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CELLULAR AUTOMATA AND SPREAD OF DAMAGE: GENERAL CONSIDERATIONS AND A RECENT ILLUSTRATION

by

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ABSTRACT

We characterize Cellular Automata (CA) within a general background of dynamical systems. We then illustrate CA by presenting various relevant properties of a stochastic one, namely an extended version of the Domany-Kinzel CA. We finally propose a quite general classification of the various types of sensitivity to initial conditions that dynamical systems might exhibit; this classification recovers, as particular cases, standard discussions related to the Hamming distance and the Lyapunov exponent.

Key-words: Cellular Automata; Dynamical Systems; Spread of Damage; Domany-Kinzel Cellular Automaton.

1 INTRODUCTION

Phenomena in Nature occur somewhere and at a given time. Consequently, the Theoretical Physics basic mathematical object for studying dynamical systems is a "*field*" $\phi(x,t)$ defined in a *space-time* (x,t). The field can be *continuous* ($\phi \in \mathbb{R}^{\bar{n}}$, where $\bar{n}$ is the *field dimension*) or *discrete* ($\phi \in \mathbb{N}$ or, equivalently, ϕ is isomorphic to $\mathbb{N}$ or to a part of $\mathbb{N}$); the space can be *continuous* ($x \in \mathbb{R}^d$, where d is the *space dimension*) or *discrete* ($x \in \mathbb{N}$); finally, the time can be *continuous* ($t \in \mathbb{R}$) or *discrete* ($t \in \mathbb{N}$). As a whole, we have 2^3 different cases, which are illustrated in Table I. It is worthy to emphasize the fact that the (discrete field) - (discrete space) - (discrete time) case is the only one which is strictly tractable in a real computer. Cellular automata (CA) belong to this category. Furthermore, the denomination CA is normally reserved to those dynamical systems which present a *time-layered* architecture in the sense that we can *simultaneously* update all the elements of the system (thus being susceptible of *parallel* processing in computers). A very general definition for CA is outlined in what follows.

Consider a *denumerable* set of positions $\{x_i\}$ $(i = 1,2,\ldots,N)$; they could be the sites of a Bravais lattice (e.g., a linear chain) or an hierarchical lattice (e.g., a Sierpinski gasket) or any other spatial array. With each position we associate a variable ϕ_i which can take q_i different values (i.e., $\phi_i = 1,2,\ldots,q_i$), the most frequent case being $q_i = q$, $\forall i$, and typically $q = 2$. At time $t = 0$ we must provide the set $\left\{\phi_i^{(0)}\right\}$; the set $\left\{\phi_i^{(1)}\right\}$ at time $t = 1$ is either directly given as part of the initial conditions, or is determined by giving the set of (deterministic or stochastic) rules $\left\{\rho_i^{(1)}\right\}$, i.e., $\phi_i^{(1)} = \rho_i^{(1)}\left(\left\{\phi_i^{(0)}\right\}\right)$. Generally speaking, the initial conditions are given through the set $\left(\left\{\phi_i^{(0)}\right\},\left\{\phi_i^{(1)}\right\},\ldots,\left\{\phi_i^{(\tau)}\right\}\right) \equiv \left\{\phi_i^{(0\to\tau)}\right\}$, the most frequent case being $\tau = 0$. The set of rules $\left\{\rho_i^{(t)}\right\}$ at time t might themselves evolve, and can in general be expressed as follows:

$$
\begin{aligned}
\phi_i^{(\tau+1)} &= \rho_i^{(\tau+1)}\left(\left\{\phi_i^{(0\to\tau)}\right\}\right) \\
\phi_i^{(\tau+2)} &= \rho_i^{(\tau+2)}\left(\left\{\phi_i^{(0\to\tau+1)}\right\}\right) \\
&\vdots \\
\phi_i^{(t)} &= \rho_i^{(t)}\left(\left\{\phi_i^{(0\to t-1)}\right\}\right) \qquad (t = \tau+1, \tau+2, \ldots)
\end{aligned}
\tag{1}
$$

The most frequent case is

$$\rho_i^{(t)} = \rho\left(\left\{\phi_i^{((t-1-\tau)\to(t-1))}\right\}\right) \ , \ \forall (i,t).$$

Moreover, the most commonly used rules are the homogeneous *local* ones, in which $\phi_i^{(t)}$ is determined, assuming that $\tau = 0$, by the values $\left\{\phi_i^{(t-1)}\right\}$ where i runs over λ neighbors of site i (including the site i itself). For such CA, the rules are established by giving, for *deterministic* CA, a correspondence of q^λ states into q states (q^{q^λ} possibilities), or, more generally for *stochastic* (also referred to as *probabilistic or random*) CA, by giving the probability set

