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CONVOLUTIONS OF PARTICLE GREEN FUNCTIONS

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ABSTRACT

The use of Feynman causal function in the perturbative treatment of S-matrix made the computation of convolutions an easy and well known procedure for free particle propagators. But the convolution of its components, like the δ and Principal values among themselves is very rarely looked upon. In field theories with higher order equations of motion some of these convolutions appear as the fundamental ingredients. A discussion of these convolutions is explicitly done in the simplest examples.

Key-words: Convolution; Green functions; Self-energy; Higher order derivatives; Field theory.

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1 INTRODUCTION

The universal use of the Feynman causal function in the perturbative development of the S-matrix, made the computation of convolution of propagators a standard procedure. But if we remember the relation

$$\frac{1}{p^2+m^2-io} \equiv F = P + i\pi\delta(p^2+m^2) \tag{1}$$

(where P means principal value of $(p^2+m^2)^{-1}$); we see that the convolution F_*F of two causal propagators could also be done through the previous determination of P_*P, $P_*\delta$ and $\delta_*\delta$. Of course, this latter procedure is not necessary for ordinary computations, nevertheless it seems interesting to have its results and to the best of our knowledge they have not appear in the literature, at least explicitly and with generality. Also, it was shown elsewhere[1] that the propagator for the tachyon should be P (not F). Further, if we study a fourth-order equation[2] with the propagator $(p^4-m^4)^{-1} = \frac{1}{2m^2}(p^2-m^2)^{-1} - \frac{1}{2m^2}(p^2+m^2)^{-1}$, then with the adoption of Feynman propagator for the bradyon and Cauchy's principal value P for the tachyon, unitarity is preserved in the lowest perturbative approximation (at least) (See ref. [23]). In this case we are naturally led to compute F_*P and P_*P.

The situation just pointed out could be general for higher order differential equations of motion and this is the motivation of the present work.

We shall divide the exposition of computations in two parts. In the first one we shall only discuss ordinary particles (bradyons). In the second part we study the modifications to be introduced in the convolutions when one or both involved particles have complex masses.

PART I. Convolutions of bradyon Green functions

2 $\delta_* \delta$

This convolution can be computed in an easier way in the system in which $p^2 = -p_0^2$ if $p^2<0$, or in the system $p^2=p_1^2$ if $p^2>0$. In the first case:

$$\delta(p^2+m_1^2)_*\delta(p^2+m_2^2) \equiv \delta_{1*}\delta_2 = \int d^\nu q \ \delta(q^2+m_1^2)\delta((p-q)^2+m_2^2) =$$

$$=\int d^\nu q \ \delta(q^2+m_1^2)\delta(-p_0^2+2p_0q_0+m_2^2-m_1^2) = \int \frac{d^{\nu-1}q}{2|p_0|} \ \delta(\vec{q}^2+m_1^2-q_0^2), \text{ with } q_0=\frac{p_0^2+m_1^2-m_2^2}{2p_0} \tag{2}$$

From now on, when we integrate over the angles in an euclidean ν-dimensional space, we use the formula:

$$\int d^{\nu-1}q \ f(\vec{q}^2) = \frac{\pi^{\frac{\nu-1}{2}}}{\Gamma(\frac{\nu-1}{2})} \int_0^\infty dq^2 \ q^{2\,\frac{\nu-3}{2}} \ f(q^2) \tag{3}$$

From (2) and (3) we get:

$$\delta_{1*}\delta_2 = \frac{\pi^{\frac{\nu-1}{2}}}{2\Gamma(\frac{\nu-1}{2})}\int_0^\infty \frac{dq^2}{|p_0|}\, q^{2^{\frac{\nu-3}{2}}}\,\delta(q^2+m_1^2-q_0^2) =$$

$$= \frac{\pi^{\frac{\nu-1}{2}}}{2|p_0|\Gamma(\frac{\nu-1}{2})}\left[(p_0^2+m_1^2-m_2^2)^2-m_1^2\right]^{\frac{\nu-3}{2}} \qquad (4)$$

Where we can put $p_0^2 = -p^2(>0)$ and $[\chi]_+^\alpha = \chi^\alpha$ when $\chi>0$ and zero otherwise. When p^2 is space-like ($p^2>0$) we have:

$$\delta_{1*}\delta_2 = \int d^\nu q\;\delta(q^2+m_1^2)\delta(p_1^2-2p_1q_1+m_2^2-m_1^2)$$

$$= \frac{1}{2|p_1|}\int dq_0 \int d^{\nu-2}q\;\delta(-q_0^2+q_1^2+q_t^2+m_1^2) \qquad (5)$$

where q_t means the component of $\vec{q}$ transversal to q_1 and $q_1 = \dfrac{p_1^2+m_2^2-m_1^2}{2p_1}$

