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Exponential Stability in Two-dimensional Magneto-Elasticity: A proof on a dissipative medium

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Abstract

We give a proof of the exponential uniform decay of magneto-elasticity waves in a compact medium.

1 The Proof

One interesting questions in the magneto-elasticity properties of an isotropic (incompressible) medium vibrations interacting with an external magnetic is to show that the energy of the total system should decay with an exponential uniform bound as time goes to infinity ([1]).

We aim at this note to give a rigorous proof of this exponential bound for a model of imaginary medium electric conductivity. (See appendix A for a complementary analysis on this problem).

The governing differential equations for the electric-magnetic medium displacement vector $U = (U^1, U^2, 0) \equiv \vec{U}(x,t)$ depending on the time variable $t \in [0,\infty)$ with $x \in \Omega$ and the intrinsic two-dimensional magnetic field $\vec{h}(x,t) = (h^1, h^2, 0)$ are given by

$$\frac{\partial^2 \vec{U}(x,t)}{\partial^2 t} = \Delta \vec{U}(x,t) + ([\vec{\nabla} \times \vec{h}](x,t) \times \vec{B}) \tag{1}$$

$$\beta \frac{\partial \vec{h}(x,t)}{\partial t} = \Delta \vec{h}(x,t) + \beta \left(\vec{\nabla} \times \left[\frac{\partial U}{\partial t}(x,t) \times \vec{B} \right] \right). \tag{2}$$

Here $\vec{B} = (B,0,0)$ is a constant vector external magnetic field and β the (constant) medium electric conductivity.

Additionally, one has the initial conditions

$$\vec{U}(0,x) = \vec{U}_0(x); \quad \vec{U}_t(0,x) = \vec{U}_1(x), \quad \vec{h}(0,x) = h_0(x) \tag{3}$$

and the (physical) Dirichlet-type boundary conditions with $\vec{n}(x)$ being the normal of the boundary of the medium Ω ([2])

$$\vec{U}(\vec{x},t)|_\Omega = 0; \quad (\vec{n}\times(\vec{\nabla}\times\vec{h}))(\vec{x},t)|_\Omega = 0 \tag{4}$$

Let us consider the contraction semi-group defined by the following essentially self-adjoint operators in the $\hat{H}_\Omega$-energy space.

$$\hat{H}_\Omega = \{(\vec{U},\vec{\pi},\vec{h}) \in (L^2(\Omega))^3 \mid (\nabla\vec{U},\nabla\vec{U})_{L^2(\Omega)} + (\vec{\pi},\vec{\pi})_{L^2(\Omega)} + (\vec{h},\vec{h})_{L^2(\Omega)} < \infty\}.$$

Namely

$$\mathcal{L}_0 = \begin{bmatrix} 0 & i & 0 \\ i\Delta & 0 & 0 \\ 0 & 0 & -\Delta \end{bmatrix} \tag{5}$$

and

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & [\vec{\nabla}\times(\quad)]\times\vec{B} \\ 0 & -(\vec{\nabla}\times[(\quad)\times\vec{B}]) & 0 \end{bmatrix}. \tag{6}$$

We, thus, have the contractive C_0-semigroup acting on $\hat{H}_\Omega$ (see appendix A for the technical proof details)

$$T_t(\vec{U},\vec{\pi},\vec{h}) = \exp(t(+i\mathcal{L}_0 - V))(\vec{U},\vec{\pi},\vec{h})(0). \tag{7}$$

It is worth call the reader attention that we have proved the essential self-adjointness of the operator V by using explicitly the boundary-conditions on the explicitly relationships below

$$([\vec{\nabla}\times(\vec{h})]\times\vec{B}) = (B, B(\partial_1 h_2 - \partial_2 h_1)) \tag{8}$$

$$([\vec{\nabla}\times[\vec{h}\times B)) = (-B\partial_2\pi^2, B\partial_1\pi^2, 0))) \tag{9}$$

By standard theorems on Contraction Semi Group theory ([8]), on has that the left-hand side of eq.(7) satisfies the magneto-elasticity equations in the strong sense with $\beta = i = \sqrt{-1}$, a pure imaginary electric conductivity medium and physically meaning that the medium has “electromagnetic dissipation” and in the context that the initial-values belong to the sub-space $(H^2(\Omega)\cap H_0^1(\Omega))^2 \oplus (L^2(\Omega))^2 \oplus (L^2(\Omega))^2 \subset \hat{H}_\Omega$.

Let us note that eq.(7) still produces a weak-integral solution on the full Hilbert space $\hat{H}_\Omega = (H_0^1(\Omega))^2 \oplus (L^2(\Omega))^2 \oplus (L^2(\Omega))^2$, result suitable when one has initial random conditions sampled on $L^2(\Omega)$ by the Minlos theorem.