$\{p(\phi_1,\phi_2,\ldots,\phi_\lambda/1), p(\phi_1,\phi_2,\ldots,\phi_\lambda/2),\ldots,p(\phi_1,\phi_2,\ldots,\phi_\lambda/q)\}$ satisfying

$$\sum_{\phi=1}^{q} p(\phi_1,\phi_2,\ldots,\phi_\lambda/\phi) = 1 \qquad (2)$$

with $\phi_j = 1,2,\ldots,q$ ($j = 1,2,\ldots,\lambda$), where $p(\phi_1,\phi_2,\ldots,\phi_\lambda/\phi)$ is the probability of having, at a given site, state ϕ at time t if the states, at time (t-1), of its λ neighbors are $(\phi_1,\phi_2,\ldots,\phi_\lambda)$. This type of model is characterized by giving $(q-1)q^\lambda$ independent probabilities, say $\{p(\phi_1.\phi_2,\ldots,\phi_\lambda/1),\ldots,p(\phi_1,\phi_2,\ldots,\phi_\lambda/(q-1)\}$ for each one of the q^λ states of $(\phi_1,\phi_2,\ldots,\phi_\lambda)$. Its physical space is, consequently, a hyperpolyhedron in $(q-1)q^\lambda$ dimensions which recovers, at each one of its q^{q^λ} corners, deterministic CA. The extended Domany-Kinzel CA discussed in Section 2 is an example of the $\lambda = q = 2$ class and, as we shall see, it is characterized by giving, for instance, p(00/1), p(01/1), p(10/1) and p(11/1); it recovers, as particular cases, 16 deterministic CA.

What are the most relevant properties that can be studied for a particular CA? First of all the attractors ($t \to \infty$) at the thermodynamic limit ($N \to \infty$): they can present spatial and/or temporal modulations of various kinds as well as spatial and/or temporal chaos. Also, the influence of the initial conditions on the attractors often is interesting. Order parameters characterizing the various possible attractors ("phases") can be studied as well, thus enabling the establishment of the CA phase diagram with all sorts of critical phenomena, critical exponents and universality classes . Various types of susceptibilities and relaxation times can also be studied. Finally, the spread of damage (sensitivity to initial conditions) often exhibits interesting peculiarities.

The study of CA is a very interesting one. On one hand they provide simple models for a great variety of systems, including chemical reactions, crystal growth models, artificial intelligence, turbulence, computers, cybernetics, biological systems, various other non-linear processes far from equilibrium, phase transitions (see, for instance, Refs. [1] for reviews, [2] for chemical reactions, [3] for the Q2R CA, [4] for spin glasses and [5] for various other spin systems). On the other hand, CA act as prototypes for a "finite difference equations" Physics in opposition to the traditional "differential equations" Physics. Indeed, if the deep nature of space-time turns out to be discrete (which we believe to be the case, essentialy due to quantum-like fluctuations effects), differential equations such as those of Maxwell, Schroedinger, Einstein, etc. will necessarily become limiting cases of finite difference equations to be found.

In the next Section we present results of a phase diagram study[6] of an extended version of the Domany-Kinzel CA[7].

2 EXTENDED DOMANY-KINZEL CA

We consider a one-dimensional chain of N lattice sites $(i = 1,2,\dots,N)$ with periodic boundary conditions. Each site has two possible states

$\phi_i \equiv \sigma_i = 0,1$ (hence $q = 2$). The state of the system at time t is given by the set $\left\{\sigma_i^{(t)}\right\}$. At the next time step, the state $\sigma_i^{(t+1)}$ of a given site equals 0 or 1 according to the conditional probabilities $\left\{p\left(\sigma_{i-1}^{(t)},\sigma_i^{(t)}/\sigma_i^{(t+1)}\right)\right\}$, namely $p(00/1) = 1 - p(00/0)$, $p(01/1) = 1 - p(01/0)$, $p(10/1) = 1 - p(10/0)$ and $p(11/1) = 1 - p(11/0)$ (hence $\lambda = 2$). This CA is closely related to directed percolation[7] as well as to directed compact percolation[8]. It is possible to define at least two relevant time-dependent order-like parameters, namely $M \equiv$ (fraction of sites with value 1) and $\psi \equiv$ Hamming distance (i.e., fraction of sites which exhibit different values on two replicas of the system while using the same sequence of random numbers). The equilibrium values (i.e., in the $t \to \infty$ limit) of M and ψ in the $(p(00/1), p(01/1), p(10/1), p(11/1))$ space enable the characterization of the "phase diagram" of the system. No analytical results are available, excepting for the $(p(00/1), p(11/1)) = (0,1)$ critical line which, due to duality arguments, is given by[8,9]

$$p(01/1) + p(10/1) = 1 \tag{3}$$

In the present study, a Monte Carlo technique has been used by always starting, at $t = 0$, with half of the sites with value 1, randomly chosen. After arrival to equiblibrium (where M is conveniently determined), a damage is produced and the two replicas (damaged and undamaged) are followed in time until stationnarity is achieved (and ψ is then determined). Let us first present the p(00/1) = 0 (*legal* rules) phase diagram: see Fig. 1 (from [6]), where three phases are present, namely the *frozen*, *active* and *chaotic* ones. M equals unity on half of the p(11/1) = 1 square (in particular at the p(01/1) = p(10/1) = p(11/1) = 1 corner) and vanishes on the frozen-active critical surface; ψ equals 1/2 at the (p(01/1), p(10/1), (11/1)) = (1,1,0) corner and vanishes on the active-chaotic critical surface.

If we now consider p(00/1) $\neq$ 0, the frozen-active critical surface disappears (since p(00/1) acts, on M, as an external conjugated field) but the active-chaotic one remains: see Fig. 2 (from [10]).

