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ABSTRACT

The usual FRW hot big-bang cosmologies have been generalized by considering the equation of state $\rho = Anm + (\gamma-1)^{-1}p$, where m is the rest mass of the fluid particles and A is a dimensionless constant. Explicit analytic solutions are given for the flat case ($\varepsilon = 0$). For large cosmological times these extended models behave as the standard Einstein-de Sitter universes regardless of the values of A and γ. Unlike the usual FRW flat case the deceleration parameter q is a time-dependent function and its present value, $q \simeq 1$, obtained from the luminosity distance versus redshift relation, may be fitted by taking, for instance, A = 1 and $\gamma = 5/3$ (monatomic relativistic gas with $m \gg k_B T$). In all cases the universe cools obeying the same temperature law of the FRW models and it is shown that the age of the universe is only slightly modified.

Key-words: Cosmology; General relativity; Thermodynamics; Equations of state.

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Observations have revealed that the universe is very close to a Friedmann-Robertson-Walker (FRW) cosmological model, whose line element may be cast in the following form[1] ($c=1$):

$$ds^2 = dt^2 - R^2(t)\left[(1-\varepsilon r^2)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right], \quad (1)$$

where $\varepsilon = 0,\pm 1$ is the curvature parameter of the maximally symmetric spatial sections $t=\text{const}$.

In that background, the nontrivial Einstein field equations for a comoving perfect fluid as source of gravitation, and the particle number conservation law are given by ($8\pi G = 1$)

$$\rho = \frac{3}{R^2}(\dot{R}^2 + \varepsilon) \quad , \quad (2)$$

$$p = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{\varepsilon}{R^2} \quad , \quad (3)$$

$$\frac{\dot{n}}{n} = -3\frac{\dot{R}}{R} \quad , \quad (4)$$

where an overdot means a time derivative, and ρ, p and n are the energy density, the thermostatic pressure and the particle number density, respectively.

As it stands, the above system is underdetermined since there are four unknowns and only three equations. Thus it is necessary to provide one more relation between the variables, which is usually supplied by thermodynamical considerations. The expression ordinarily employed in the cosmological context is the well-known "gamma-law" equation of state

$$p = (\gamma-1)\rho \quad , \tag{5}$$

where the constant parameter γ lies on the interval $[0,2]$. With this choice, the subsystem (2),(3) and (5), in the unknowns R,ρ and p is already determined, independently of (4); accordingly, the usual procedure in the literature is to discard (4), thereby ignoring the true dynamic degree of freedom related to the variable n. Such an approach is in fact justifiable only in four distinct cases: (i) a vacuum-like stage ($\gamma = 0$); (ii) a radiative (ultrarelativistic) fluid ($\gamma = 4/3$); (iii) an incoherent fluid (dust) ($\gamma = 1$), or (iv) stiff matter[2] ($\gamma = 2$).

In this work, our aim is to examine the role of the variable n upon the behavior of the model (evolution of the scale factor, thermodynamical and observational aspects, etc). To that end, we consider an equation of state more general than (5), which holds for an ideal relativistic Maxwell-Boltzmann gas subject to a polytropic process, that is, one for which

$$p = Kn^{\gamma} \tag{6}$$

where K is an integration constant and γ is the polytropic index. In this case, it may be shown that, for $\gamma \neq 1$, the new relation between ρ,p and n is of the form[2-4]

$$\rho = Amn + (\gamma-1)^{-1}p \quad , \tag{7}$$

where A is a dimensionless constant and m is the (rest) mass of the constituent particles of the fluid. If $A=0$, the usual "gamma-law" is recovered. For a monatomic ($\gamma = 5/3$) relativistic gas at temperature T, as first shown by Jüttner[5-7] from a statistical point of view, one may take $A=1$, providing $m \gg k_B T$, where k_B is Boltzmann's constant. Notice also that, as concerns Eqs.(2)+(4), one may interpret (7) just as the energy density of a noninteracting two-fluid mixture in the old scheme ("gamma-law"): for $A=1$ and $\gamma = 4/3$, we have dust plus radiation. From now on, we consider arbitrary constant values of A and γ, and search out analytical solutions to the full system of equations (2)-(4) and (7).

