\nu\mu}(s,r)$ $G_{\mu\nu}(x) = G_{\mu\nu}(s,r)$ $= s_{\mu\nu 0} u^*_0 + \vec{s}_{\mu\nu} . \vec{u}^* + r_{\mu\nu 0} u_0 + \vec{r}_{\mu\nu} . \vec{u}$ $s_{\mu\nu 0} = r_{\mu\nu 0}$
Orthogonality relations	$h^\mu_a h^b_\mu = \delta^a_b$ $h^\mu_a h^a_\nu = \delta^\mu_\nu$	$e^{*\mu}_a e^b_\mu = \delta^a_b$ $e^{*\mu}_a e^a_\nu = \delta^\mu_\nu$	$E^b_\mu E^{\dagger\mu}_c = E^\mu_c E^{\dagger b}_\mu = \delta^b_c w_0$ $E^{\dagger a}_\alpha E^\nu_a = E^{\dagger\nu}_a E^a_\alpha = \delta^\nu_\alpha w_0$	$H^b_\mu H^{\dagger\mu}_c = H^\mu_c H^{\dagger b}_\mu = \delta^b_c (u_0 + u^*_0)$ $H^{\dagger a}_\alpha H^\nu_a = H^{\dagger\nu}_a H^a_\alpha = \delta^\nu_\alpha (u_0 + u^*_0)$
Tangent connection	$\Lambda_{\mu ab} = -\Lambda_{\mu ba}$	$\Lambda_{\mu ab} = -\Lambda_{\mu ba}$ (*a real number*)	$\Lambda_{\mu ab} = (\lambda_{\mu ab}) w_0$ $\lambda_{\mu ab} = -\lambda_{\mu ba}$	$\Lambda_{\mu ab} = (q_{\mu ab})(u_0 + u^*_0)$ $q_{\mu ab} = -q_{\mu ba}$
Internal connection	(*none*)	$C_\mu = i A_\mu$ (*a pure imaginary number*)	$\Gamma_\mu = \vec{B}_\mu . \vec{w}$	$\Gamma_\mu = \vec{L}_\mu . \vec{u}^* + \vec{K}_\mu . \vec{u}$
Tangent curvature	$S_{\mu\nu} = \Lambda_{\mu,\nu} - \Lambda_{\nu,\mu} - [\Lambda_\mu, \Lambda_\nu]$	$S_{\mu\nu} = \Lambda_{\mu,\nu} - \Lambda_{\nu,\mu} - [\Lambda_\mu, \Lambda_\nu]$ (*a real number*)	$S_{\mu\nu} = (\Lambda_{\mu,\nu} - \Lambda_{\nu,\mu} - [\Lambda_\mu, \Lambda_\nu]) w_0$	$S_{\mu\nu} = (\Lambda_{\mu,\nu} - \Lambda_{\nu,\mu} - [\Lambda_\mu, \Lambda_\nu])(u^*_0 + u_0)$
Internal curvature	(*none*)	$P_{\mu\nu} = C_{\mu,\nu} - C_{\nu,\mu}$ (*an imaginary number*)	$P_{\mu\nu} = \Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} - [\Gamma_\mu, \Gamma_\nu]$	$P_{\mu\nu} = \Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} - [\Gamma_\mu, \Gamma_\nu]$

Table I. Principal geometrical properties for tangent spaces local to the space-time manifold, using R, C, Q, and split-O algebras.

	The Riemannian Theory (R-algebra)	The non-Riemannian Theory C-algebra
metric equations vierbeins equations	$g_{\mu\nu;\alpha} = 0 \longleftrightarrow h^a_{\mu;\alpha} = 0$, $h^a_{\mu;\alpha} = h^a_{\mu,\alpha} - \{{}^{\rho}_{\mu\alpha}\} h^a_\rho + \Lambda_\alpha{}^a{}_b h^b{}_\mu = 0$	$g_{\underset{+-}{\mu\nu\vert\alpha}} = 0 \longleftrightarrow e^a_{\underset{+}{\mu\vert\alpha}} = (e^{*a}_{\underset{-}{\mu\vert\alpha}})^* = 0$, $e^a_{\underset{+}{\mu\vert\alpha}} = e^a_{\mu,\alpha} - \Theta^\rho_{\mu\alpha} e^a_\rho + \Lambda_\alpha{}^a{}_b e^b_\mu + C_\alpha e^a_\mu = 0$,
metric equations vierbeins equations	$g^{\mu\nu}{}_{;\alpha} = 0 \longleftrightarrow h^\mu_{a;\alpha} = 0$, $h^\mu_{a;\alpha} = h^\mu_{a,\alpha} + \{{}^{\mu}_{\rho\alpha}\} h^\rho_a - \Lambda_\alpha{}^b{}_a e^\mu_b = 0$	$g^{\underset{+-}{\mu\nu}}{}_{\vert\alpha} = 0 \longleftrightarrow e^{*\overset{\mu}{+}}_{a\vert\alpha} = (e^{\overset{\mu}{-}}_{a\vert\alpha})^* = 0$, $e^{*\overset{\mu}{+}}_{a\vert\alpha} = e^{*\mu}_{a,\alpha} + \Theta^\mu_{\rho\alpha} e^{*\rho}_a - \Lambda_\alpha{}^b{}_a e^{*\mu}_b - C_\alpha e^{*\mu}_a = 0$ $\Theta^\mu_{\rho\alpha} =$ Schroedinger connection $\Theta_\nu \equiv \Theta^\rho_{\underset{\vee}{\rho\nu}} = 0$