The results we have presented up to now have been obtained by using *independent* random numbers for updating each one of the N sites at time t. Let us now generalize this in the sense that the *same* random number will be used to update n ($1 \leq n \leq N$) neighboring sites (the same set of groups of n sites each for all times). The n = 1 model recovers the previous one; the n = N model is an extreme case for which a single random number is used for updating the entire generation. The n-evolution of the phase diagram is indicated in Fig. 3 (from [10]). In the $n = N \to \infty$ limit, the p(00/1) = 0 phase diagram exhibits a frozen phase almost everywhere since the frozen-active and the active-chaotic critical surfaces have collapsed onto the p(11/1) = 1 plane and/or onto the p(01/1) = 1 and the p(10/1) = 1 planes. This fact cannot be considered as surprising since, in the $n = N \to \infty$ limit, the system becomes one-dimensional-like in space-time (whereas it is two-dimensional for finite n and $N \to \infty$). It is worth stressing that, for the (p(00/1) = 0, p(01/1) = p(10/1)) phase diagram (Fig. 3(a)), the frozen area A_f tends to unity whereas the active area A_a as well as the chaotic area A_c

tend to zero when n increases from 1 to infinity; in addition to that, it can be shown that the ratio A_a/A_c decreases with increasing n. Hence, tendency towards a "totalitarian" limit (same random number for *all* the elements of a given generation) decreases chaos, but decreases even more (certain type of) activity!

3 SPREAD OF DAMAGE: A NEW CLASSIFICATION

There are dynamical systems (deterministic or stochastic) which can be very sensitive to small numerical departures of the quantities involved in the determination of the actual trajectory. These quantities include the initial conditions, the roundings of the real numbers (say 6,8,16 algarisms) at every calculational step, the particular sequence of random numbers that might be used, the parameters which are fixed along the evolution, etc. Whenever a system is sensitive to the initial conditions, it is necessarily sensitive to the numerical roundings adopted for calculating its state at a given time from the previous time(s) by using (among other possible ingredients) a finite-difference or differential equation: this is the globally so called *sensitivity to initial conditions*, and constitutes one of the two essential properties that qualify the use of the word "chaos" (the other property being the existence of a large attractor). This sensitivity is checked on an actual system by introducing a "damage" in it and following its spreading. It is now known[11] that the spread of damage in various spin systems has a deep relation with thermodynamical properties. For these systems, the quantity whose time evolution is followed (in order to characterize the damage) typically is the Hamming distance (ψ introduced in Section 2). By comparing the *initial* Hamming distance between two replicas A and B of the system and the *final* (after a long time) Hamming distance, Herrmann presented[12] various typical situations to which he referred to as *Chaotic* I (e.g., in Barber and Derrida 1988 in [5]), *Chaotic* II (e.g., in Derrida and Weisbuch 1987 in [3]), *Frozen* I and *Frozen* II (e.g., in Boissin and Herrmann 1991 in [5]). By following, in this Section, along this line we define a generalized Hamming distance in order to cover both discrete and continuous systems (unifying, in particular, the Hamming distance and the Lyapunov exponent), and then propose a quite general classification (based on the sensitivity to initial conditions) of dynamical systems.

Consider a discrete or continuous "field" $\phi_i(t)$ defined on a discrete space ($i = 1,2,\ldots,N$) and a discrete or continuous time t (everything that follows can be trivially adapted to a continuous space, but we speak here of a discrete space in order to be adapted to the most frequent systems on which spread of damage is studied). We construct, at $t = t_\infty$ (typically $t_\infty \gg 1$ and corresponds to the time necessary for the system to practically arrive to its attractor or equilibrium), two replicas A and B of the system (typically, one or both of the replicas are damaged versions of the original system at time t_∞). And we define the following normalized *generalized Hamming distance*:

$$D(\bar{t}) \equiv \frac{\left\langle \left\langle \sum_{i=1}^{N} \left| \phi_i^A(t_\infty+\bar{t}) - \phi_i^B(t_\infty+\bar{t}) \right| \right\rangle_{\substack{\text{time}\\ \text{sequences}}} \right\rangle_{\substack{\text{initial}\\ \text{configurations}}}}{\sup\left(\sum_{i=1}^{N} \left| \phi_i^A - \phi_i^B \right| \right)} \qquad (4)$$

$<\ldots>_{\substack{time\\ sequences}}$ refers to only one trajectory if the system is deterministic, and refers to an average using a sufficiently large set of random number sequences (the *same* for both replicas) if the system is stochastic; $<\ldots>_{\substack{initial\\ configurations}}$ refers to an average using a sufficiently large set of initial configurations (at $t = 0$) satisfying the external parameters that are fixed; sup refers to the maximal value its argument can achieve at any conditions (sup $(\ldots) = N$ for binary variables taking values 0 or 1, or $\pm 1/2$: $D = \psi$ for the case discussed in Section 2), hence $D(\bar{t}) \in [0,1]$. We have defined "distance" in the traditional way, i.e., by using the modulus, but, clearly, other definitions (e.g., $|\ldots|^k$ with $k > 0$) could be as well used; if $\phi_i(t)$ is a cyclic or angular-like variable, the *smallest* angle can be conveniently used. It follows, from definition (4), that

$$D(0) = 0 \Rightarrow D(\bar{t}) = 0,\ \forall \bar{t} \geq 0 \tag{5}$$

We assume the quite frequent case satisfying:

