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HUYGHENS' PRINCIPLE IN (2n+1) DIMENSIONS FOR NON LOCAL PSEUDODIFFERENTIAL OPERATORS OF THE TYPE $\Box^{\alpha}$

by

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Abstract

The validity of the Huyghens principle for half interger powers of the D'Alembertian in odd number of dimensions is discussesd. It is shown in particular that for $\Box^{1/2}$ in 2 + 1 the dynamic of $\Box$ in 3 + 1 is obtained.

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It is a well known result (see Ref. [1]) that Huyghens Principle (HP) is not valid for the D'Alembertian operator when the number of dimensions (space time) is odd. This is equivalent to say that the Green function is different from zero only on the surface of the light cone.

Recently, E. Marino and R. de Amaral[1] found a solution in 2 + 1 dimensions, satisfying HP in spite of being of odd dimensionality [2].

In this note we give a proof of this result and generalize it to any odd space-time dimensions. The Quantum Green functions are also given.

We shall use for that purpose, the Riesz method of Analytical continuation [3] and some results of distribution theory [4].

Marcel Riesz generalized the Rieman Lionville formula for integration and derivation of fractional order to parallel expressions for the D'Alembertian. He gave the solution to the equation

$$\Box U = f \tag{1}$$

by the expression:

$$u = \Box^{-\alpha} f \Big|_{\alpha=1} = \frac{\int \cdots \int f\left(x_1 \ldots x_{d-1}, t\right) \overline{P}_+^{\alpha - \frac{d}{2}} dx_1 \ldots dx_{d-1} dt}{Hm(\alpha)} \tag{2}$$

where $Hm(\alpha) = \Pi^{\frac{d-2}{2}} 2^{2\alpha-1} \Gamma\left(1 + \alpha - \frac{d}{2}\right)$ $\overline{P}_+^{\lambda} = \begin{cases} (t^2-r^2)^{\lambda} & \text{if } t > r \\ 0 & \text{Otherwise} \end{cases}$

But if we put $\alpha = \frac{1}{2}$, we get the solution for the eq.

$$\Box^{1/2} u = f \tag{3}$$

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[1] Private communication to be published. See also [4].

We see from (2) that for ordinary $\Box$, ($\alpha = 1$) the zeros of the argument of the Γ functions garantee the validity of HP for even $n^{\underline{o}}$ of dimensions, and excludes the odds one, while for $d = 2n + 1$, we have HP for the following values of α.

$d = 2 + 1$	$\alpha = \frac{1}{2}$
$4 + 1$	$\alpha = \frac{1}{2}, \frac{3}{2}$
....	....
$2n + 1$	$\alpha = \frac{1}{2} \ldots \frac{2n-1}{2}$

Table 1

The result can also be seen from a more general distribution theory point of view which will allow us to get the Green functions obeying the quantum causality requirements. We want to prove the following result.

The function

$$G = \frac{\Pi^{-\frac{d}{2}}\, 4^{-\alpha}}{\Gamma(1+\lambda+\alpha)\Gamma(\alpha)}\, \Gamma\left(1 - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)P_+^{\alpha+\lambda} \tag{4}$$

$$P_+ = \begin{cases} t^2-r^2 & \text{when } t^2 > r^2 \\ o & t^2 < r^2 \end{cases}$$

with d = odd and $\lambda = -\frac{d}{2}$ is the Green function of the operator $\Box^{\alpha}$; i.e.:

$$\Box^{\alpha} G = \delta(x) \tag{5}$$

In page 274 (ref. [3]) one finds for k = integer

$$\Box^k (P+i0)^{\lambda+k} = 4^k \frac{\Gamma(1+\lambda+k)\Gamma\left(\lambda+\frac{d}{2}+k\right)(P+i0)^{\lambda}}{\Gamma(\lambda+1) \quad \Gamma\left(\lambda+\frac{d}{2}\right)} \tag{6}$$

where $(P+i0) = \ell im\ \varepsilon \to 0\ (t^2-r^2+i\varepsilon)$.

An analogous definition for P-i0 is also valid.

