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ALGEBRAIC PROPERTIES OF THE DIRAC EQUATION IN THREE DIMENSIONS

by

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Abstract

We argue that, in three dimensions, spinors should have four components as a consequence of the algebraic structure realised from the Clifford algebra related to the Dirac equation. As an example, we show then that no induced mass appears in vacuum polarisation at 1-loop in 3D quantum electrodynamics.

Key-words: Clifford algebra; Differential form; Three-dimensional field theory.

1 Introduction

In this article we analyse the kinematics of the Dirac equation in three dimensions. Our interest is concentrated on the spacetime with Minkowski metric $(+--)$, but our results are easily extended to Euclidean space as well.

In recent years, we have studied the four-dimensional Dirac equation [1] and discovered that there is a spacetime $SU(4)$ symmetry closely related to the formulation of the Dirac equation in terms of the Clifford algebra either of the Dirac matrices or of differential forms (the Dirac–Kähler equation). Actually, the former are a representation of the latter [2,3]. An analogous structure is easily verified for the Dirac (or Dirac–Kähler) equation in two dimensions, with $SU(2)$ being the relevant group. This work in three dimensions is a step forward in the foundation of such ideas.

Our point is that the Lie-algebraic structure related to the Dirac equation, or its differential-form Dirac–Kähler counterpart, follows directly from the Clifford algebra structure (endowed on spacetime for the latter). We show in what follows that the corresponding symmetry in three dimensions is $SU(2) \times SU(2)$.

In section 2, we construct the $SU(2) \times SU(2)$ Lie algebra as derived from the Clifford generators and their products. We thus show that a 4×4 representation follows naturally. It contains the two 2×2 inequivalent irrducible representations obtained treating the Clifford algebra as a finite group [4].

Subsequently, in section 3, working with differential forms in three dimensions including, besides the usual Grassmann exterior product, a Clifford product between two 1-forms previously introduced by Kähler [5], we reproduce the results for matrices as should be expected from the isomorphism between the two sets demonstrated by Graf [6].

In section 4, we sketch the consequences of the formalism heretofore developed for the Dirac equation and its differential-form version.

In section 5, we show that the action of discrete symmetry operators such as parity and time reversal precludes the use of a two-component formalism for spinors in three dimensions.

Further, we give in section 6 an example of how physical predictions may be affected through the application of the (Clifford-algebra consistent) 4×4 formalism: the 1-loop vacuum polarisation for quantum electrodynamics in three dimensions has no induced topological mass term (independently of regularisation procedures).

Finally, in section 7 we present our conclusions.

2 The Clifford algebra of Dirac matrices and its Lie algebraic structure

Let us consider the Clifford algebra obtained from three (matrix) generators, γ^μ $(\mu = 0,1,2)$. The well-known defining relationship is

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \tag{1}$$

These generators may be made hermitian by suitable multiplication by a factor of i. The same can be done for their (matrix) products, and all together form a commutator algebra of the form

$$[\zeta^K, \zeta^L] = c^{KLM} \zeta^M, \tag{2}$$

where $K, L, M = 0,1,2,01,02,12,012$. Notice that the product of the three Clifford generators commutes with them and all their products. Besides, notice that this structure is contained in the subsets of the 4-dimensional Clifford algebra obtained when one discards there a given generator and the products including it.

For the sake of definiteness, let us consider the set formed by γ^0, $i\gamma^1$ and the matrix product $\gamma^0\gamma^1$, in order to work with hermitian matrices. Let us define

$$X_1 = \frac{1}{2}\gamma^0, \qquad X_2 = \frac{1}{2} i\gamma^1, \qquad X_3 = \frac{1}{2}\gamma^0\gamma^1. \tag{3}$$

Because of the defining relationship, eq. (1), one finds

$$[X_i, X_j] = i\varepsilon_{ijk}X_k. \tag{4}$$

This is an $SU(2)$ structure, like the one appearing in two dimensions. Consider now the set formed by the remaining generator and its products, $i\gamma^2$, $\gamma^0\gamma^2$ and $i\gamma^1\gamma^2$. Define

$$Y_1 = \frac{1}{2}i\gamma^1\gamma^2, \qquad Y_2 = -\frac{1}{2}\gamma^0\gamma^2, \qquad Y_3 = \frac{1}{2}i\gamma^2. \tag{5}$$

