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SUM RULES, ASYMPTOTIC BEHAVIOUR AND (MULTI)BARYON STATES IN THE SKYRME MODEL

by

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Abstract

We obtain sum rules that should be satisfied by the solutions of the Euler-Lagrange equation for the chiral angle in the Skyrme model in the hedgehog representation. The sum rules allow to determine the existence of solutions with integer baryon number for well determined values of a relevant dimensionless parameter Φ only. For all other values, there are no solutions with integer baryon number, in particular for the pure non-linear sigma model.

Key-words: Baryons; Chiral solitons; Non-linear sigma model; Skyrme model.

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1 Introduction

We derive sum rules for the solutions of the Euler-Lagrange equations for the hedgehog in the Skyrme model[1] at the origin and at infinity and study with them the solutions corresponding to integer baryon number. These turn out to lie on straight lines for particular values (Φ_1,Φ_2,...) of the ratio, Φ, between two parameters: the value of the slope at the origin of a regular solution for the chiral angle and the Skyrme parameter, e. The baryon number increases with Φ by one unit for each straight line (1,2,...). We find evidence for a new kind of solution for baryons that interpolate between half integer values of π. For all other values of these parameters, the solutions are of two types: one, regular at the origin (we take its initial value as $-\pi$), tends asymptotically through a damped oscillation to half integer multiples of π (that is, carry half integer baryon number); the other starts oscillating wildly at a half integer multiple of π and goes smoothly to a regular solution at infinity, where it is an integer multiple of π. As a particular case, we recover the results of Iwasaki and Ohyama for the pure non-linear sigma model [2]. The values of physical quantities like the mass of the soliton [3] vary along the straight line corresponding to a given integer baryon number.

In short, our results are that there exists a denumerable infinity of possible sum rules and that the possible baryon soliton solutions group along straight lines in the plane of parameters mentioned above, each line being labelled by the integer value of the baryon number of the solution, which grows with their slope. Between these lines, the regular solutions at the origin for the chiral angle tend asymptotically to intermediate multiples of $\frac{\pi}{2}$. The choice of the initial integer multiple of π at the origin turns out to be irrelevant for the relative asymptotic behaviour.

For the regular solutions at infinity, the sum rules indicate a kind of mirror behaviour, as a half integer value of π at the origin is attained through a wild damped oscillation for values of Φ between integer baryon lines. We shall show below that this is essentially controlled by the Skyrme term in the lagrangean.

In any case, as the Skyrme parameter is let to increase indefinitely, the results of Iwasaki and Ohyama for the non-linear sigma model [2](which, in another context, were exhibited some time ago by Itzykson [4]) are naturally reproduced.

2 The Sum Rules

We give now the derivation of a set of sum rules, along the lines developed by [2]. The sum rules in our version cover the cases of the Euler-Lagrange equations at the origin and infinity simultaneously. Our starting point is the Euler-Lagrange equation for the hedgehog in the Skyrme model written in terms of the dimensionless variable $\tilde{r} = ef_\pi r$ introduced by Adkins, Nappi and Witten [5]. In terms of this variable the parameter Φ mentioned above is the slope at the origin of the chiral angle. The Euler-Lagrange equation is:

$$\left(\tilde{r}^4 + 8\tilde{r}^2\sin^2 F(\tilde{r})\right)\frac{d^2F(\tilde{r})}{d\tilde{r}^2} + 2\tilde{r}^3\frac{dF(\tilde{r})}{d\tilde{r}}$$
$$+4\tilde{r}^2\sin 2F(\tilde{r})\left[\frac{dF(\tilde{r})}{d\tilde{r}}\right]^2 - \tilde{r}^2\sin 2F(\tilde{r}) - 4\sin^2 F(\tilde{r})\sin 2F(\tilde{r}) = 0\ . \qquad (1)$$