First, we integrate (7) to obtain

$$n = n_o \left(\frac{R_o}{R}\right)^3 \quad , \tag{8}$$

where n_o is the particular value of n at $R = R_o$. Then, substituting for n from (8) into (7), it follows that the variables ρ and p may be eliminated from (2) and (3), yielding a generalized second-order Friedmann equation for the scale factor R(t),

$$R\ddot{R} + \frac{3\gamma-2}{2}\dot{R}^2 + \frac{3\gamma-2}{2}\varepsilon - (\gamma-1)\frac{B}{R} = 0 \quad , \tag{9}$$

a first integral of which is

$$\dot{R}^2 = \left(\frac{R_o}{R}\right)^{3\gamma-2} - \varepsilon + \frac{2B}{3R} \quad , \tag{10}$$

where $B = Amn_o R_o^3/2$. So we can express the energy density and the pressure as functions solely of R:

$$\rho = \frac{3}{R_o^2}\left(\frac{R_o}{R}\right)^{3\gamma}\left[1 + \frac{2B}{3R_o}\left(\frac{R}{R_o}\right)^{3(\gamma-1)}\right] \quad , \tag{11}$$

$$p = \frac{3(\gamma-1)}{R_o^2}\left(\frac{R_o}{R}\right)^{3\gamma} \quad . \tag{12}$$

Notice that the functional dependence of p on R remains the same as in the usual FRW models, as should be expected from (6) and (8) which are valid (at least implicitly) on both formulations. If B = 0, Eqs. (9)-(12) reduce to those of the standard FRW models. In this case, the unified solutions of (9) for all values of ε and γ have recently been found by Assad and Lima[8] in terms of hypergeometric functions. Here, for the sake of simplicity, we consider only the flat models ($\varepsilon = 0$). Following the procedure carried out in Ref. (8), it is straightforward to show that the solution of (9) or, equivalently, (10) is given by

$$t - t_o = \frac{2R_o}{3(\gamma-1)}\left(1 + \frac{2B}{3R_o}\right)^{1/2} f(R_o) + - \frac{2R_o}{3(\gamma-1)}\left(\frac{R}{R_o}\right)^{3\gamma/2}\left[1 + \frac{2B}{3R_o}\left(\frac{R}{R_o}\right)^{3(\gamma-1)}\right]^{1/2} f(R) \tag{13}$$

where $t_o = t(R_o)$ and $f(R)$ is the hypergeometric function

$$f(R) = F\left[\frac{2\gamma-1}{2(\gamma-1)}, 1; 3/2; 1+\frac{2B}{3R_o}\left(\frac{R}{R_o}\right)^{3(\gamma-1)}\right] . \qquad (14)$$

It should be stressed that, taking the limit $B \to 0$ and using the identity[9] $F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, Eq. (13) furnishes the expected result,

$$R = R_o\left[1 + \frac{3\gamma}{2}\left(\frac{t-t_o}{R_o}\right)\right]^{2/3\gamma} , \qquad (15)$$

of the usual FRW models with the "gamma-law" equation of state.[8]

Parametric solutions. Defining a dimensionless quantity τ by

$$dt = R_o\left(\frac{R}{R_o}\right)^{3/2} d\tau , \qquad (16)$$

and an auxiliary scale factor y by

$$y = R_o\left(\frac{R}{R_o}\right)^{3(\gamma-1)/2} , \qquad (17)$$

it follows that (9) may be rewritten (for $\varepsilon = 0$) as

$$y'' - \omega^2 y = 0 , \qquad (18)$$

where a prime denotes a derivative with respect to τ, and we have introduced the parameter

$$\omega = \left(\frac{3B}{2R_o}\right)^{1/2} |\gamma-1| . \qquad (19)$$

For $\gamma \neq 1$, the general solution of (18), valid also in the limit $\omega \to 0$, is

$$Y = Y_0 \frac{\sinh\omega(\tau+\delta)}{\omega} , \qquad (20)$$

where Y_0 and δ are integration constants. Now, using the first integral (10), one finds $Y_0 = 3R_0|\gamma-1|/2$, and choosing $\delta=0$, (17) gives