They satisfy

$$[Y_i, Y_j] = i\varepsilon_{ijk}X_k. \tag{6}$$

Also, we have

$$[X_i, Y_j] = \epsilon_{ijk}Y_k. \tag{7}$$

These two sets of commutators point to an $SU(2)\times SU(2)$ structure, which is readily brought to light by defining

$$W_k^{\pm} = \frac{1}{2}(X_k \pm Y_k). \tag{8}$$

These objects obey separate $SU(2)$ algebras,

$$\left[W_i^{\pm}, W_j^{\pm}\right] = i\varepsilon_{ijk}W_k^{\pm} \tag{9}$$

$$\left[W_i^{+}, W_j^{-}\right] = 0. \tag{10}$$

The matrices representing these $W^{\pm}$ generators can be taken to be block matrices, namely,

$$W_k^+ = \frac{1}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad W_k^- = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_k \end{pmatrix}, \tag{11}$$

where σ_k denote the Pauli matrices. It is now easy to reconstruct backwards the generators and their products. We get

$$\gamma^0 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \gamma^1 = -i\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \gamma^2 = -i\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}. \tag{12}$$

Notice the all-important minus sign in the lower block of γ^2. This is a consequence of the fact that $i\gamma^0\gamma^1\gamma^2$ commutes with every other matrix, and should naturally have the structure

$$i\gamma^0\gamma^1\gamma^2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{13}$$

The particular form of this matrix agrees with the arguments in the classical work by Brauer and Weyl [10]. (In fact, Brauer and Weyl even introduced the concept analogous to Hodge duality for Dirac matrices, but did not completely finalize the arguments leading to the actual form of the algebra.)

As a starting point, we omitted from the original set the generator γ^2. We could as well omit any one of the others. The results are shown below for γ^0 omitted (analogous to the Kramers–Weyl picture in four dimensions):

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = -i \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \gamma^2 = -i \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{14}$$

Instead, to get the analogous of the Dirac–Pauli picture in four dimensions, it is needed a symmetric substitution in the example with γ^2 omitted. We get

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \gamma^1 = -i \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \gamma^2 = -i \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \tag{15}$$

The reason for the proposed names of these two pictures seems now evident: γ^0 is diagonal and has the sign of the energy in the rest frame for the Dirac–Pauli picture, while γ^0 and $i\gamma^1\gamma^2$ exchange places when going to the Kramers–Weyl picture.

The main point for these results is to take the product of two Clifford generators as a different object as compared to the third generator. When analysing the problem using differential forms we shall see that this comes from a geometrical consideration.

Let us now compare these results with the ones currently used in the literature, coming from the consideration of a Clifford algebra as a finite group under multiplication [4]. This is what results from looking at the generators $W_k^{\pm}$ of the two-block $SU(2)$ group matrices. They satisfy

$$2W_k^{\pm}\, 2W_\ell^{\pm} + 2W_\ell^{\pm}\, 2W_k^{\pm} = \delta_{k\ell} I \pm i\gamma^0\gamma^1\gamma^2 \tag{16}$$

by construction. (Notice the same feature for Pauli matrices and $SU(2)$ generators, which differ by a factor 2.) Given that $i\gamma^0\gamma^1\gamma^2$ is of the diagonal form showed in eq. (13), it is clear that these generators satisfy a Clifford defining algebraic relation at each of the single subspaces corresponding to the eigenvalues of $i\gamma^0\gamma^1\gamma^2$.

The generators $W^{\pm}$ constitute then two irreducible representations for the finite group obtainable from the Clifford generators and their products which are inequivalent. The geometrical meaning of the inequivalence will be displayed with differential forms.