From (5)

$$\delta_1 * \delta_2 = \frac{1}{2|p_1|} \frac{\pi^{\frac{\nu-2}{2}}}{\Gamma(\frac{\nu-2}{2})} \int dq_0 \int_0^\infty dq_t^2 \, {q_t^2}^{\frac{\nu-4}{2}} \, \delta(-q_0^2+q_1^2+q_t^2+m_1^2)$$

$$\delta_1 * \delta_2 = \frac{\pi^{\frac{\nu-2}{2}}}{2|p_1|\Gamma(\frac{\nu-2}{2})} \int_{-\infty}^{\infty} dq_0 (q_0^2-q_1^2-m_1^2)_+^{\frac{\nu-4}{2}} =$$

$$= \frac{\pi^{\frac{\nu-2}{2}}}{2|p_1|\Gamma(\frac{\nu-2}{2})} 2\int_{q_1^2+m_1^2}^{\infty} \frac{dq_0^2}{2q_0} \, (q_0^2-q_1^2-m_1^2)^{\frac{\nu-4}{2}}$$

$$= \frac{\pi^{\frac{\nu-2}{2}}}{2|p_1|\Gamma(\frac{\nu-2}{2})} \int_0^\infty \frac{dx \; x^{\frac{\nu-4}{2}}}{\sqrt{x+q^2+m^2}} \tag{6}$$

We use now Ref. [3], p.285

$$\int_0^\infty dx \; x^{\lambda-1}(x+A)^{-\mu} = A^{\lambda-\mu} \frac{\Gamma(\mu-\lambda)\Gamma(\lambda)}{\Gamma(\mu)} \tag{7}$$

and (6) gives

$$\delta_1 * \delta_2 = \frac{\pi^{\frac{\nu-2}{2}}}{2|p_1|\Gamma(\frac{\nu-2}{2})} \frac{\Gamma(\frac{3-\nu}{2})\Gamma(\frac{\nu-2}{2})}{\Gamma(\frac{1}{2})} \left[\frac{(p_1^2+m_2^2-m_1^2)^2}{4p_1^2} + m_1^2\right]^{\frac{\nu-3}{2}} \quad (p^2>0) \tag{8}$$

Note that for ν=odd ≥ 3, (8) is divergent.

In order to compare with (4) we use:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{9}$$

When ν is even (8) and (9) give:

$$\delta_1 * \delta_2 = \frac{\pi^{\frac{\nu-1}{2}}(-1)^{\frac{\nu-2}{2}}}{2(p^2)^{1/2}\Gamma(\frac{\nu-1}{2})}\left[\frac{(p^2+m_2^2-m_1^2)^2}{4p^2} + m_1^2\right]^{\frac{\nu-3}{2}}$$

$$= \frac{\pi^{\frac{\nu-1}{2}}}{2(-p^2)^{1/2}\Gamma(\frac{\nu-1}{2})}\left[\frac{(p^2+m_2^2-m_1^2)^2}{-4p^2} - m_1^2\right]^{\frac{\nu-3}{2}}$$

which is formally similar to (4) with $p_0^2=-p^2$.

Comment. For the bracket in (4) to be positive it is necessary that $(p_0^2+m_1^2-m_2^2)^2 > 4p_0^2m_1^2$. It is easy to see that then, either $p_0^2>(m_1+m_2)^2$, or $p_0^2<(m_1-m_2)^2$. We may call "physical cut" the region $(m_1+m_2)^2<p_0^2<\infty$, and "low energy cut" the interval $0<p_0^2<(m_1-m_2)^2$. When the masses are equal the low-energy cut desappears. Formula (4) shows that the convolution $\delta_1 * \delta_2$ has support in both cuts (when $p^2<0$).

3 $P_* P$

With the usual definition of Hilbert transform:

$$H_0(f) = \frac{1}{\pi} \int_{-\infty}^{\infty} P \frac{1}{x-y} f(x)\, dx ,$$

the following theorem holds

$$\mathcal{H}(\mathcal{H}(f)) = -f$$

We can then write

$$\int_{-\infty}^{\infty} dy\ P \frac{1}{x-y}\ P \frac{1}{y-z} = -\ \pi^2 \delta(x-z) \qquad (10)$$