Multiplying this equation by $\tilde{r}^n dF(\tilde{r})/d\tilde{r}$ and integrating by parts with respect to $\tilde{r}$ from $\tilde{r}_1$ to $\tilde{r}_2$, we get:

$$\frac{1}{2}\tilde{r}^{n+4}\left[\frac{dF(\tilde{r})}{d\tilde{r}}\right]^2\Bigg|_{\tilde{r}_1}^{\tilde{r}_2} - \frac{1}{2}n\int_{\tilde{r}_1}^{\tilde{r}_2} d\tilde{r}\ \tilde{r}^{n+3}\left[\frac{dF(\tilde{r})}{d\tilde{r}}\right]^2 + \tilde{r}^{n+2}\sin^2 F(\tilde{r})\Big|_{\tilde{r}_1}^{\tilde{r}_2}$$
$$+ (n+2)\int_{\tilde{r}_1}^{\tilde{r}_2} d\tilde{r}\ \tilde{r}^{n+1}\sin^2 F(\tilde{r}) + 4\ \tilde{r}^{n+2}\sin^2 F(\tilde{r})\left[\frac{dF(\tilde{r})}{d\tilde{r}}\right]^2\Bigg|_{\tilde{r}_1}^{\tilde{r}_2}$$
$$- 4\ (n+2)\int_{\tilde{r}_1}^{\tilde{r}_2} d\tilde{r}\ \sin^2 F(\tilde{r})\left[\frac{dF(\tilde{r})}{d\tilde{r}}\right]^2 - 2\tilde{r}\sin^4 F(\tilde{r})\Big|_{\tilde{r}_1}^{\tilde{r}_2}$$
$$+ 2n\int_{\tilde{r}_1}^{\tilde{r}_2} d\tilde{r}\ \tilde{r}^{n-1}\ \sin^4 F(\tilde{r}) = 0. \qquad (2)$$

As n is arbitrary we can get from the last expression an infinite number of sum rules. We are going to discuss mainly in this work only two of them, those that in the limit $e \longrightarrow \infty$, correspond to the sum rules of ref. [2].

We first notice a few basic points. We look for the solution that satisfies:

$$F(0) = -\pi \tag{3}$$

in agreement with the requirement of regularity of the Euler-Lagrange equation at the origin [2,3]. We have also

$$\frac{dF(\tilde{r})}{d\tilde{r}} \overset{\tilde{r}\to 0}{\longrightarrow} const. \tag{4}$$

Asymptotically, we demand only that

$$F(\tilde{r}) \overset{\tilde{r}\to\infty}{\longrightarrow} const. \tag{5}$$

though strict regularity of the solution would require the constant to be an integer multiple of π. Finally, we have to require that

$$\frac{dF(\tilde{r})}{d\tilde{r}} \overset{\tilde{r}\to\infty}{\longrightarrow} 0. \tag{6}$$

This to have finite results for the integrals in the sum rules. Then, for $n = -2$ and $n = -4$ we have, respectively,

$$\sin^2 F(\tilde{r} \to \infty) = \int_0^\infty d\tilde{r}\, \tilde{r} \left[\frac{dF(\tilde{r})}{d\tilde{r}}\right]^2 - \int_0^\infty d\tilde{r}\, \frac{4}{\tilde{r}^3} \sin^4 F(\tilde{r}) \ , \tag{7}$$

$$\begin{aligned} &\left[\frac{dF(\tilde{r}\to 0)}{d\tilde{r}}\right]^2 + 8\left[\frac{dF(\tilde{r}\to 0)}{d\tilde{r}}\right]^4 \\ &= \int_0^\infty d\tilde{r}\, \frac{4}{\tilde{r}} \left[\frac{dF(\tilde{r})}{d\tilde{r}}\right]^2 + \int_0^\infty d\tilde{r}\, \frac{16}{\tilde{r}^3} \sin^2 F(\tilde{r}) \left[\frac{dF(\tilde{r})}{d\tilde{r}}\right]^2 \\ &\quad - \int_0^\infty d\tilde{r}\, \frac{4}{\tilde{r}^3} \sin^2 F(\tilde{r}) \ - \ \int_0^\infty d\tilde{r}\, \frac{16}{\tilde{r}^5} \sin^4 F(\tilde{r}) \ . \end{aligned} \tag{8}$$