The fact that inversion of an axis transforms W^+ and W^- between themselves will be evident in the subsequent study of discrete symmetry operations. Anticipating the treatment through differential forms with Clifford product, let us state that the signs in the diagonal-block matrix $i\gamma^0\gamma^1\gamma^2$ may be related precisely to the handedness, or chirality (giving this word its precise Greek meaning), of the spacetime reference frame.

A comparison between these matrices and those quoted in the literature for the four-dimensional representation [7] shows a substantial difference: there is always a relative minus sign for one of the lower blocks.

3 Clifford algebra with differential forms in 2+1 dimensions and Lie algebra

The content of this section is an explicit construction of the isomorphism proven by Graf [6] for all dimensions between the matrices of the Dirac algebra and differential forms, provided that a Clifford product is defined for the latter.

The space of differential forms in three dimensions has eight components,

$$1,\, dx^\mu,\, dx^\mu \wedge dx^\nu,\, dx^0 \wedge dx^1 \wedge dx^3 \equiv \varepsilon,$$

where μ and ν run from 0 to 2 and $\wedge$ denotes the usual representation of the exterior product of differential forms, implying a Grassmann algebra between 1-forms,

$$dx^\mu \wedge dx^\nu + dx^\nu \wedge dx^\mu = 0. \tag{17}$$

The duality $*$ operator defined by Hodge links the subspace of forms with degree p with that of those with degree $3-p$; thus, the set of four components $(1, dx^\mu)$ maps into the remaining four, that is, $(dx^\mu \wedge dx^\nu, \varepsilon)$.

We assume further, following Kähler [5,8], that a Clifford product between 1-forms is defined such that

$$dx^\mu \vee dx^\nu = dx^\mu \wedge dx^\nu + g^{\mu\nu}. \tag{18}$$

The associative properties of this product follow from the above definitions and are summarised at the simplest level by the expressions

$$\begin{aligned} (dx^\mu \vee dx^\nu) \wedge dx^\rho &= (g^{\mu\nu} + dx^\mu \wedge dx^\nu) \wedge dx^\rho \\ &= g^{\mu\nu} dx^\rho + dx^\mu \wedge dx^\nu \wedge dx^\rho \qquad (19) \\ dx^\mu \vee (dx^\nu \wedge dx^\rho) &= (dx^\mu \vee dx^\nu) \wedge dx^\rho. \qquad (20) \end{aligned}$$

With this operation, we endow the space of differential forms with a Clifford algebra:

$$dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2g^{\mu\nu}. \tag{21}$$

We now use the Clifford commutator between differential forms [8]:

$$\left[dx^K, dx^L\right]_\vee = dx^K \vee dx^L - dx^L \vee dx^K, \tag{22}$$

where dx^K and dx^L represent two out of the eight basic differential forms.

In order to be able to exhibit the Lie-algebraic structure relevant to three dimensions, we give explicitly the Clifford commutators for all forms:

$$[dx^0, dx^1]_\vee = 2dx^0 \wedge dx^1 \quad [dx^1, dx^1 \wedge dx^2]_\vee = -2dx^2$$

$$[dx^0, dx^2]_\vee = 2dx^0 \wedge dx^2 \quad [dx^2, dx^0 \wedge dx^2]_\vee = 2dx^0$$

$$[dx^0, dx^0 \wedge dx^1]_\vee = 2dx^1 \quad [dx^2, dx^1 \wedge dx^2]_\vee = 2dx^1$$

$$[dx^0, dx^0 \wedge dx^2]_\vee = 2dx^2 \quad [dx^0 \wedge dx^1, dx^0 \wedge dx^2]_\vee = -2dx^1 \wedge dx^2$$

$$[dx^1, dx^2]_\vee = 2dx^1 \wedge dx^2 \quad [dx^0 \wedge dx^1, dx^1 \wedge dx^2]_\vee = -2dx^0 \wedge dx^2$$

$$[dx^1, dx^0 \wedge dx^1]_\vee = 2dx^0 \quad [dx^0 \wedge dx^2, dx^1 \wedge dx^2]_\vee = 2dx^0 \wedge dx^1,$$

all others being zero. We notice that the volume form ε commutes with all the remaining forms. As for matrices, this indicates the algebraic structure that is expected to arise.