So we have:

$$P_1 * P_2 = \int d^\nu q \; P \frac{1}{q^2+m_1^2} \; P \frac{1}{(p-q)^2+m_2^2} =$$

$$= \int \frac{d^{\nu-1}}{4\omega_1\omega_2} \int dq_0 \left[P \frac{1}{q_0+\omega_1} - P \frac{1}{q_0-\omega_1} \right] \left[P \frac{1}{p_0-q_0+\omega_2} - P \frac{1}{p_0-q_0-\omega_2} \right]$$

$$P_1 * P_2 = -\pi^2 \int \frac{d^{\nu-1}q}{4\omega_1\omega_2} \left[\delta(p_0-\omega_1-\omega_2) + \delta(p_0+\omega_1+\omega_2) - \delta(p_0+\omega_1-\omega_2) - \delta(p_0-\omega_1+\omega_2) \right] \tag{11}$$

with

$$\omega_1 = +\sqrt{\vec{q}^{\,2}+m_1^2} \;\; ; \;\; \omega_2 = \sqrt{(\vec{p}-\vec{q})^2+m_2^2} \tag{12}$$

The supports of these δ-functions do not overlap. They are mutually exclusive. For the first two δ's in (11):

$$\omega_1+\omega_2 = \pm p_0$$
$$\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 = p_0^2$$

and for the last two:

$$\omega_1-\omega_2 = \pm p_0$$
$$\omega_1^2+\omega_2^2 - 2\omega_1\omega_2 = p_0^2$$

The four cases are contained in

$$(2\omega_1\omega_2)^2 = (p_0^2-\omega_1^2-\omega_2^2)^2$$

For the rest system (when $0>p^2 = -p_0^2$) we get from (12).

$$\vec{q}^2 = \frac{1}{4p_0^2}\left[(p_0^2-m_1^2-m_2^2)^2 - 4m_1^2m_2^2\right] \tag{13}$$

The conditions for the bracket to be positive are the same as those for (4). When p_0 belongs to the physical cut ($p_0^2 > (m_1+m_2)^2$ we have

$$\omega_1 = \frac{1}{2|p_0|}(p_0^2+m_1^2-m_2^2) \text{ and } \omega_2 = \frac{1}{2|p_0|}(p_0^2+m_2^2-m_1^2)$$

(we take $m_1 \geq m_2$)

$$\omega_1 + \omega_2 = |p_0|$$

and only the first two δ's can contribute to (11).

If instead $p_0^2<(m_1-m_2)^2$, we get:

$$\omega_1 = \frac{1}{2|p_0|}\,(p_0^2+m_1^2-m_2^2) \text{ and } \omega_2 = \frac{-1}{2|p_0|}\,(m_1^2-m_2^2-p_0^2)$$

$$\omega_1 - \omega_2 = |p_0|$$

and only the last two δ's can contribute in (11). In any case it is easy to perform the integration by a change of variables. For example, for the first δ-function we have to divide by

$$\frac{d}{dq^2}\,(p_0-\omega_1-\omega_2) = -\frac{1}{2\omega_1} - \frac{1}{2\omega_2} = -\frac{2p_0}{4\omega_1\omega_2}$$

In this way we obtain, after integration over the angles:

$$P_1{*}P_2 = -\,\varepsilon\,\pi^2\,\frac{\pi^{\frac{\nu-1}{2}}}{2|p_0|\Gamma(\frac{\nu-1}{2})}\left\{\frac{1}{4p_0^2}\left[(p_0^2-m_1^2-m_2^2)\right]^2 - 4m_1^2m_2^2\right]\right\}_+^{\frac{\nu-3}{2}} \tag{14}$$

where

$$\varepsilon = +1 \text{ on the physical cut } p_0^2 \geq (m_1+m_2)^2 \tag{15}$$

$$\varepsilon = -1 \text{ on the low-energy cut } p_0^2 \leq (m_1-m_2)^2 \tag{16}$$

A comparison with (4) shows that

$$P_{1}*P_{2} = -\varepsilon\ \pi^{2}\ \delta_{1}*\delta_{2} \quad \text{if} \quad p^{2}<0 \tag{17}$$

When p^2 is spacelike we go back to (11) assuming that the only component of p_μ is p_1. In this case we have

$$P_{1}*P_{2} = 2\pi^{2} \int \frac{d^{\nu-1}q}{4\omega_{1}\omega_{2}}\ \delta(\omega_{1}-\omega_{2}) \tag{18}$$

as the first two δ-functions do not contribute (p_0=0) and the last two are equal.