(i) $D(\infty) \equiv \lim_{\bar{t}\to\infty} D(\bar{t})$ exists and depends on $\{\phi_i^A(t_\infty), \phi_i^B(t_\infty)\}$ *only* through $D(0)$; (6.a)

(ii) The only finite-cycle attractors are fixed points. (6.b)

We follow, on a $D(\bar{t})$ vs. $D(0)$ representation, the time evolution of the generalized Hamming distance. By assuming the frequent case in which

$$\Delta \equiv \lim_{D(0)\to 0} \frac{D(\infty)}{D(0)} \tag{7}$$

is well defined, we have the following possibilities (see Fig. 4):

i) *Strongly sensitive*: $\Delta = \infty$ and $D(\infty)$ is a discontinuous function of $D(0)$ at $D(0) = 0$;

ii) *Sensitive*: $\Delta = \infty$ and $D(\infty)$ is a continuous function of $D(0)$ at $D(0) = 0$;

iii) *Marginal*: Δ is finite ($\infty > \Delta > 0$);

iv) *Nonsensitive*: $\Delta = 0$

The generic strongly sensitive case (Fig. 4(a)) corresponds to Herrmann's Chaotic II situation, and its particular case for which $D(\infty)$ is a nonvanishing constant for all $D(0) \in (0,1]$ corresponds to Herrmann's Chaotic I situation (the chaotic phase of the CA discussed in Section 2 as well as the *positive* Lyapunov exponent cases of say the logistic equation are typical illustrations). Although we are not aware of an illustration for the sensitive case (Fig. 4(b)), there is no reason for its non existence. The marginal case (Fig. 4(c)) corresponds to Herrmann's Frozen II situation (the *zero* Lyapunov exponent case can belong to this class). The generic nonsensitive case could in principle exist (as for the sensitive case, we are not aware of any example at the present moment); its particular case for which $D(\infty) = 0$ for any $D(0) \in [0,1]$ corresponds to Herrmann's Frozen I situation (the *negative* Lyapunov exponent case of say the logistic case is a typical illustration).

4 CONCLUSION

We have defined CA within an unified picture for dynamical physical systems, and discussed the phase diagram of a recent extension of the Domany Kinzel stochastic CA. One of the relevant phenomena is the spread of damage characterizing its chaotic phase. We have introduced a generalized Hamming distance which enables, along Herrmann's lines, a convenient classification of dynamical systems. This classification is based on the type of sensitivity to the initial conditions and recovers the Lyapunov exponent concept as a particular case. Specific illustrations of the *sensitive* and the generic *nonsensitive* classes would be very welcome.

I am very indebted to H.J. Herrmann for fruitful remarks concerning the present proposal for classification of the spread of damage behavior, as well as to E.M.F. Curado and J.S. Helman for useful remarks on Table I. Finally, I am thankful to my collaborators M.L. Martins and T.J.P. Penna for allowing me to use, in the present talk, Figs. 2 and 3 prior to publication.

CAPTION FOR FIGURES AND TABLE

Fig. 1 - (a) p(00/1) = 0 phase diagram of the extended Domany-Kinzel CA; the solid line belongs to the critical surface separating the *frozen* (M = 0 and ψ = 0) and *active* (M ≠ 0 and ψ = 0) phases; the dashed lines belong to the boundary between the active and *chaotic* (M ≠ 0 and ψ ≠ 0) phases. (b) p(00/1) = 0 and p(01/1) = p(10/1) phase diagram. The data correspond to simulations with 3200 sites; transients of 10000 (3000) time steps were used for the frozen-active (active-chaotic) phase transitions. The damage was averaged over another 3000 time steps.

Fig. 2 - Phase diagram, for p(01/1) = p(10/1), of the extended Domany-Kinzel CA.

Fig. 3 - p(00/1) = 0 and p(01/1) = p(10/1) phase diagram of the CA for n ≥ 1 (n = 1 recovers the extended Domany-Kinzel CA). (a) Full p(01/1) = (10/1) space; (b) n-evolution of the p(01/1) = p(10/1) = 1 critical point; (c) n-evolution of the (p(01/1) = p(10/1); (p/11/1) = 0) critical point. The dashed lines are guides to the eye.

Fig. 4 - Final (D(∞)) vs. initial (D(0)) possible behaviors for the spread of damage. (a) strongly sensitive; (b) sensitive; (c) marginal; (d) nonsensitive.

Table I - Dynamical physical systems: (a) continuous time ($t \in \mathbb{R}$); (b) discrete time ($t \in \mathbb{N}$). *In this case trajectories are defined in space-time, consequently a *binary* field can be defined (nonvanishing on the trajectory, and vanishing out of it); **for example, a localized random binary variable which, at arbitrary continuous times, can be zero or one; ***each consecutive iteration can be considered as a discrete "time", ϕ being one (zero) for all values of $x \in \mathbb{R}$ corresponding to present (absent) points.