We now construct the analogous to the example developed in the previous section. Let us consider the three forms dx^0, idx^1 and $dx^0 \wedge dx^1$. Notice that all have in common that the Clifford product of each one of them with itself is 1. The Clifford commutators are

$$\left[dx^0, idx^1\right]_\vee = 2idx^0 \wedge dx^1$$

$$\left[idx^1, dx^0 \wedge dx^1\right]_\vee = 2idx^0$$

$$\left[dx^0 \wedge dx^1, dx^0\right]_\vee = 2i(idx^1).$$

Defining

$$X_1 = \frac{1}{2}dx^0, \quad X_2 = \frac{1}{2}idx^1, \quad X_3 = \frac{1}{2}dx^0 \wedge dx^1, \tag{23}$$

the above commutators can be synthesised as

$$[X_k, X_\ell]_\vee = i\epsilon_{k\ell m} X_m, \tag{24}$$

with $\epsilon_{k\ell m}$ the usual totally antisymmetric symbol. The dual Hodge forms are $dx^1 \wedge dx^2$, $dx^0 \wedge dx^2$ and dx^2 for the original set. Consider now the commutators among the set $idx^1 \wedge dx^2$, $-dx^0 \wedge dx^2$ and idx^2. One finds

$$\left[idx^1 \wedge dx^2, -dx^0 \wedge dx^2\right]_\vee = 2idx^0 \wedge dx^1$$

$$\left[idx^2, idx^1 \wedge dx^2\right]_\vee = 2i(idx^1)$$

$$\left[-dx^0 \wedge dx^2, idx^2\right]_\vee = 2idx^0 \wedge dx^1,$$

so that by defining

$$Y_1 = \frac{1}{2} idx^1 \wedge dx^2, \quad Y_2 = -\frac{1}{2} dx^0 \wedge dx^2, \quad Y_3 = \frac{1}{2} idx^2, \tag{25}$$

the above commutators can be summarised in the relation

$$[Y_k, Y_\ell]_\vee = i\epsilon_{k\ell m} X_m. \tag{26}$$

Analogously to the matrix case, define now

$$W_k^+ = \frac{1}{2}(X_k + Y_k) \tag{27}$$

$$W_k^- = \frac{1}{2}(X_k - Y_k). \tag{28}$$

Application of the rules of the game results in

$$\left[W_k^+, W_\ell^+\right]_\vee = i\epsilon_{k\ell m} W_m^+ \tag{29}$$

$$\left[W_k^+, W_\ell^-\right]_\vee = 0 \tag{30}$$

$$\left[W_k^-, W_\ell^-\right]_\vee = i\epsilon_{k\ell m} W_m^-. \tag{31}$$

Just as for the matrices, we get the structure of an $SU(2) \times SU(2)$ algebra for the differential forms. That is, conversely, any differential form can be represented as a linear combination of $SU(2) \times SU(2)$ generators, the unit matrix and the volume form. This is precisely the content of the Graf isomorphism [6],

$$\gamma^\mu \leftrightarrow dx^\mu \vee . \tag{32}$$

Moreover, the following properties are valid:

$$*W_k^\pm \;=\; \pm i W_k^\mp \quad (k = 1, 2) \tag{33}$$

$$*W_3^\pm \;=\; \mp i W_3^\mp \tag{34}$$

and, subsequently,

$$**W_k^\pm = W_k^\pm \quad (k = 1, 2, 3). \tag{35}$$

We see that the prominent role of the matrix $i\gamma^0\gamma^1\gamma^2$, the product of the three generators of the Clifford algebra, corresponds in this language to the volume form of spacetime. The handedness or chirality of the volume element labels precisely the two blocks in the corresponding matrix description.

Our emphasis on the consideration that the product of two generators of the Clifford algebra be a different quantity than the other generator translates here in the well-known fact that the exterior product of two line elements is an element of area, not another line element.