Besides, when $\omega_1 = \omega_2$ (Cf. (12)):

$$q_{1} = \frac{p_{1}^{2}+m_{2}^{2}-m_{1}^{2}}{2p_{1}}$$

and

$$\left.\frac{d(\omega_{1}-\omega_{2})}{dq_{1}}\right|_{\omega_{1}=\omega_{2}} = \left.\frac{q_{1}(\omega_{2}-\omega_{1}) + p_{1}\omega_{1}}{\omega_{1}\omega_{2}}\right|_{\omega_{1}=\omega_{2}} = \frac{p_{1}}{\omega_{2}}$$

Integration of (18) over the angles (in the $(\nu-2)$-dimensional space of $\vec{q}_t$) gives:

$$P_{1*}P_2 = 2\pi^2 \frac{\pi^{\frac{\nu-2}{2}}}{\Gamma(\frac{\nu-2}{2})} \int_0^{\infty} dq_t^2\, q_t^{2\,\frac{\nu-4}{2}} \int_{-\infty}^{\infty} \frac{dq_1}{4\omega_1\omega_2}\, \delta(\omega_1-\omega_2)$$

$$= \frac{\pi^2}{2|p_1|} \frac{\pi^{\frac{\nu-2}{2}}}{\Gamma(\frac{\nu-2}{2})} \int_0^{\infty} \frac{dq_t^2 (q_t^2)^{\frac{\nu-4}{2}}}{\omega} \quad , \qquad \omega = \left[q_t^2 + \frac{p_1^2+m_2^2-m_1^2}{4p_1^2} + m_1^2\right]^{\frac{1}{2}}$$

And, using (7)

$$P_{1*}P_2 = \pi^2 \pi^{\frac{\nu-3}{2}}\, \Gamma(\tfrac{3-\nu}{2}) \left[\frac{(p^2+m_2^2-m_1^2)^2}{4p^2} + m_1^2\right]^{\frac{\nu-3}{2}} \quad (p^2>0) \tag{19}$$

Comparing with (8) we see that

$$P_{1*}P_2 = \pi^2 \delta_{1*}\delta_2 \;\; (p^2>0) \tag{20}$$

Comment. It is perhaps unexpected to find out that $P_{1*}P_2$ have, up to a sign, the same value that $\pi^2\delta_{1*}\delta_2$ on the same cuts. Further, on the physical cut $P_{1*}P_2 = (i\pi)^2\delta_{1*}\delta_2$. This has relation with unitarity.

4 $\delta_* F$

Here we shall use the decomposition:

$$\delta(p^2+m^2) = \frac{1}{2\omega}\left[\delta(p_0-\omega) + \delta(p_0+\omega)\right] \tag{21}$$

and

$$F = \frac{1}{p^2+m^2} = \frac{1}{2\omega}\left[\frac{1}{\omega-p_0} + \frac{1}{\omega+p_0}\right] \tag{22}$$

Where in (22) the mass is supposed to have a small negative imaginary part.

$$\delta_{1*}F_2 = \int\frac{d^\nu q}{4\omega_1\omega_2}\left[\delta(q_0-\omega_1) + \delta(q_0+\omega_1)\right]\left[\frac{1}{\omega_2-p_0+q_0} + \frac{1}{\omega_2+p_2-q_0}\right]$$

$$= \int\frac{d^{\nu-1}q}{4\omega_1\omega_2}\left[\frac{1}{\omega_1+\omega_2-p_0} + \frac{1}{\omega_1+\omega_2+p_0} + \frac{1}{\omega_2-\omega_1-p_0} + \frac{1}{\omega_2-\omega_1+p_0}\right]$$

$$= \int\frac{d^{\nu-1}q}{4\omega_1\omega_2}\left[\frac{2(\omega_1+\omega_2)}{(\omega_1+\omega_2)^2-p_0^2} + \frac{2(\omega_2-\omega_1)}{(\omega_2-\omega_1)^2-p_0^2}\right] \tag{23}$$

$$\delta_{1*}F_2 = \int\frac{d^{\nu-1}q}{4\omega_1\omega_2}\left[\frac{4\omega_2(\omega_2^2-\omega_1^2-p_0^2)}{(\omega_1^2+\omega_2^2-p_0^2)^2 - 4\omega_1^2\omega_2^2}\right] \tag{24}$$

where ω_1 and ω_2 are given by (12). In the rest system ($p^2<0$) we can use (3).

$$\delta_1 * F_2 = \frac{\pi^{\frac{\nu-1}{2}}}{\Gamma(\frac{\nu-1}{2})} \int_0^\infty \frac{dq^2 q^{2\frac{\nu-2}{2}}}{\omega_1} \frac{(m_2^2-m_1^2-p_0^2)}{p_0^4-2p_0^2(2q^2+m_1^2+m_2^2) + (m_1^2-m_2^2)^2} \tag{25}$$