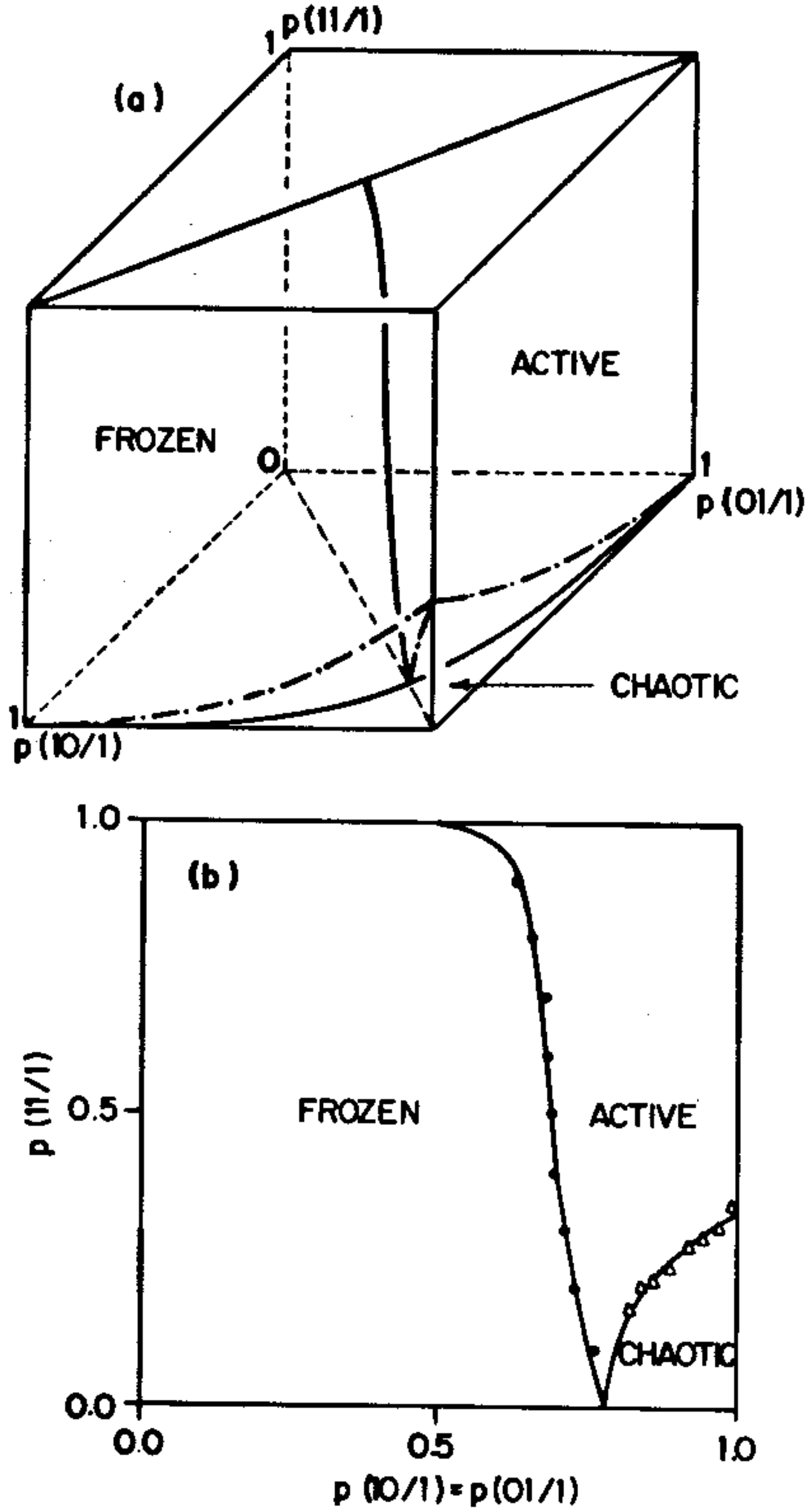
p(11/1)
1
(a)
ACTIVE
FROZEN
O
1
p(01/1)
CHAOTIC
1
p(10/1)
1.0
(b)
p(11/1)
0.5
FROZEN
ACTIVE
CHAOTIC
0.0
0.0
0.5
1.0
p(10/1) = p(01/1)