We see from the above equations for the $SU(2)$ generators $W^{\pm}$ under a duality (Hodge) transformation that the irreducible inequivalent 2×2 representations coming from the analysis of the Clifford algebra as a finite group form in fact a dual closed set. From this follow their transformation under an inversion of any axis. In terms of the volume form, they transform from one chirality into the opposite.

As a final interesting observation, let us remark that

$$2iW_1^+ \vee 2iW_2^+ \vee 2iW_3^+ = \frac{1}{2}(1 + i\varepsilon) \tag{36}$$

$$2iW_1^- \vee 2iW_2^- \vee 2iW_3^- = \frac{1}{2}(1 - i\varepsilon), \tag{37}$$

which are properly the chiral projections on the 2×2 block spaces.

4 The Dirac equation in three dimensions

The explicit construction of the representation for the Dirac matrices confirms that the four components of a 4-spinor couple in pairs in the differential equation.

This implies that physically the world in three dimensions is made of two blocks labelled by the eigenvalues of $i\gamma^0\gamma^1\gamma^2$, which could perhaps be qualified as right handed or left handed, according to the way coordinate axes are oriented. Particles and antiparticles are partners with the same handedness, as we shall see below.

However, as demonstrated for two and four dimensions of spacetime [1,2,3], the discrete transformations of space inversion and time reversal are crucial to the complete understanding of the underlying algebraic structure. In our case, they forbid one to isolate a single handedness and force the description of spin-$\frac{1}{2}$ particles in three dimensions by four-component spinors.

We now refer briefly to the formalism which deals with spin-$\frac{1}{2}$ particles in terms of differential forms. Further references can be found in our previous work [1,2,3] and in the work of Becher and Joos [8].

The differential operators of exterior differentiation, d, and its adjoint (with respect to the "usual" scalar product), δ, combine to form the Dirac–

Kähler operator, $i(d-\delta)$.[1] This operator leaves invariant the minimal left ideals of the Clifford algebra [3,8] and any ideal can be associated to a Dirac spinor.[2]

Since the work with differential forms for spin-$\frac{1}{2}$ particles is less familiar to theoretical physicists, we shall only state here that one can construct the set of four coupled linear differential equations of the first degree from the application of the Dirac–Kähler operator to a minimal left ideal, Ψ,

$$i(d-\delta)\Psi = m\Psi. \tag{38}$$

The four equations couple by pairs, as expected from the algebraic structure described above. The properties under discrete transformations can be discussed in the same way we shall proceed for spinors.

5 Discrete transformations: C, P, T and CPT

5.1 Charge conjugation

Returning to the matrix formalism, let us begin by considering charge conjugation. Applying the standard procedures from textbooks [11], the matrix

[1] We have slightly changed our conventions with respect to previous work, following now those of Curtis and Miller [9].

[2] We remind the reader that a left ideal is a subset of an algebra which is invariant under left multiplication; it is minimal when it is the smallest subset of this class.

that implements the charge conjugation operator, C, should satisfy

$$C^{-1}\gamma^{\mu}C = -\gamma^{\mu t}, \tag{39}$$

where the superscript t denotes transposition. For three dimensions, it turns out that the Hodge duality properties of differential forms translates into the existence of two matrices in the matrix formalism as the representatives of differential forms related by Hodge duality. The matrices also depend on the picture. For the pictures considered above, we have

$$\psi^c = \gamma^0\gamma^1\psi^* \quad \text{or} \quad -i\gamma^2\psi^* \quad \text{in the first example, Eqs. (3) and (5),}$$

$$\psi^c = i\gamma^1\psi^* \quad \text{or} \quad -\gamma^0\gamma^2\psi^* \quad \text{for the Dirac–Pauli and Kramers–Weyl pictures.}$$

The difference between the two alternatives is a relative sign for the lower pair of components.

Since the matrices at work belong always to the class of block-diagonal 4×4 matrices, the charge-conjugate spinor features only a reshuffling within the blocks corresponding to each of the eigenvalues of the matrix $i\gamma^0\gamma^1\gamma^2$.