The integration can be carried out with the aid of:

$$\int_0^\infty ds\ s^{b-1}(1+s)^{a-c}(1+sz)^{-a} = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ z^{-a}\ F(a,c-b,c;1-\frac{1}{z}) \tag{26}$$

The result of (25) is:

$$\delta_1 * F_2 = \pi^{\frac{\nu-2}{2}}\ \Gamma(\frac{4-\nu}{2})\ \frac{(m_1^2)^{\frac{\nu-4}{2}}}{2p_0^2}\ (p_0^2+m_1^2-m_2^2)F(1,\ \frac{4-\nu}{2};\ \frac{3}{2};\ \ \frac{(p_0^2+m_1^2-m_2^2)^2}{4p_0^2m_1^2}) \tag{27}$$

($F(a,b,c;z)$ is the hypergeometric function).

Comment. According to the remark made just after (22), the mass m_2 in (27) has a small negative imaginary part. The argument of the hypergeometric function in (27) has then a small positive imaginary component. As $F(s,b;c,z)$ has a cut for z real >1 (Cf.), the real axis should be approached from above. If the limit is taken from below, the convolution of δ_1 with the anticausal Green-function $\overline{F}_2$, is obtained.

5 P_*F

We shall use (22) and, instead of (21), the principal value decomposition:

$$P_1 = \frac{1}{2\omega_1}\left[P\,\frac{1}{\omega_1-p_0} + P\,\frac{1}{\omega_1+p_0}\right] \tag{28}$$

which was already utilized in (11).

$$P_{1*}F_2 = \int \frac{d^\nu q}{4\omega_1\omega_2}\left[P\,\frac{1}{\omega_1-q_0} + P\,\frac{1}{\omega_1+q_0}\right]\left[\frac{1}{\omega_2+p_0+q_0} + \frac{1}{\omega_2+p_0-q_0}\right]$$

Consulting now a table of Hilbert transforms, taking into account the small imaginary part of ω_2 (due to that of m_2), we obtain:

$$P_{1*}F_2 = i\pi \int \frac{d^{\nu-1} q}{4\omega_1\omega_2}\left[\frac{1}{\omega_1+\omega_2-p_0} + \frac{1}{\omega_1+\omega_2+p_0} - \frac{1}{\omega_2-\omega_1-p_0} - \frac{1}{\omega_2-\omega_1+p_0}\right]$$

$$= i\pi \int \frac{d^{\nu-1} q}{4\omega_1\omega_2}\left[\frac{2(\omega_1+\omega_2)}{(\omega_1+\omega_2)^2-p_0^2} - \frac{2(\omega_2-\omega_1)}{(\omega_2-\omega_1)^2-p_0^2}\right] \tag{29}$$

$$P_{1*}F_2 = i\pi\int\frac{d^{\nu-1}q}{4\omega_1\omega_2}\left[\frac{4\omega_1(\omega_1^2-\omega_2^2-p_0^2)}{(\omega_1^2+\omega_2^2-p_0^2)^2-4\omega_1^2\omega_2^2}\right] \tag{30}$$

A comparison of (29) with (23), or (30) with (24), shows that we have the equality:

$$P_{1*}F_2 = \left.\begin{array}{l} i\pi\ \delta_{2*}F_1 \text{ on the physical cut} \\ i\pi\ \delta_{2*}\bar{F}_1 \text{ on the low energy cut}\end{array}\right\} \tag{31}$$

So we can immediatly write (Cf. (27)):

$$P_{1*}F_2 = i\pi^{\frac{\nu}{2}}\ \Gamma(\frac{4-\nu}{2})\left(\frac{m_2^2}{2p_0^2}\right)^{\frac{\nu-4}{2}}(p_0^2+m_2^2-m_1^2)F\left(1,\frac{4-\nu}{2};\ \frac{3}{2};\ \frac{(p_0^2+m_2^2-m_1^2)^2}{4p_0^2m_2^2}\right) \tag{32}$$

Comment. The relation (31) can also be deduced from the identity $P_{1*}F_2=i\pi F_{1*}\delta_2+P_{1*}P_2+\pi^2\delta_{1*}\delta_2$. Also, in (32) the limit of the argument of the hypergeometric function should be taken from above on the physical cut and from below in the low energy cut. Of course, $P_{1*}\tilde{F}_2$ can be obtained by complex conjugation.