FIG.1

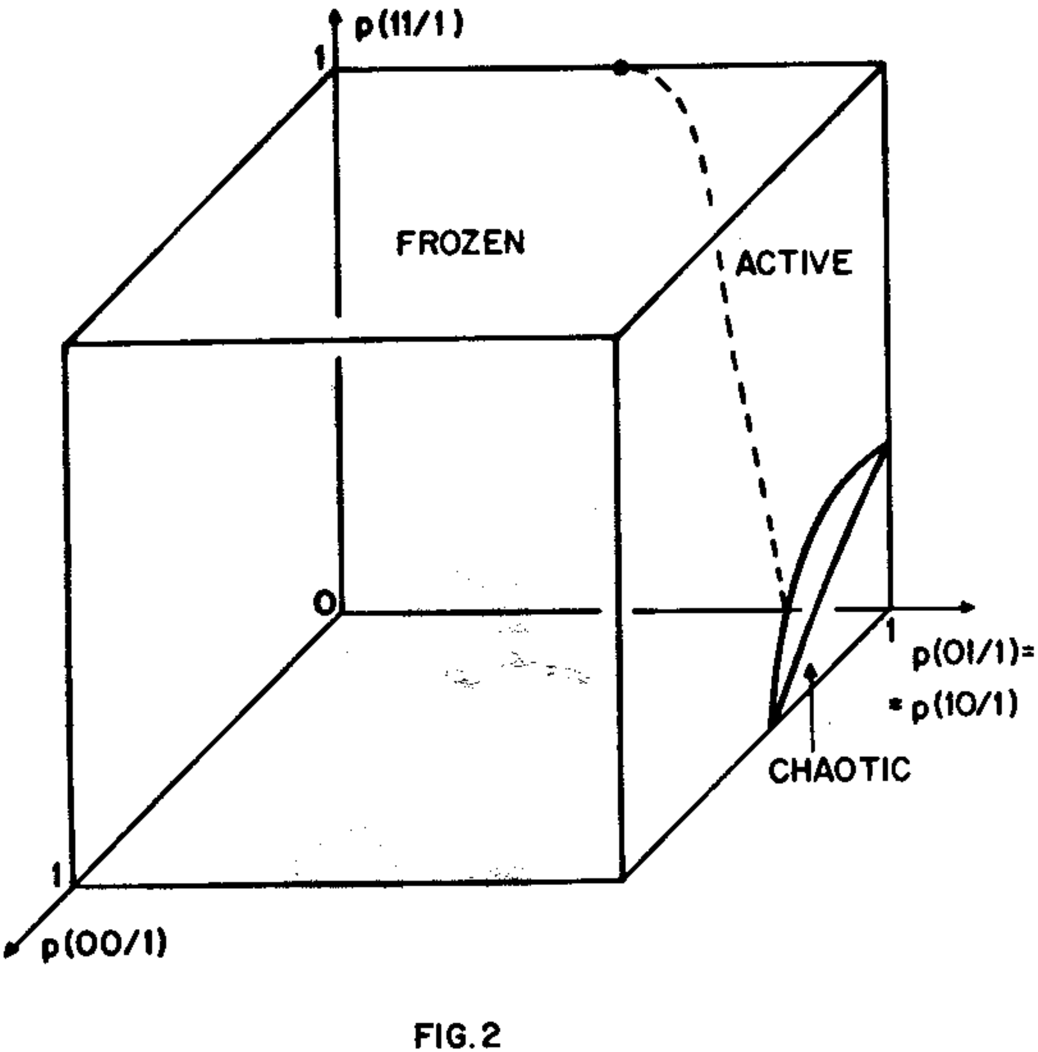

FIG. 2

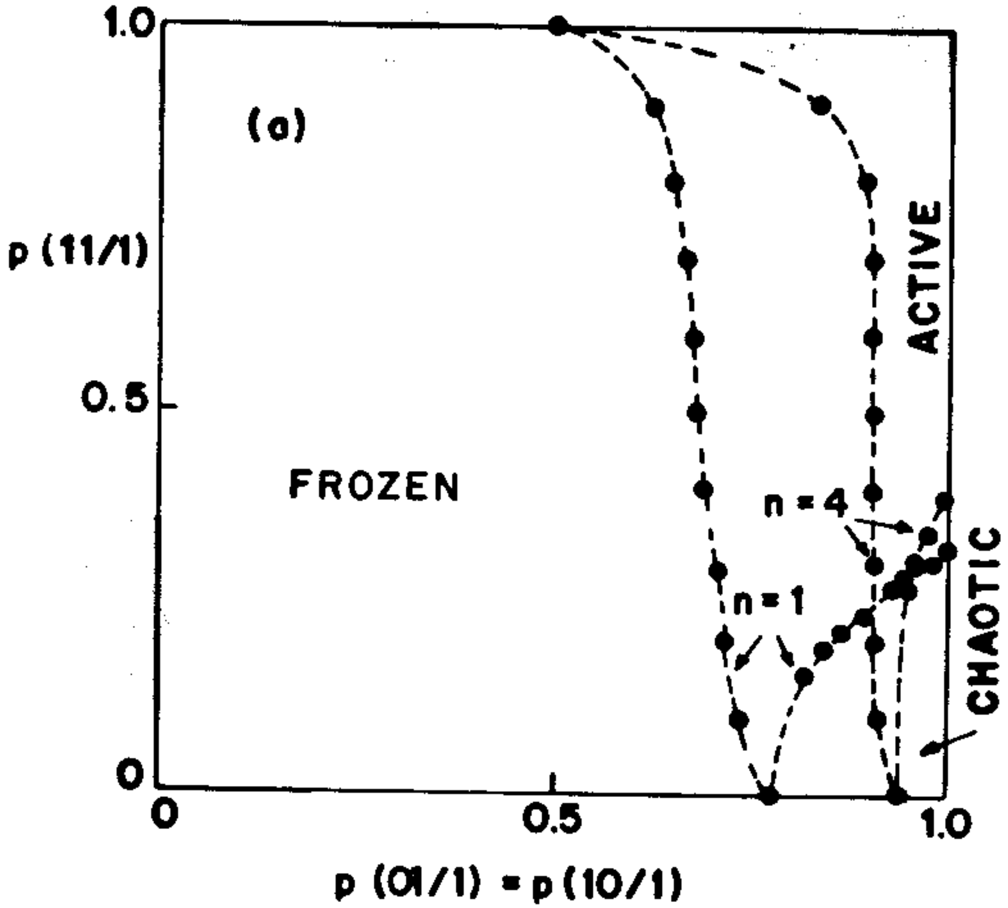

FIG. 3

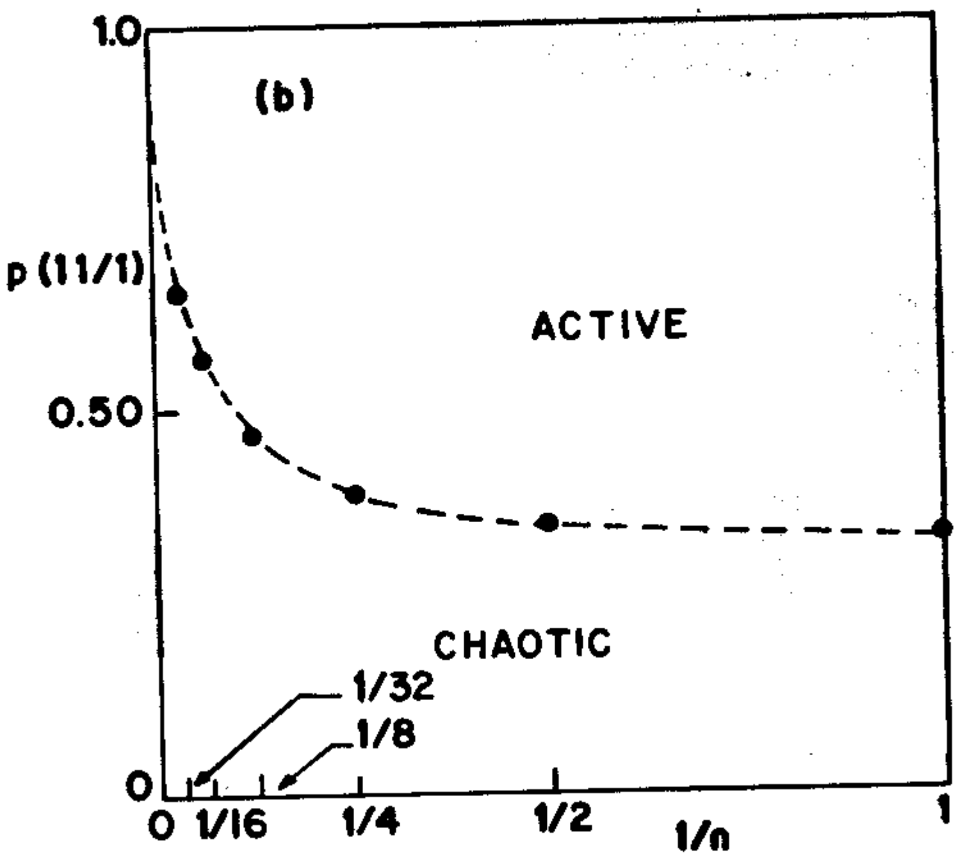

FIG. 3

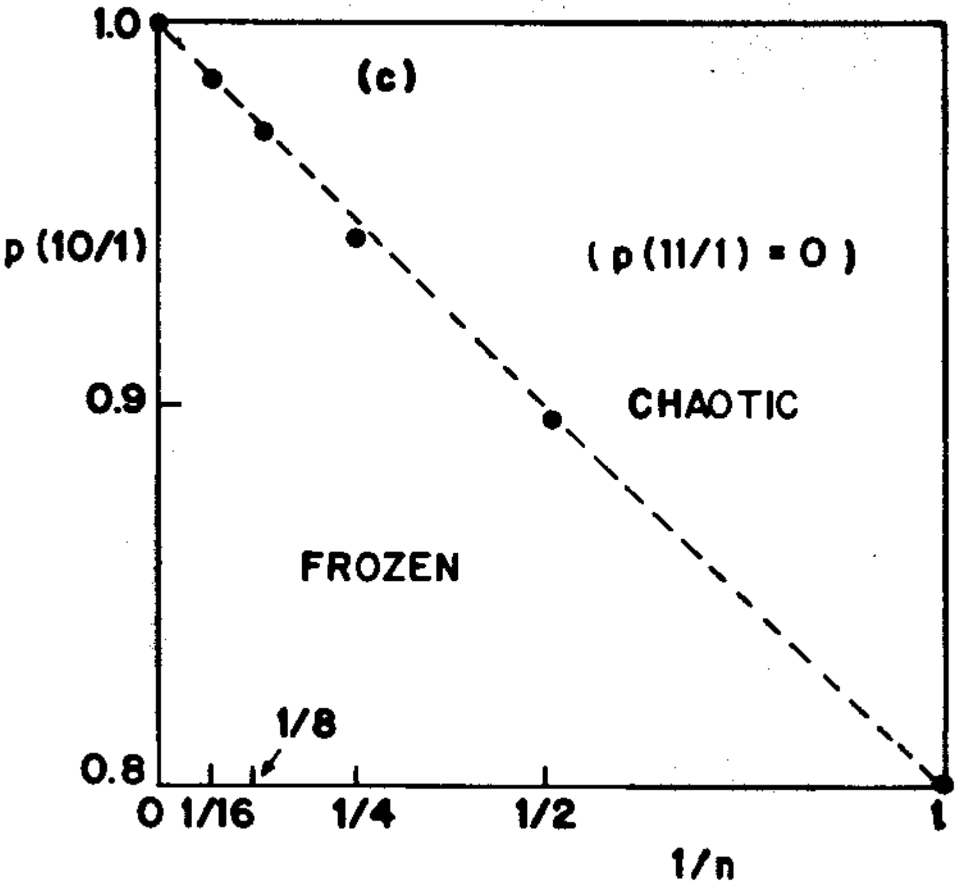

FIG. 3

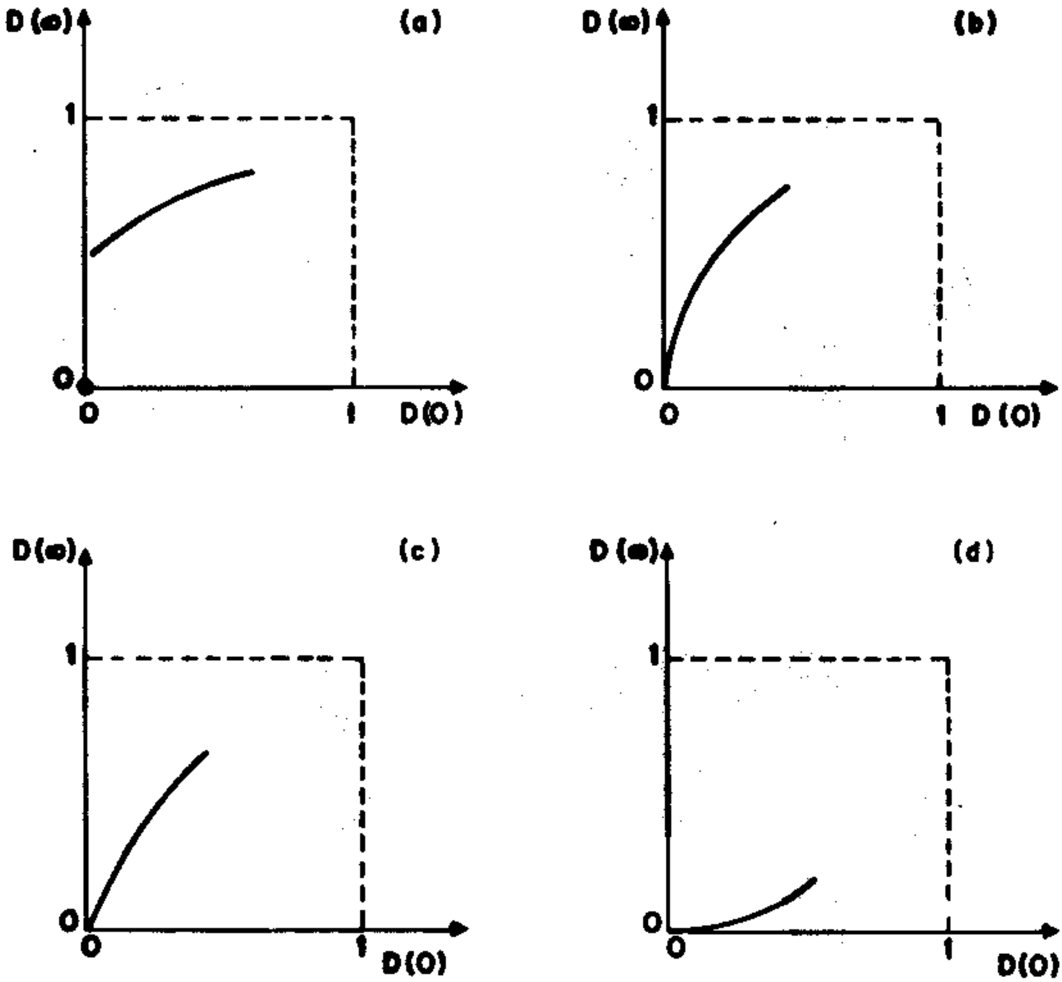

FIG.4

TABLE I

(a) Continuous time ($t \in \mathbb{R}$)

ϕ \ x	continuous ($x \in \mathbb{R}^d$)	discrete ($x \in \mathbb{N}$)
continuous ($\phi \in \mathbb{R}^n$)	Maxwell equations Schroedinger equation Navier-Stokes equation Field theory	Classical phonons in a Bravais lattice Lattice field theory
discrete ($\phi \in \mathbb{N}$)	Newton equation* Special relativity equation*	Random binary noise**

(b) Discrete time ($t \in \mathbb{N}$)

ϕ \ x	continuous ($x \in \mathbb{R}^d$)	discrete ($x \in \mathbb{N}$)
continuous ($\phi \in \mathbb{R}^n$)	Discrete-time field theory	Real and complex logistic map Maps coupled on a lattice Network of continuous neurons
discrete ($\phi \in \mathbb{N}$)	Cantor fractal dust***	Hopfield neuronal model Cellular automata

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