5.2 Space inversion

In three dimensions, space inversion is different when compared to the neighbouring cases of even total dimensions $(2, 4)$. It is a simple drawing exercise to check that the simultaneous inversion of both space axes corresponds to a rotation through an angle π about the time axis. Accordingly, spinors are related by a simple rotation operator. There is again an equivalent action by the dual operator. For all pictures, one has

$$\psi'(x' = -x, t) = i\gamma^1\gamma^2\psi(x, t),$$

or

$$\psi'(x' = -x, t) = \gamma^0\psi(x, t).$$

Thus, the blocks with different handedness are not exchanged.

If it is desirable to exchange handedness (or chiralities), this can be performed by inversion of a single space axis. The matrix representing this is no longer of the block-diagonal class, but the ambiguity concerning the relative sign for the pair of lower components in the spinor persists. Calling $P_{(k)}$ the matrix for inversion of the k space axis, the candidates, for the pictures considered, are

$$\text{The first example:}\quad P_{(1)} = \begin{pmatrix} 0 & \pm\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \qquad P_{(2)} = \begin{pmatrix} 0 & \pm I \\ I & 0 \end{pmatrix}$$

$$\text{Dirac–Pauli:}\quad P_{(1)} = \begin{pmatrix} 0 & \pm\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad P_{(2)} = \begin{pmatrix} 0 & \pm I \\ I & 0 \end{pmatrix}$$

$$\text{Kramers–Weyl:}\quad P_{(1)} = -i\begin{pmatrix} 0 & \pm\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad P_{(2)} = \begin{pmatrix} 0 & \pm\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}.$$

This clearly shows the inadequacy of the representation by two-component spinors.

5.3 Time reversal

This transformation exchanges handedness by necessity. The novel feature in three dimensions is the link between the components of the complex-conjugate spinor and a spinor transformed from the original one. The results for the time-reversed spinors in the pictures considered are

Dirac–Pauli

$$\psi'(x, t' = -t) = \begin{pmatrix} 0 & \sigma_1 \\ \pm\sigma_1 & 0 \end{pmatrix} \psi(x, -t) = \begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix} \psi^*(x, -t); \quad (40)$$

Kramers–Weyl

$$\psi'(x, t' = -t) = \begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix} \psi(x, -t) = \begin{pmatrix} 0 & \sigma_1 \\ \pm\sigma_1 & 0 \end{pmatrix} \psi^*(x, -t). \quad (41)$$

The operation for the first example is the same as in eq. (40). Again, we see that the matrices concerned are outside the realm of diagonal-block matrices.

It is interesting to remark that, when a relative sign appears between the non-diagonal blocks, $T^2 = -1$. For matrices with blocks of the same sign, $T^2 = 1$.

5.4 CPT

As follows from the considerations above, there are two classes of results for the combined operations. When space inversion is meant as a simultaneous reversal of both space axes, the pair of upper components takes the place of the pair of lower components. In general, there is, in addition, an exchange inside each pair, with relative phases being introduced. There is one particular exception, which occurs for the Kramers–Weyl picture, with

$$C = \gamma^2, \qquad P = i\gamma^1\gamma^2, \qquad T = \begin{pmatrix} 0 & \sigma_1 \\ \pm\sigma_1 & 0 \end{pmatrix}, \quad (42)$$

we have

$$(CPT)\psi = \begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix} \psi. \tag{43}$$

When space inversion means the reversal of a single space axis, CPT always results in a block-diagonal matrix acting on the original spinor. This block matrix turns out to be the gamma matrix with the spatial index that is not inverted, or its "dual". For $P_{(1)}$ it is γ^2 or $i\gamma^0\gamma^1$, and for $P_{(2)}$ the result is similar. The geometrical meaning of this result is not yet fully appreciated.

6 A physical application: non-existence of topological mass in QED_3

In this section we illustrate how our prescription for 4×4 gamma matrices lead to different physical results when compared to the usual representation in terms of Pauli matrices, namely, $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_1$, $\gamma^2 = i\sigma_2$. For this, we compute the 1-loop vacuum polarisation in three-dimensional electrodynamics, commonly pictured by the Feynman graph of Figure 1.