6 $\delta_* P$

For this convolution we shall follow the same steps as with $\delta_* F$. We take (21) and (22), but this time the mass m_2 will be taken as real, and the principal value will be adopted at the singularity. Formulae (23), (24) and (25) are still valid, but for the principal value evaluation of (25) we go again to the table of Hilbert transforms and find: See ref. [6]

$$\frac{1}{\pi}\int_0^\infty dx\; x^{\alpha-1}(x+a)^{1-\beta}\; P\,\frac{1}{x-y} = y^{\alpha-1}(y+a)^{1-\beta}\,\mathrm{ctn}\,[(\beta-\alpha)\pi] -$$

$$-\frac{\Gamma(\beta-\alpha-1)\Gamma(\alpha)a^{1-\beta+\alpha}}{\pi\,\Gamma(\beta-1)(y+a)}\,F(2-\beta,\;1;\;2-\beta+\alpha;\;\frac{a}{y+a}) \qquad y>0$$

or

$$\frac{\Gamma(\beta-\alpha)\Gamma(\alpha)(-y)^{\alpha-1}}{\pi\,\Gamma(\beta)\;a^{\beta-1}}\,F(\beta-1,\alpha,\beta;1+\frac{y}{a}) \qquad y<0 \qquad (33)$$

Now we write (25) in the form:

$$\delta_{1*}P_2 = \frac{\pi^{\frac{\nu-1}{2}}}{\Gamma(\frac{\nu-1}{2})}\,\frac{m_1^2-m_2^2+p_0^2}{4p_0^2}\int_0^\infty dx\,\frac{x^{\frac{\nu-3}{2}}}{\sqrt{x+m^2}}\,P\left[x-\frac{p_0^4-2p_0^2(m_1^2+m_2^2)+(m_1^2-m_2^2)^2}{4p_0^2}\right]^{-1} \tag{34}$$

To use (33) we chose $\alpha = \frac{\nu-1}{2}$, $\beta = \frac{3}{2}$, $a = m_1^2$, $y = \frac{1}{4p_0^2}\left[(p_0^2-(m_1^2+m_2^2))^2-4m_1^2m_2^2\right]$,

$y+a = \frac{1}{4p_0^2}\,(p_0^2-m_2^2+m_1^2)^2$.

For positive y we have:

$$\delta_{1*}P_2 = \frac{\pi^{\frac{\nu+1}{2}}}{\Gamma(\frac{\nu-1}{2})}\,(y+a)\left\{y^{\frac{\nu-3}{2}}\,(y+a)^{-\frac{1}{2}}\,\mathrm{ctn}\left[\frac{4-\nu}{2}\,\pi\right]-\right.$$
$$\left.-\frac{\Gamma(\frac{2-\nu}{2})\,\Gamma(\frac{\nu-1}{2})\,a^{\frac{\nu-3}{2}}}{\pi\,\Gamma(\frac{1}{2})(y+a)}\,F(\tfrac{1}{2},\ 1;\ \tfrac{\nu}{2};\ \tfrac{a}{y+a})\right\} \tag{35}$$

While for negative y we get:

$$\delta_{1*}P_2 = \frac{\pi^{\frac{\nu+1}{2}}}{\Gamma(\frac{\nu-1}{2})}\,(y+a)\frac{\Gamma(\frac{4-\nu}{2})\Gamma(\frac{\nu-1}{2})}{\pi\,\Gamma(\frac{3}{2})}\,\frac{(-y)^{\frac{\nu-3}{2}}}{a^{1/2}}\,F\left(\frac{1}{2},\ \frac{\nu-1}{2};\ \frac{3}{2};\ \frac{y+a}{a}\right) \tag{36}$$

Comment. In paragraph -9- we will learn how to evaluate the imaginary part of the hypergeometric function. With this knowledge it is possible to find

δ_*P by taking the real part of δ^*F or also by taking the imaginary part of P_*F.

7 F_*F

The convolution of two Feynman propagators can be computed in the usual way by means of dimensional regularization:

$$F_{1*}F_2 = \int d^\nu q \; \frac{1}{q^2+m_1^2} \, \frac{1}{(p-q)^2+m^2} = \int d^\nu q \int_0^1 dx \left[(q^2+m_1^2)x + ((p-q)^2+m_2^2)(1-x)\right]^{-2} =$$

$$= i\pi^{\frac{\nu}{2}} \; \Gamma(\tfrac{4-v}{2}) \int_0^1 dx \; \left[p^2x(1-x)+m_1^2x+m_1^2(1-x)\right]^{\frac{\nu-4}{2}} \qquad (37)$$