	The Quaternionic Theory Q-algebra	The Octonionic Theory split O-algebra
metric equations vierbeins equations	$G_{\underset{+-}{\mu\nu\vert\alpha}} = 0 \longleftrightarrow E^a_{\underset{+}{\mu\vert\alpha}} = (E^{\dagger a}_{\underset{-}{\mu\vert\alpha}})^\dagger = 0$, $E^a_{\underset{+}{\mu\vert\alpha}} = E^a_{\mu,\alpha} - E^a_\rho \Gamma^\rho{}_{\mu\alpha} + \Lambda_\alpha{}^a{}_b E^b_\mu$, $\Gamma^\rho{}_{\mu\alpha} = \Theta^\rho{}_{\mu\alpha} w_0 + \delta^\rho_\mu \Gamma_\alpha$, $\Gamma_\alpha = -\vec{B}_\alpha \cdot \vec{w} = -\Gamma^\dagger_\alpha$, $\Lambda_\alpha{}^a{}_b = (\Lambda_\alpha{}^a{}_b + \delta^a_b C_\alpha) w_0 + \delta^a_b \Gamma_\alpha$	$G_{\underset{+-}{\mu\nu\vert\alpha}} = 0 \longleftrightarrow H^a_{\underset{+}{\mu\vert\alpha}} = (H^{\dagger a}_{\underset{-}{\mu\vert\alpha}})^\dagger = 0$, $H^a_{\underset{+}{\mu\vert\alpha}} = H^a_{\mu,\alpha} - H^a_\rho \Gamma^\rho{}_{\mu\alpha} + \Lambda_\alpha{}^a{}_b H^b_\mu = 0$, $\Gamma^\rho{}_{\mu\alpha} = \Theta^\rho{}_{\mu\alpha}(w^*_0 + w_0) + \delta^\rho_\mu \Gamma_\alpha$, $\Gamma_\alpha = -\vec{L}_\alpha \cdot \vec{w}^* - \vec{K}_\alpha \cdot \vec{w}$, $\Lambda_\alpha{}^a{}_b = (\Lambda_\alpha{}^a{}_b + \delta^a_b C_\alpha)(w^*_0 + w_0) + \delta^a_b \Gamma_\alpha$
metric equations vierbeins equations	$G^{\underset{+-}{\mu\nu}}{}_{\vert\alpha} = 0 \longleftrightarrow E^{\dagger\overset{\mu}{+}}_{a\vert\alpha} = (E^{\dagger\overset{\mu}{-}}_{a\vert\alpha})^\dagger = 0$, $E^{\dagger\overset{\mu}{-}}_{a\vert\alpha} = E^\mu_{a,\alpha} + E^\rho_a \Gamma^{\dagger\mu}_{\rho\alpha} - \Lambda^\dagger{}_\alpha{}^b{}_a E^\mu_b = 0$, $\Gamma^{\dagger\mu}_{\rho\alpha} = \Theta^\mu_{\alpha\rho} w_0 - \delta^\mu_\rho \Gamma_\alpha$, because $\Theta^{*\mu}_{\rho\alpha} = \Theta^\mu_{\alpha\mu}$, $\Lambda^\dagger{}_\alpha{}^a{}_b = (\Lambda_\alpha{}^a{}_b - \delta^a_b C_\alpha) w_0 - \delta^a_b \Gamma_\alpha$	$G^{\underset{+-}{\mu\nu}}{}_{\vert\alpha} = 0 \longleftrightarrow H^{\dagger\overset{\mu}{+}}_{a\vert\alpha} = (H^{\dagger\overset{\mu}{-}}_{a\vert\alpha})^\dagger = 0$, $H^{\dagger\overset{\mu}{-}}_{a\vert\alpha} = H^\mu_{a,\alpha} + H^\rho_a \Gamma^{\dagger\mu}_{\rho\alpha} - \Lambda^\dagger{}_\alpha{}^b{}_a H^\mu_b = 0$, $\Gamma^{\dagger\mu}_{\rho\alpha} = \Theta^\mu_{\alpha\rho}(w^*_0 + w_0) - \delta^\mu_\rho \Gamma_\alpha$, because $\Theta^{*\mu}_{\rho\alpha} = \Theta^\mu_{\alpha\mu}$, $\Lambda^\dagger{}_\alpha{}^a{}_b = (\Lambda_\alpha{}^a{}_b - \delta^a_b C_\alpha)(w^*_0 + w_0) - \delta^a_b \Gamma_\alpha$

Table II Free field equations for the metric and the corresponding equations for the vierbeins

References

[1] See K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov, and A.I. Shirshov, *Rings that are Nearly Associative*, chapter II, (Academic Press, 1982). See also, M. Gunaydin, F. Gursey, J. of Math. Phys. 14: 1651 (1973), and references therein.

[2] A. Einstein, Rev. Mod. Phys. **10**, 35 (1048); A. Einstein, *The Meaning of Relativity* (Princeton U.P., Princeton, NJ, 1955), Appendix 2.

[3] M.A. Tonnelat, *Les Theories Unitaires de l'Électromagnétisme e de la Gravitation* (Gauthier-Villars, Paris, 1965).

[4] J.W. Moffat, D.L. Boal, Phys. Rev. D **11**, 1375 (1975); W.B. Bonnor, Proc. R. Soc. London, Ser. A **226**, 336 (1954). Also, see J.W. Maluf and S. Okubo in Phys. Rev D **31**, 1327 (1985).

[5] K. Borchsenius, Phys. Rev. D **13**,3554 (1976).

[6] S. Marques, C.G. Oliveira, J. of Math. Phys. 26: 3131 (1985).

[7] S. Marques, C.G. Oliveira, Phys. Rev. D36: 1716 (1987).

[8] S. Marques, J. of Math. Phys. 29: 2127 (1988); Fermilab preprint pub-88/50-A; Fermilab preprint (pub-89).