Thus, for any $(\vec{U}_0,\vec{\pi}_0,\vec{h}_0) \in \hat{H}_\Omega$ we have the following energy estimate

$$\begin{aligned}
&||(\vec{U},\partial_t\vec{U},\vec{h})||_{\tilde{H}_\Omega} \equiv \int_\Omega dx(|\nabla\vec{U}|^2 + |\partial_t\vec{U}|^2 + |\vec{h}|^2)(x,t) \\
&\leq ||(\exp(t(i\mathcal{L}_0 - V)))(\vec{U}(0),\partial_t\vec{U}(0),\vec{h}(0))||_{\tilde{H}_\Omega} \\
&\leq \left\| S - \lim_{n\to\infty}\left\{\left[\exp\left(\frac{it}{n}\mathcal{L}_0\right)\exp\left(-\frac{t}{n}V\right)\right]^n (\vec{U}_0,\vec{U}_1,\vec{h}_0)\right\}\right\|_{\tilde{H}_\Omega} \\
&\leq \exp(-t\omega(V))||\vec{U}_0,\partial_t\vec{U}(0),\vec{h}(0)||_{\tilde{H}_\Omega}
\end{aligned} \tag{10}$$

where $\omega(V)$ is the infimum of the spectrum of the self-adjoint operator V on the space $\hat{H}$.

In order to determine the spectrum of the self-adjoint operator V, which is a discret set since Ω is a compact region of R^2, we consider the associated V-eigenfunctions problem

$$[\nabla \times (\vec{h}_n)] \times \vec{B} = \lambda_n \vec{\pi}_n \tag{11}$$

$$-\nabla \times [(\vec{\pi}_n) \times \vec{B}] = \lambda_n \vec{h}_n \tag{12}$$

which, by its turn, leads to the usual spectral problem for the Laplacean with the usual Dirichlet boundary conditions on Ω

$$\Delta \pi_2^n = -\left(\frac{\lambda_n^2}{B}\right) \pi_2^n \tag{13}$$

$$\pi_2^n(x)|_{\partial\Omega} = 0. \tag{14}$$

As a consequence of equations (12) and (13), one finally gets the exponential bound for the total magneto-elastic energy eq.(10)

$$\begin{aligned} ||(\vec{U}, \partial_t \vec{U}, \vec{h})||_{\tilde{H}_\Omega}(t) \leq \exp\left(-t\sqrt{B\lambda_0(\Omega)}\right) \\ \times ||(\vec{U}_0, \partial_t \vec{U}(0), \vec{h}(0)||_{\tilde{H}_\Omega} \end{aligned} \tag{15}$$

where $\lambda_0(\Omega) = \inf\{\text{spec}(-\Delta)\}$ em $H^2(\Omega) \cap H_0^1(\Omega)$.

Appendix A

Exponential stability in two-dimensional magneto-elastic: Another proof

In this complementary technical appendix on the exponential decay of the Magneto-Elastic energy associated to the (imaginary electric medium conductivity) magneto-elastic wave

$$\begin{pmatrix} U \\ \pi \\ h \end{pmatrix}(t) = \exp\{it(+\mathcal{L}_0 + iV)\}\begin{pmatrix} U \\ \pi \\ h \end{pmatrix}(0) \tag{A-1}$$

with the essential self-adjointness operators on the Hilbert Energy space $\tilde{H}^1(\Omega)$

$$\mathcal{L}_0 = \begin{bmatrix} 0 & +i & 0 \\ +i\Delta & 0 & 0 \\ 0 & 0 & \Delta \end{bmatrix} \tag{A-2}$$

and

$$iV = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i[\vec{\nabla}\times(\ )\times\vec{B} \\ 0 & i\vec{\nabla}\times[(\ )\times\vec{B}] & 0 \end{bmatrix}, \tag{A-3}$$

we have used the fact that the operator $-V + i\mathcal{L}_0$ is the generator of a contraction semi group on this energy Hilbert space $\tilde{H}^1(\Omega) = (H^1(\Omega))^3 \oplus (L^2(\Omega))^3 \oplus (L^2(\Omega))^3$ in order to write eq.(A-1) in a mathematically rigorous way.

Let us give a proof of this mathematical result by means of a direct application of the Hille-Yosida theorem ([2]) to the operator $(-V + i\mathcal{L}_0)$.

Firstly, the domain of $(-V + i\mathcal{L}_0)$ is everywhere dense on $\tilde{H}^1(\Omega)$ as a consequence of the self-adjointeness of the operators V and $\mathcal{L}_0$ on $\tilde{H}^1(\Omega)$.