Let us make a few comments on these expressions. The first is the analog of Eq. (14) of Iwasaki and Ohyama for the pure non-linear sigma model, and it fundamentally informs about the asymptotic behaviour of the solution for the chiral angle. Going back to the dimensional radial variable one recovers their result as the Skyrme parameter tends to infinity since the second term in the right hand side vanishes at this limit.

The second expression corresponds to Eq. (15) of Iwasaki and Ohyama, and we interpret it as a kind of consistency condition on the solution for the values at the origin. Notice the nice cancellation at the lowest limit of integration. Again, we may recover the results for the pure non-linear sigma model, since in this limit the second term at the left hand side and the second and fourth terms in the right hand side vanish.

In the same way these sum rules were obtained for the chiral angle, analogous expressions result for the transformed differential equation after inverting the independent variable $\tilde{r}$. After elementary manipulations , it turns out that the new set of sum rules is the same of the Eq. (2), the difference being of two units in the power of the independent variable. As we shall discuss below, the solutions in this case are essentially different to those of the non-linear sigma-model, which when regular at infinity diverge at the origin.

3 Numerical Study and Results

We now pass to the study of the solutions of the Euler-Lagrange equation regular at the origin with the help of Eq. (7). In this case, we take profit from the fact that there is no intermediate situation for the left hand side: either it is zero or one. When solving the equation numerically, and at the same time performing the integrals on the right hand side, it is easy to follow the asymptotic behaviour since sufficiently far from the origin the numerical values are close to one or really quite small.

We have applied this to study the behaviour of the regular solutions of Eq. (1) as a function of the relevant parameter of the problem: the slope of the solution at the origin, Φ. As anticipated previously [3] it can be written in terms of the dimensionless Skyrme parameter, e, and the slope at the origin of the chiral angle as a function of the original radial variable:

$$\Phi = \frac{F_1}{e f_\pi} \ , \tag{9}$$

$$F_1 = \left.\frac{dF(r)}{dr}\right|_{r=0} \ . \tag{10}$$

Our results are the following: starting from $\Phi = 0$. the solutions asymptotically tend to $-\frac{\pi}{2}$ up to a value $\Phi_1 \simeq 1.00376\ldots$. For this value, the solutions tend to 0., whatever the values of the parameters determining Φ_1. In the plane F_1 *vs* e, Φ_1 is the slope of a straight line along which are located the solutions with unit baryon number. The values of physical quantities like the classical or quantum mass vary along the line, and the latter has apparently a single quantum minimum for a given angular momentum.

As the value of Φ increases, the solutions now tend asymptotically to $\frac{\pi}{2}$, until a value $\Phi_2 \simeq 1.9650\ldots$ is attained. For this value, the solutions go to π, that is, have baryon number 2. Again, physical quantities vary along this straight line in the F_1 *vs* e plane. Proceeding further, the solutions tend now to $\frac{3\pi}{2}$ up to a value of Φ corresponding to baryon number 3 ($\Phi_3 \simeq 2.8882\ldots$) and the picture continues evolving this way, apparently indefinitely. In general, the approach to integer values of π happens uniformly in configuration space, whereas for the other situation the solution ripples around the limiting value. The "wavelength" of the oscillation increases as Φ approaches from below a noteworthy value corresponding to integer baryon number. These oscillations are of the form found by Iwasaki and Ohyama [2] for the irregular solutions at infinity in the non-linear sigma model.