We find ref. [3] p. 270.

$$\operatorname{res}(P+i0)^{\lambda}\Big|_{\lambda=-\frac{d}{2}} = \Pi^{\frac{d}{2}} \frac{\exp\left(-i\frac{q}{2}\right)}{4^k\Gamma\left(\frac{d}{2}\right)\sqrt{\Delta}}\ \delta(x) \tag{7}$$

Where q is the number of negative coefficients in the quadratic form

$$p^2 = x_1^2 + \ldots \qquad x_p^2 - x_{p+1}^2 \ldots - x_{p+q}^2 \tag{8}$$

In our case, p will always be one (only one time).

Δ is the determinant of the coefficient of the quadratic form. (In our examples $\Delta = 1$).

We see from (6) and (7) that

$$(P+i0)^{\lambda+k} \text{ is up to a constant, } \left(\text{for } \lambda = -\frac{d}{2}\right) \tag{9}$$

the Green function of $\Box^k$. See p. 273 ref. [3]. (d = odd)

In ref. [3] p. 280 one finds

$$P_+^{\lambda} = \frac{e^{-i\pi\lambda}(P+i0)^{\lambda} - e^{i\pi\lambda}(P-i0)^{\lambda}}{-2i\operatorname{sen}\pi\lambda} \tag{10}$$

From (6) and the analogous for (P_-i0) we have

$$\Box^k \left(P_+\right)^{\lambda+k} = 4^k \frac{\Gamma(1+\lambda+k)}{\Gamma(\lambda+1)} \frac{\Gamma\left(\lambda+\frac{d}{2}+k\right)}{\Gamma\left(\lambda+\frac{d}{2}\right)} P_+^{\lambda}$$

We extend it for complex k, which variable we call α, and divide by $\Gamma(1+\lambda+\alpha)$.

So, we get

$$\Box^{\alpha} \left.\frac{P_+^{\alpha+\lambda}}{\Gamma(1+\lambda+\alpha)}\right|_{\lambda=-\frac{d}{2}} = 4^{\alpha} \frac{\Gamma\left(\lambda+\alpha+\frac{d}{2}\right)}{\Gamma(\lambda+1)} \lim_{\lambda\to-\frac{d}{2}} \frac{P_+^{\lambda}}{\Gamma\left(\lambda+\frac{d}{2}\right)} \qquad (11)$$

Remembering that d = odd and also ref. [3], p 252

$$\text{resid. } \left. P_+^{\lambda}\right|_{\lambda\to-\frac{d}{2}} = (-1)^{\frac{q}{2}} \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \delta(x) \qquad (12)$$

From 11 and 12, we have:

$$\Box^{\alpha} \frac{P_+^{\alpha-\frac{d}{2}}}{\Gamma\left(1+\alpha-\frac{d}{2}\right)} = 4^{\alpha} \frac{\Gamma(\alpha)(-1)^{\frac{q}{2}}\pi^{\frac{d}{2}}}{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)} \delta(x) \qquad (13)$$

From which we get results (4) and (5). (We are interested in half intergers values of α).

We confirm the results of Table 1, as we see that we have poles (d.h.s) for the Γ function in the denominator of 13 for the values specified of d(odd) and α.

So, we see that in spite of having the Green function of the operator $\Box$ disobeying the H.P., fractional powers of this operators (non-local operators) obeys it.

In particular, it is easy to see that the Green function for d = 2 + 1 and $\alpha = 1/2$.

(See ref. [3], p. 252).

$$\lim_{\alpha\to\frac{1}{2}} \frac{P_+^{\alpha-\frac{3}{2}}}{\Gamma\left(1+\alpha-\frac{3}{2}\right)} = \delta(P) \tag{14}$$

So, we get for (4)

$$G_{\frac{1}{2}} = \frac{1}{2}\,\frac{1}{\sqrt{\pi}}\,\delta(P) \tag{15}$$

$$G_{\frac{1}{2}}(xyt) = K\,\frac{\delta(t+r)+\delta(t-r)}{r}$$

It means that we have in (2+1) dimensions, retarded and advanced potentials with a dynamic $\frac{1}{r}$, typical of 3 dimensions.

The same happens for $\alpha = 3/2$ and $d = 4 + 1$, $\alpha = 5/2$ $d = 6 + 1$. etc.

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