Applying the Feynman rules for a 3-dimensional spacetime, we obtain, for the polarisation tensor,

$$\Pi^{\mu\nu}(p^2) = -\int \frac{d^3q}{(2\pi)^3} \frac{\mathrm{tr}\,\{\gamma^\nu(\not q + m)\gamma^\mu[(\not q - \not p) + m]\}}{[(q-p)^2 - m^2](q^2 - m^2)}. \qquad (44)$$

For comparison, we compute the trace of gamma matrices appearing in the above expression in the 4-dimensional representation for which we have given arguments to support its use as natural in three dimensions and a 2-dimensional one, currently found in the literature, through the use of the following properties:

- Algebraic representation (4×4)

$$\begin{aligned} \mathrm{tr}\,\gamma^\mu\gamma^\nu &= 4g^{\mu\nu} \\ \mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda &= 0 \\ \mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho &= 4(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho} + g^{\mu\rho}g^{\nu\lambda}) \end{aligned}$$

- Finite-group representation (2×2)

$$\begin{aligned} \mathrm{tr}\,\gamma^\mu\gamma^\nu &= 2g^{\mu\nu} \\ \mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda &= -2i\epsilon^{\mu\nu\lambda} \\ \mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho &= 2(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho} + g^{\mu\rho}g^{\nu\lambda}). \end{aligned}$$

It is mainly the clearly different trace of three gamma matrices that leads to physically inequivalent results for either type of representation, as Jackiw emphasises in recent lectures [12].

Thus, the traces in the polarisation tensor formula becomes

- 4×4 representation:

$$4[-p^{\mu}q^{\nu} - p^{\nu}q^{\mu} + (p \cdot q)g^{\mu\nu} + 2q^{\mu}q^{\nu} - q^2 g^{\mu\nu} + m^2 g^{\mu\nu}] \tag{45}$$

- 2×2 representation:

$$2[-p^{\mu}q^{\nu} - p^{\nu}q^{\mu} + (p \cdot q)g^{\mu\nu} + 2q^{\mu}q^{\nu} - q^2 g^{\mu\nu} - im\epsilon^{\mu\nu\lambda}p_{\lambda} + m^2 g^{\mu\nu}]. \tag{46}$$

After introducing a Feynman parameter x, making a shift $q^{\mu} \rightarrow q^{\mu} + p^{\mu}x$ in the loop momentum, passing to Euclidian space according to $x^3 = ix^0$, $q^0 = -iq^3$, so that $d^3q = -i|\vec{q}|^2 \sin\theta d|\vec{q}|d\theta d\phi$ in spherical coordinates, and performing symmetrical integration, we get for the polarisation tensor,

$$\Pi^{\mu\nu}(p^2) = g^{\mu\nu}\Pi_1(p^2) + \frac{p^{\mu}p^{\nu}}{p^2}\Pi_2(p^2) + im\epsilon^{\mu\nu\lambda}p_{\lambda}\Pi_3(p^2), \tag{47}$$

with

$$\begin{aligned}
\Pi_1(p^2) &= \frac{2i}{\pi^2}\int_0^1 dx \int_0^{\infty} |\vec{q}|^2 d|\vec{q}| \frac{p^2x(1-x) + \frac{1}{3}|\vec{q}|^2 + m^2}{[|\vec{q}|^2 - p^2x(1-x) + m^2]^2} && (48) \\
\Pi_2(p^2) &= -\frac{4i}{\pi^2}\int_0^1 dx \int_0^{\infty} |\vec{q}|^2 d|\vec{q}| \frac{p^2x(1-x)}{[|\vec{q}|^2 - p^2x(1-x) + m^2]^2} && (49) \\
\Pi_3(p^2) &= 0, && (50)
\end{aligned}$$

for the 4×4 representation, while for the the 2×2 case, $\Pi_1(p^2)$ and $\Pi_2(p^2)$ get a factor of $1/2$ with respect to the above expressions and $\Pi_3(p^2)$ becomes

$$\Pi_3(p^2) = \frac{i}{\pi^2}\int_0^1 dx \int_0^{\infty} |\vec{q}|^2 d|\vec{q}| \frac{1}{[|\vec{q}|^2 - p^2x(1-x) + m^2]^2}. \tag{51}$$