Secondly, the existence of a solution for the elliptic problem

$$(-V + i\mathcal{L}_0)x = \alpha y \tag{A-4}$$

for $x \in D(\mathcal{L}_0) \subset D(V)$ and every $y \in \tilde{H}^2(\Omega)$ with $\alpha > 0$ is a standard result even if Ω has a non-trivial topology (holes inside!), i.e.: Ω is a multiply-connected planar region ([2]).

The unicity of the solution x is a straightforward consequence of the fact the spectrum $\{\lambda_n\}$ of the self-adjoint operator V coincides with the positive Laplacean $\omega_n(-\Delta)$, i.e.: $\lambda_n^2/B = \omega_n(-\Delta)$. As a consequence, the solution of the equation

$$(-V + i\mathcal{L}_0)x = 0 \tag{A-5}$$

leads to the relation below due to the self-adjointness of the operators V and $\mathcal{L}_0$ on $\tilde{H}^1(\Omega)$

$$\overline{\langle Vx, x\rangle}_{\tilde{H}^1(\Omega)} = \langle Vx, x\rangle_{\tilde{H}^1(\Omega)} = i\langle \mathcal{L}_0 x, x\rangle_{\tilde{H}^1(\Omega)} = i\overline{\langle \mathcal{L}_0 x, x\rangle}_{\tilde{H}^1(\Omega)} \tag{A-6}$$

from which we conclude that

$$\langle Vx, x\rangle_{\tilde{H}^1(\Omega)} = 0 \tag{A-7}$$

$$\langle \mathcal{L}_0 x, x\rangle_{\tilde{H}^1(\Omega)} = 0 \tag{A-8}$$

or, equivalently

$$x \in \text{Ker}\{V\} \cap \text{Ker}\{\mathcal{L}_0\} \tag{A-9}$$

which is zero if Ω is *simply* connected planar domain as supposed in the text.

Finally, for any $x > 0$, we have that

$$||(\alpha 1 - (-V + i\mathcal{L}_0))^{-1}||_{\tilde{H}^1(\Omega)} \leq \frac{1}{\alpha} \tag{A-10}$$

since for every $z \in \text{Dom}(\mathcal{L}_0) \subset \text{Dom}(V)$

$$\begin{aligned}\alpha^2||z||^2_{\tilde{H}^1(\Omega)} &\leq ||((\alpha 1 - (-V + i\mathcal{L}_0))z||^2_{\tilde{H}^1(\Omega)} \\ &= \alpha^2||z||^2_{\tilde{H}^1(\Omega)} + ||\mathcal{L}_0 z||^2_{\tilde{H}^1(\Omega)} \\ &\quad + ||Vz||^2_{\tilde{H}^1(\Omega)} + 2\alpha(z, Vz)_{\tilde{H}^1(\Omega)}.\end{aligned} \tag{A-11}$$

Note that we have used the fact that V is positive operator on $\tilde{H}^1(\Omega)$.

As a consequence of the above exposed results we have that $-V + i\mathcal{L}_0$ is a generator of a contractive semi-group on the space $(1 - P_{\text{Ker}(V)\cap\text{Ker}(\mathcal{L}_0)})\tilde{H}^1(\Omega)$, where $P_{\text{Ker}(V)\cap\text{Ker}(\mathcal{L}_0)}$ is the orthogonal projection on the Kernel sub-spaces of the self-adjoint operators V and $\mathcal{L}_0$.

As a last final remark to be made in this appendix A, let us call the physicist oriented reader attention for the following (somewhat formal) abstract Lemmas, alternatives to the Banach space methods used in ref. [2].

Lemma 1. Suppose that the matrix-valued operator with self-adjoint operators entries on suitable Hilbert Spaces H_1, H_2

$$L_0 = \begin{bmatrix} 0 & A & 0 \\ B & 0 & 0 \\ 0 & 0 & C \end{bmatrix}. \tag{A-12}$$

Then, it will be a self-adjoint on the Hilbert Space $\tilde{H} = H_1 \oplus H_1 \oplus H_2$ with an inner product given by (D is a positive self-adjoint operator on H_1)

$$\langle (U, \pi, h); (U', \pi', h')\rangle_{\tilde{H}} = (D^* DU, U')_{H_1} + (\pi, \pi)_{H_1} + (h, h)_{H_2} \tag{A-13}$$

if we have the constraints below for the operators on the Hilbert components spaces:

$$\begin{gathered} C = C^* \quad \text{on} \quad H_2 \\ A^* D^* D = B \quad \text{on} \quad H_1 \\ \text{Range}(A) \subset \text{Dom}(D) \end{gathered} \tag{A-14}$$