The masses are both supposed to have a small negative imaginary part. The roots of the quadratic form in (37) are

$$x_1 = \frac{1}{2}\left(1+ \frac{m_1^2-m_2^2}{p^2}\right) \mp \sqrt{\frac{1}{4}\left(1+ \frac{m_1^2-m_2^2}{p^2}\right)^2 + \frac{m_2^2}{p^2}} \qquad (38)$$

$$F_{1*}F_2 = i\pi^{\frac{\nu}{2}} \; \Gamma(\tfrac{4-\nu}{2}) \int_0^1 dx \; \left[-p^2(x-x_1)(x-x_2)\right]^{\frac{\nu-4}{2}} =$$

$$= i\pi^{\frac{\nu}{2}}\,\Gamma\left(\frac{4-\nu}{2}\right)\left\{\int_0^{x_1} dx\left[-p^2(x_1-x)(x_2-x)\right]^{\frac{\nu-4}{2}} + \int_{x_1}^{x_2} dx\,\left[p^2(x-x_1)(x_2-x)\right]^{\frac{\nu-4}{2}} + \right.$$

$$\left. + \int_{x_2}^{1} dx\left[-p^2(x-x_1)(x-x_2)\right]^{\frac{\nu-4}{2}}\right\} \tag{39}$$

Each of the integrals in (39) can be written down with the aid of a table.

$$F_1 * F_2 = i\pi^{\frac{\nu}{2}}\,\Gamma\left(\frac{4-\nu}{2}\right)\left\{\frac{\Gamma(\frac{\nu-2}{2})\Gamma(\frac{\nu-2}{2})}{\Gamma(\nu-2)}\,(p^2)^{\frac{\nu-4}{2}}\,(x_2-x_1)^{\nu-3} + 2\,\frac{(-p^2)^{\frac{\nu-4}{2}}\,x_1^{\frac{\nu-2}{2}}\,x_2^{\frac{\nu-4}{2}}}{\nu-2}\right.$$

$$\left. F\left(1,\ \frac{4-\nu}{2};\ \frac{\nu}{2};\ \frac{x_1}{x_2}\right) + 2\,\frac{\left(-p^2\right)^{\frac{\nu-4}{2}}\left(1-x_2\right)^{\frac{\nu-2}{2}}\left(1-x_1\right)^{\frac{\nu-4}{2}}}{\nu-2}\,F\left(1,\ \frac{4-\nu}{2};\frac{\nu}{2},\frac{1-x_2}{x_2-x_1}\right)\right\} \tag{40}$$

And, after a quadratic transformation of the hypergeometric function:

$$F_1 * F_2 = i\pi^{\frac{\nu}{2}}\,\frac{\Gamma(\frac{4-\nu}{2})\Gamma(\frac{\nu-2}{2})\Gamma(\frac{\nu-2}{2})}{\Gamma(\nu-2)(p^2)^{1/2}}\left[p^2\left(1+\frac{m_1^2-m_2^2}{p^2}\right)^2 + 4m_2^2\right]^{\frac{\nu-3}{2}} +$$

$$+i\pi^{\frac{\nu}{2}}\,\Gamma(\tfrac{4-\nu}{2})\;\frac{2}{\nu-2}\;\frac{(m_2^2)^{\frac{\nu-2}{2}}}{(-p^2)}\left[\left(1+\frac{m_1^2-m_2^2}{p^2}\right)^2+\frac{4m_2^2}{p^2}\right]^{-\frac{1}{2}}F\left(\frac{1}{2},\frac{\nu-2}{2};\frac{\nu}{2},\frac{4m_2^2p^2}{(p^2+m_1^2-m_2^2)^2+4m_2^2p^2}\right)$$

$$+i\pi^{\frac{\nu}{2}}\Gamma(\tfrac{4-\nu}{2})\;\frac{2}{\nu-2}\;\frac{(m_1^2)^{\frac{\nu-2}{2}}}{(-p^2)}\left[\left(1+\frac{m_1^2-m_2^2}{p^2}\right)^2+\frac{4m_2^2}{p^2}\right]^{-\frac{1}{2}}F\left(\frac{1}{2},\frac{\nu-2}{2};\frac{\nu}{2},\frac{4m_1^2p^2}{(p^2+m_1^2-m_2^2)^2+4m_2^2p^2}\right)$$

(41)

Comment. The conditions for the roots (38) to be real, are the same as those for the square bracket in (4) or (8), to be positive.