$$R(\tau) = R_0\left(\frac{3|\gamma-1|}{2}\right)^{\frac{2}{3(\gamma-1)}}\left(\frac{\sinh\omega\tau}{\omega}\right)^{\frac{2}{3(\gamma-1)}} , \qquad (21)$$

whereas the cosmic time t is readily obtained inserting (21) into (23),

$$t(\tau) - t_0 = \frac{2R_0}{3(\gamma-1)}\left[1 + \frac{4\omega^2}{9(\gamma-1)^2}\right]^{1/2} f(R_0) +$$

$$- \frac{2R_0}{3(\gamma-1)}\left(\frac{3|\gamma-1|}{2}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{\sinh\omega\tau}{\omega}\right)^{\frac{\gamma}{\gamma-1}} \cosh\omega\tau\, g(\tau) , \qquad (22)$$

with $g(\tau) = f(R(\tau))$ as given by (14) and (21). We remark that, by taking the limit $B \to 0$ in (21) and (22), the parametric solutions of the standard FRW models are established in an unusual (nonconformal) time coordinate.

Some thermodynamical aspects and observable quantities. The temperature of an expanding perfect fluid evolves according to[10-12]

$$\frac{\dot{T}}{T} = \left(\frac{\partial p}{\partial \rho}\right)_n \frac{\dot{n}}{n} . \qquad (23)$$

Now, using (7) and (8), we get, for any value of ε,

$$T = T_o\left(\frac{R_o}{R}\right)^{3(\gamma-1)} \quad , \tag{24}$$

where $T_o = T(R_o)$. Thus, the temperature of the model scales with $R^{-3(\gamma-1)}$, just as in the standard FRW case.[12] However, it is easy to show that the speed of sound v_s is now a time-dependent quantity, as it should be:

$$v_s^2 = \gamma \frac{p}{\rho+p} = \frac{\gamma(\gamma-1)p}{(\gamma-1)Bmn+\gamma p} \tag{25}$$

If B = 0, the constant speed $v_s^2=\gamma-1$ of the standard FRW models is recovered.

To illustrate some observational predictions of the present model, we compute first the deceleration parameter.[1] Using Eqs. (10) and (11), and taking $\varepsilon=0$, we find that $q = -\ddot{R}R/\dot{R}^2$ is given by

$$q = q_{FRW} - \frac{(\gamma-1)B}{R_o\left[(R_o/R)^{3(\gamma-1)} + 2B/3R_o\right]} \tag{26}$$

where $q_{FRW} = (3\gamma-2)/2$ is just the value of the parameter q in the standard FRW flat case ($B=0$) for an arbitrary value of γ. Using the time coordinate τ, in which $R(\tau)$ assumes the form (21), an explicit expression for $q(\tau)$ may be obtained:

$$q = \frac{3\gamma-2}{2} - \frac{3(\gamma-1)}{2}\tanh^2\omega\tau \tag{27}$$

In Fig. 1 we plot q as a function of τ for several representative values of γ. Notice that the present value $q \simeq 1$ obtained from the luminosity distance versus redshift relation may be fitted by taking, for instance, $\gamma = 5/3$ (monatomic gas) or any of the more realistic $\gamma > 1$, for that matter.

Finally, we remark that, since the metric evolves to the standard dust-filled FRW universe for large cosmic time, one would expect that the age of the universe is near the value of the age computed by using the Einstein-de Sitter model. In fact, from Eq. (10) with $\varepsilon = 0$, one finds for the present value of the Hubble parameter.

$$H_0^2 = H_{FRW,0}^2 (1 + \Delta) \quad , \qquad (28)$$

where $H_{FRW,0} = 1/R_0$ is the Hubble parameter of the usual FRW model ($B = 0$) at $R = R_0$, and $\Delta = 2B/3R_0$. Since $B = mn_0 R_0^3/2$, we may take $mn_0 \simeq \rho_G \simeq 3,1.10^{-31}$ g/cm^3 (galactic mass density) and $1/R_0 \simeq$ 75.Km/s.Mpc to loosely estimate $\Delta \simeq 0,03$.