Proof: Let us consider the inner product on $\tilde{H}$

$$\left\langle L_0 \begin{pmatrix} U \\ \pi \\ h \end{pmatrix} ; (U', \pi', h') \right\rangle_{\tilde{H}} = (DA\pi, DU')_{H_1} + (BU, \pi')_{H_1} + (Ch, h')_{H_2} \tag{A-15}$$

and

$$\langle (U, \pi, h); L_0(U', \pi', h') \rangle_{\tilde{H}} = (DU, DA\pi')_{H_1} + (\pi, BU')_{H_1} + (h, Ch')_{H_2}. \tag{A-16}$$

On basis of eq.(A-15) and eq.(A-16) one can see that L_0 is a closed symmetric operator- Besides, if there is a $(U_0, \pi_0, h_0) \in \tilde{H}$ such that for every (U, π, h) we have the orthogonality relation

$$\langle (U_0, \pi_0, h_0); (i + L_0)(U, \pi, h) \rangle_{\tilde{H}} = 0. \tag{A-17}$$

Then

$$\langle (U_0, \pi_0, h_0); (A\pi, +iU, BU + i\pi, Ch + ih) \rangle_{\tilde{H}} = 0 \tag{A-18}$$

As a result

$$\begin{aligned} (D^* DU_0, (A\pi + iU))_{H_1} &= 0 \\ (\pi_0, BU + i\pi)_{H_1} = (h_0, Ch + ih)_{H_2} &= 0. \end{aligned} \tag{A-19}$$

We thus have by self-adjointness of the operators A, B and C that $U_0 = \pi_0 = h_0 = 0$. As a consequence the deficiencies indices of L_0 vanishe. What formally concludes the self-adjointness of L_0 on that vectors of $\tilde{H}_1$ such that (the domain of L_0 since L_0 is a closed symmetric operator):

$$\langle L_0(U, \pi, h), (U, \pi, h) \rangle_{\tilde{H}} < \infty \tag{A-20}$$

Lemma 2. Let U be the matrix valued operator acting on $\tilde{H}$ given by

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & V_1 \\ 0 & V_2 & 0 \end{bmatrix}. \tag{A-21}$$

Then V is self-adjoint on $\tilde{H}$ if and only if on H_2 we have the constraint relationship below between the adjoints of the closable symmetric operator V_1 and V_2

$$V_1^* = V_2. \tag{A-22}$$

Proof: One can use similar arguments of the Lemma $1'$ proof to arrive at such result.

Let us now take into account the case of the spectrum of L_0 contained on the negative real line $(-\infty, 0]$ and that one associated with V is by its turn, contained only on the upper bounded negative infinite interval $(-\infty, -c]$ (with $c > 0$), then we have by a straightforward application of the Trotter-Kato formulae for $t \in [0, \infty)$ the following estimate on the norm of semi-group evolution operator-generated by $L_0 + V_0$

$$\begin{aligned} ||e^{t(L_0+V)}|| &\leq \lim_n ||e^{\frac{t}{n}L_0} e^{\frac{t}{n}V}||^n \\ &\leq \lim_n ||e^{\frac{t}{n}V}||^n \leq ||e^{tV}|| \leq e^{-tc} \end{aligned} \tag{A-23}$$

In the following Magneto-elastic wave problem with *real* conductivity $\beta > 0$,

$$\begin{aligned} \partial_t \begin{pmatrix} U \\ \pi \\ h \end{pmatrix}(t) &= \begin{bmatrix} 0 & -1 & 0 \\ -\Delta & 0 & 0 \\ 0 & 0 & \frac{1}{\beta}\Delta \end{bmatrix} \begin{pmatrix} U \\ \pi \\ h \end{pmatrix}(t) \\ &\times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\vec{B} \times [\vec{\nabla} \times (\ )] \\ 0 & -\nabla \times [(\ ) \times \vec{B}] & 0 \end{bmatrix} \begin{pmatrix} U \\ \pi \\ h \end{pmatrix}(t) \end{aligned} \tag{A-24}$$

One can see that it satisfies the conditions of the above written Lemmas with the operator $D = \nabla$ and with the Hilbert space

$$\tilde{H} = (L^2(\Omega))^3 \oplus (L^2(\Omega))^3 \oplus (L^2(\Omega))^3$$

and $c = \inf\ \mathrm{spec}(-\Delta)$ an $H^2(\Omega) \cap H_0^1(\Omega)$ (see the section 5) and, thus, leading to the expected total energy exponential decay showed in section 5.

References

[1] G.P. Menzala and E. Zuazua, Asymptotic Anal. **18**, 349–367, (1998).

[2] J.E.M. Riviera and R. Racue, IMA J. Appl. Math. **66**, 269–283, (2001).