As particular cases, the pure non-linear sigma model corresponds to $\Phi = 0$. [2,6], that is, there is no classical soliton for it. The case analyzed by Adkins, Nappi and Witten [5] is located on the line of unit baryon number, though not at the corresponding quantum minimum. In fact, there are two independent minima, one for each angular momentum ($\mathbf{J} = \frac{1}{2}, \frac{3}{2}$). For the nucleon case, the minimum occurs at a value $e \simeq 7.67$ and the energy is $\simeq 0.82$ Gev for a f_π value of 0.129 Gev; for $f_\pi = 0.186$ Gev, the energy becomes $\simeq 1.18$ Gev. The axial weak coupling constant results $g_A = 0.61$ for Adkins et al. ($e = 5.45$, $f_\pi = 0.129$ Gev), and becomes 0.31 at the minimum for the energy. For the $\Delta(1232)$, the minima occur at $\simeq 5.13$ for the Skyrme parameter, and the energies for the two values of f_π are $\simeq 1.22$ and $\simeq 1.76$ Gev, respectively.

The solutions regular at infinity has a similar behaviour as the ones regular at the origin, this time in the inverted variable. In terms of $\tilde{r}$ they oscillate wildly, but with decreasing amplitude, to a half integer multiple of π at the origin. To trace this behaviour from Eq. (1) shows that it is controlled by the Skyrme term contribution. We have:

$$F(\tilde{r}) \simeq m\,\frac{\pi}{2} + \alpha\,\sqrt{\tilde{r}}\,\cos\left(\frac{\sqrt{3}}{2}\ln\gamma\tilde{r} + \delta\right) + \dots\ (\tilde{r} \to 0)\,, \tag{11}$$

where m is odd and α, γ and δ are undetermined constants. In our case the regular solutions at infinity have finite (though irregular) behaviour at the origin, whereas in the non-linear sigma model they grow indefinitely.

For the regular solutions at infinity a dimensionless parameter analogous to Φ can be formed out of the second derivative at the origin in terms of the inverse radial variable for the chiral angle and the squares of the Skyrme parameter and the pion decay constant. The solutions with baryon number 1, 2, ... correspond to the values $-17.277\ldots$, $-51.673\ldots$, ... of this parameter.

Now, to the straight lines of integer baryon number there seems to correspond solutions that at both ends of the real half line tend smoothly to half integer values of π. This behaviour we have assessed only numerically, integrating the differential equation to both ends starting from intermediate values for the chiral angle and taking a value for the slope equal to the corresponding one for the (multi)baryonic solution that tends to integer values of π. We believe that the existence of these new (multi)baryonic solutions, interpolating between imaginary values for the vacuum expectation value should result from the general symmetry of the solutions in the non-baryonic region.

4 Further Discussion and Comments

Let us make some final comments. First, there are several possibilities of writing sum rules using other functions than powers combined with derivatives of the chiral angle profile. These we have only partially explored, but have seen no advantage in its use. Second, it has not been possible for us to establish a functional dependence between the noteworthy values for the parameter Φ and the corresponding baryon number integers. Neither we have looked more carefully at the lines with baryon number larger than one for situations where the solution would correspond to an object more stable than the corresponding agglomerate formed out of unit baryon number ones.

It is worth to notice that coming back to the dimensional radial variable r we have different profiles for the chiral angle for any point of the straight line corresponding to a given baryon number. That is, to select a particular curve, one needs to specify one value for a parameter producing the relevant Φ. This may result from choosing, for instance, to locate the solution for the quantum minimum for the rest energy of the soliton.

We think that the sum rules prove useful to support the idea that the non-linear sigma model does not possess a baryonic soliton, hence there is no question of stabilizing it, and thus the Skyrme model provides baryonic solitons by its own means. The baryonic solitons seem to be of two types, accordingly to that they connect integer or half-integer multiples of π for the chiral angle at the ends of the real half-line.

The Skyrme model lagrangean seems to stand, then, as the natural starting point for a chiral low energy theory of hadrons along the lines studied by Pak and Tze [7]. Besides, it provides naturally candidates for baryons with larger baryon number, whose possible phenomenological consequences are an open question.

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