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FIGURE CAPTION

Fig. 1 - The deceleration parameter for flat models with distinct values of γ.

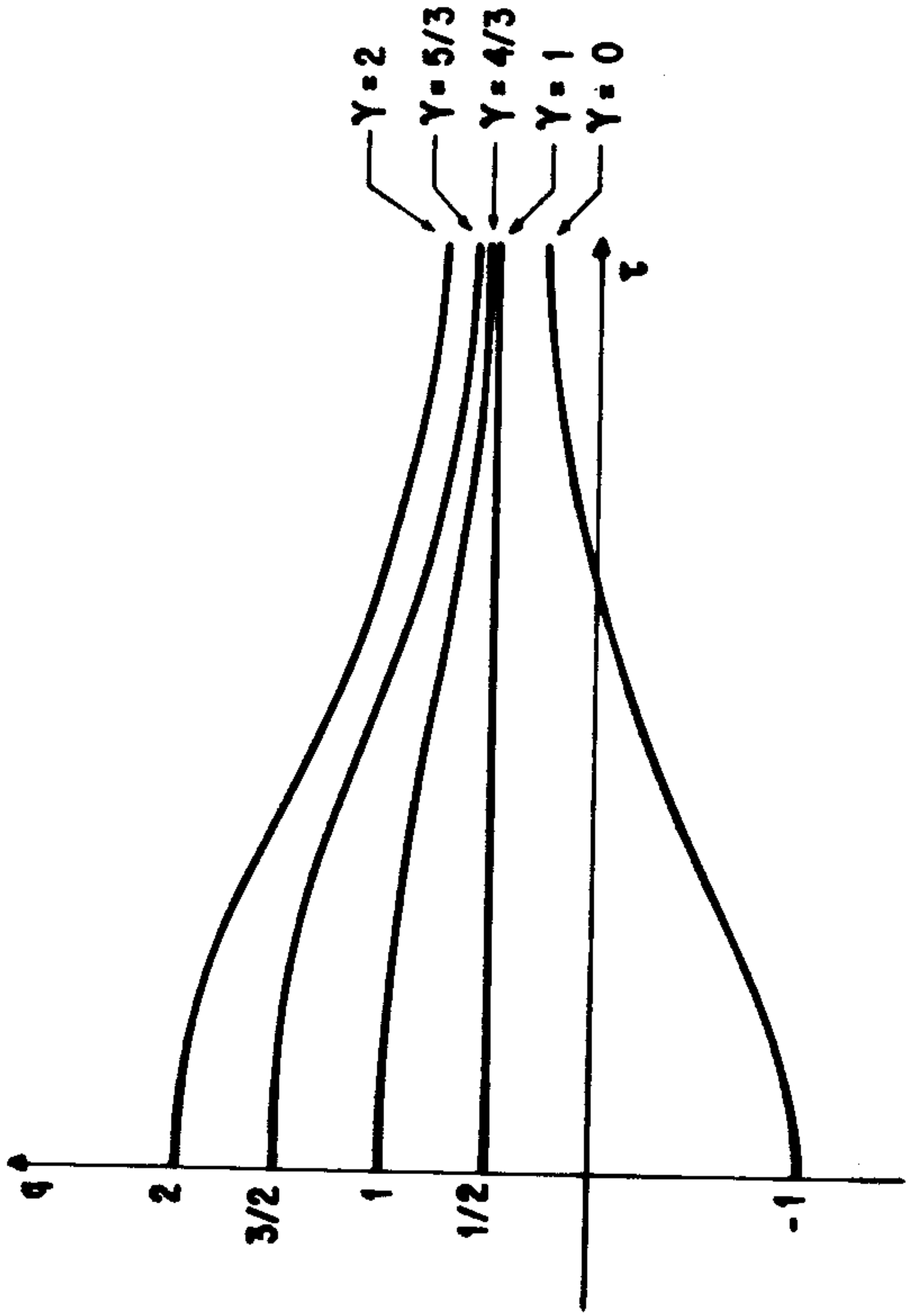

Fig. 1

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