The gauge-invariant expression for the polarisation tensor is then given by

$$\Pi^{\mu\nu}_{\rm G.I.}(p^2) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)[(1-\alpha)\Pi_1(p^2) + \alpha\Pi_2(p^2)], \tag{52}$$

where $0 \leq \alpha \leq 1$ is an arbitrary parameter (see [13]), or

$$\Pi^{\mu\nu}_{\rm G.I.}(p^2) = \frac{1}{2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Pi_{\rm G.I.}{}^{\mu}_{\mu}(p^2) + im\epsilon^{\mu\nu\lambda}p_\lambda\Pi_3(p^2). \tag{53}$$

As we can see from eq. (52) or (53), there is no induced Chern–Simons mass term in the consistent 4×4 representation.

Using the 2×2 representation, an argument was raised [14] concerning the dependence of this term on the regularisation procedure. We show that, in any case, the appearance of this term [12] is in reality an artifact of the 2×2 representation.

7 Conclusions

The main results of this work were listed in the introduction. Let us here just add a few comments.

We believe that we have unveiled a most important algebraic structure of three-dimensional spacetime. These new results exemplify and reinforce, for three dimensions, the validity of the Graf isomorphism between the matrices associated to the Dirac equation and the differential forms with Clifford product. In other words, the algebraic structure associated with spacetime follows directly from the related Clifford algebra structure.

Besides, our results support the existence, for any number of dimensions, of a Dirac ring in which the products of Dirac matrices are different from them. This is natural for differential forms, but was not appreciated earlier in three dimensions for the Dirac matrices.

We have also shown that this algebraic structure makes the discrete operations of space inversion, time reversal and charge conjugation in three dimensions rather peculiar. The understanding of this demands further investigation.

In the light of this structure, the need for a description through four-component spinors of spin-$\frac{1}{2}$ particles follows. These four-component spinors and four-dimensional representation for the gamma matrices are certainly different from the ones currently quoted in the literature, and this may induce changes in several physical results obtained for three-dimensional systems. As an example of how physical results may change, we have shown explicitly for QED_3 the non-appearance of the induced Chern–Simons mass term at one-loop approximation.

A careful treatment of space inversion for Chern–Simons theories including (three-dimensional) fermions was required in recent articles by Hagen

[15] in order to handle the possible P or T non-invariance of these theories. The need to include both spin "projections" (which correspond in our terminology to "chirality") is emphasised, and the correct understanding of fermion transformation properties under discrete symmetries seems mandatory.

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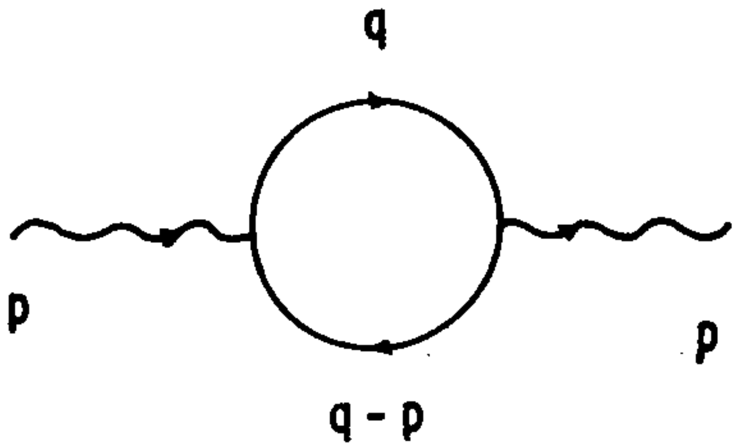

FIG.1

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