8 $F_*\bar{F}$

This convolution can be computed by using (1) and its complex conjugate.

$$F_1{}_*\bar{F}_2 = (P_1+i\pi\delta_1)_*(P_2-i\pi\delta_2) = P_1{}_*P_2+\pi^2\delta_1{}_*\delta_2 + i\pi(\delta_1{}_*P_2-\delta_2{}_*P_1) \qquad (42)$$

We see that the real part is symmetric, while the imaginary part is antisymmetric under the interchange of m_1 and m_2. Of course for $F_1{}_*F_2$ both parts are symmetric in the masses.

$$F_1 * F_2 = P_1 * P_2 - \pi^2 \delta_1 * \delta_2 + i\pi \ (\delta_1 * P_2 + \delta_2 * P_1) \tag{43}$$

The real part of (43) has support on the physical cut only, while the real part of $F_1 * \bar{F}_2$ is zero on the physical cut (has support on the low-energy cut and in p^2 space-like).

For the actual value of $F_1 * \bar{F}_2$ it is enough to use (4) or (8), (14) or (20) and (35) or (36) in (42).

Comment. It is equally possible, instead of (42), to use the equality $\bar{F}_2 = F_2 - 2i\pi\delta_2$, to obtain $F_1 * \bar{F}_2 = F_1 * F_2 - 2i\pi \ F_1 * \delta_2$. Of course, there are others equivalent expressions. If $m_1 = m_2$; $F x \bar{F} = 0$ in the physical cut
$2\pi^2 \delta_* \delta$ otherwise.

9 ABSORTIVE PARTS

The so called "absortive part" of a convolution, can easily be obtained from the hypergeometric function when its first or second parameter is equal to unity (or can be reduced to unity by an appropriate transformation). In such a case we take the integral representation

$$F(1,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{1-tz} \, dt \tag{44}$$

The physically interesting imaginary part of (44) is developed at the singularity which appears when $z=x+i\varepsilon$; $x\geq 1$, $\varepsilon\to 0$.

As $\mathrm{Im}(1-tz)^{-1} = \pm\, \pi\delta(1-tx)$, we have

$$\mathrm{Im}F(1,b;c;z) = \pm\, \frac{\pi\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\delta(1-tx)dx$$

$$\mathrm{Im}F(1,b,c;z) = \pm\, \frac{\pi\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\, x^{1-c}(x-1)_+^{c-b-1}\ ,\ z \to x\pm i\varepsilon \tag{45}$$

With (45) we can take the imaginary part of (27), which is seen to coincide with $\mathrm{Im}\delta_{1*}F_2 = \pi\delta_{1*}\delta_2$. Also, the imaginary part of $iP_{1*}F_2$ (Cf. eq. (32)), is seen to coincide with $-P_{1*}P_2$.

Eq. (45) can not be applied directly to (41), but after the transformation:

$$F(a,b;c;z) = (1-z)^{-a}\, F(a,c-b;c;z/(z-1))$$

We have

$$F\Big(\frac{1}{2},\ \frac{\nu-2}{2};\ \frac{\nu}{2};\ \frac{4m_2^2p^2}{(p^2+m_1^2-m_2^2)^2+4m_2^2p^2}\Big) = \left[\frac{(p^2+m_1^2-m_2^2)^2}{(p^2+m_1^2-m_2^2)^2+4m_2^2p^2}\right]^{-\frac{1}{2}} F\left(\frac{1}{2},1;\frac{\nu}{2};\frac{-4m_2^2p^2}{(p^2+m_1^2-m_2^2)^2}\right)$$

whose imaginary part is different from zero for p^2 time-like and $(p^2+m_1^2-m_2^2)^2<-4m_2^2p^2$.

Comment: When the first parameter of the hypergeometric function is an integer n, eq. (44) has the denominator $(1-tz)^n$ instead of just (1-tz). The imaginary part is then proportional to $\delta^{(n)}(1-tx)$ and (45) is modified by taking the (n-1)-th derivative of the x-dependent factors.

DISCUSSION

Whenever a propagator can be presented as the addition of two interesting parts, like principal value and δ-function, its convolutions can also be decomposed in pieces whose actual computation can shed some light on aspects of the original one. Moreover, in field-theories with higher order equations of motion some of these partial convolutions appear as fundamental and have to be computed independently of the more usual ones. This is the motivation behind the present work. Anyway it is interesting, or amusing, to find out that the structure of P*P is very similar to that of $\pi^2\delta*\delta$. They have the same support and the same absolute values. They have different signs outside the physical cut. In this way they only leave the physical cut in the convolution F*F. Of course, this is related to unitary as the imaginary part of F*F has to do with the scattering cross section of bradions.

Something similar happens with the convolution of δ_*F and P_*F. As a matter of facts we will see in the forthcoming second part of this work that

δ and P are distributions whose structures are similar and can both be represented as integral fucntionals that differ in the domain of integration on the energy-plane. One of them (δ) being closed and the other (P) open.

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