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CRITICALITY OF THE POTTS FERROMAGNET IN MIGDAL-KADANOFF-LIKE HIERARCHICAL LATTICES

by

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ABSTRACT

Within the real space renormalisation group framework, we discuss the critical point and exponent ν of the Potts ferromagnet in b-sized Migdal-Kadanoff-like hierarchical lattices. Both $b \to \infty$ and $b \to 1$ limits are exhibited. The important discrepancies that might exist between the exact results for d-dimensional hierarchical lattices and d-dimensional Bravais lattices are illustrated.

Key-words: Potts model; Hierarchical lattices; Renormalisation group; Criticality.

1 INTRODUCTION

The study of the criticality of magnetic models (e.g., the Potts model) on d-dimensional Bravais lattices is frequently replaced, within some real space renormalisation group (RG) techniques, by the study of d_f-dimensional hierarchical lattices ($d_f \equiv$ *intrinsic fractal dimensionality* $\equiv \ln N_b / \ln b$, where N_b is the number of bonds of the two-rooted graph whose iteration generates the hierarchical lattice, and b is the chemical distance between its roots [1,2]) which satisfy $d_f \to d$ in the limit of large cells ($b \to \infty$). It is important to evaluate, both qualitatively and quantitatively, the benefits as well as the restrictions of such procedures (see [1-3] and references therein). A very simple and commonly used framework is the Migdal-Kadanoff (or bond moving) one [4]; it is based on the so called *diamond* hierarchical lattice. Here we generalize this procedure through a comprehensive discussion of the criticality of the q-state Potts ferromagnet. This constitutes a clear illustration of analogies and discrepancies between Bravais and hierarchical lattices. Some of the results appearing in Refs. [1] and [5] (q = 2 and the $b \to 1$ limit for q = 1 respectively) are herein recovered as particular cases.

2 MODEL AND FORMALISM

We consider the b-sized d-dimensional *diamond* (*tress*) hierarchical lattice; it is defined through infinite iteration of a two-rooted graph which consists in an array of b^{d-1}(b) strings in parallel (series), each of them constituted by $b(b^{d-1})$ bonds in series (parallel). Typical such lattices are presented in Fig. 1. Two important topological properties are verified, namely: (i) for all b and d and both diamond and tress types, the intrinsic fractal dimensionality is given $d_f = \ell n b^d / \ell n b = d$; (ii) for arbitrary fixed b and d = 2, and only then, the diamond and tress hierarchical lattices are dual of each other.

Each bond of these lattices represents the elementary Potts interaction, whose Hamiltonian is given by $\mathcal{H} = -qJ\delta_{\sigma_i,\sigma_j}$ ($J>0$; the site variables σ_i and σ_j take the values $1,2,\ldots,q$). We introduce the convenient variable $t \equiv [1-\exp(-qJ/k_BT)]/[1+(q-1)\exp(-qJ/k_BT)]$ (named *thermal transmissivity* [6]). Both diamond and tress graphs are reducible in series and parallel operations; therefore the corresponding transmissivities (noted G_D and G_T respectively) can be easily calculated [6], thus yielding

$$G_D(t;b,d,q) = \frac{1-\left[\dfrac{1-t^b}{1+(q-1)t^b}\right]^{b^{d-1}}}{1+(q-1)\left[\dfrac{1-t^b}{1+(q-1)t^b}\right]^{b^{d-1}}} \tag{1}$$

and

$$G_T(t;b,d,q) = \left\{\frac{1-\left[\dfrac{1-t}{1+(q-1)t}\right]^{b^{d-1}}}{1+(q-1)\left[\dfrac{1-t}{1+(q-1)t}\right]^{b^{d-1}}}\right\}^b \tag{2}$$

Let us now focus the diamond case (the tress case is strictly analogous). We renormalise, for fixed d and q, a b-sized graph into a b'-sized one. Within this approach (hereafter referred to as $RG_{bb'}$), the recursive relation is given by

$$G_D(t';b',d,q) = G_D(t;b,d,q) \quad (3)$$

This equation admits, for all (b,b',d,q), two trivial (stable) fixed points, namely t = 0 (paramagnetic phase; P) and t = 1 (ferromagnetic phase; F), as well as a critical (unstable) fixed point noted $t^*_{bb'}$, which satisfies

$$G_D(t^*_{bb'};b',d,q) = G_D(t^*_{bb'};b,d,q) \quad (4)$$

The corresponding thermal critical exponent $\nu_{bb'}$ is given by

$$\nu_{bb'} = \frac{\ell n(b/b')}{\ell n(\lambda_b/\lambda_{b'})} \quad (5)$$

with $\lambda_b \equiv [dG_D(t;b,d,q)/dt]_{t=t^*_{bb'}}$ and $\lambda_{b'} = [dG_D(t;b',d,q)/dt]_{t=t^*_{bb'}}$

3 RESULTS

The critical point $t^*_{bb'}$ depends on (b,b'd,q). These dependences are illustrated on Fig. 2 (b-evolution of t^*_{b1} and $t^*_{b,b-1}$ for d = q = 2) and Fig. 3 (t^*_{21} as a function of (d,q)). The values obtained for t^*_{b1} are exact for the corresponding hierarchical lattices.

The critical exponent $\nu_{bb'}$ depends on (b,b',d,q), but its value is one and the same for the diamond and tress cases. These dependences are illustrated in Fig. 4 (b-evolution of ν_{b1} and $\nu_{b,b-1}$ for $d = q = 2$) and Fig. 5 (ν_{21} as a function of (d,q)). The values obtained for ν_{b1} are exact for the corresponding hierarchical lattices. The d-dependence of ν_{21} at fixed value of q deserves some comments, namely:

(i) For q high enough (q above $q_{max} = 2$), ν_{21} presents, as a function of d, a minimum at a value of d (hereafter referred to as d_{min}), and then increases again and reaches the value 1 in the $d \to \infty$ limit; d^*_{min} monotonously increases with increasing q and finally diverges in the $q \to \infty$ limit. The whole convergence is a non uniform one. We verify that

$$\lim_{q\to\infty} \nu_{2,1} = 1/d \quad (d \geq 1) \tag{6}$$

which confirms the conjecture [2] that $\lim_{q\to\infty} \nu_{b,1} = 1/d_f$. Also, $\lim_{d\to\infty} \lim_{q\to\infty} \nu_{21} = 0$ while $\lim_{q\to\infty} \lim_{d\to\infty} \nu_{21} = 1$.

(ii) For q low enough (q below $q_{min} \simeq 0.215$), ν_{21} presents, as a function of d, a local maximum at a value of d (hereafter referred to as d_{max}), and diverges in the $d \to 1$ limit; d_{max} monotonously increases from slightly below 2 to 2 while q decreases from q_{min} to 0; ν_{21} (d_{max}) monotonously increases from about 2.96 to infinity while q decreases from q_{min} to 0. Consistently with these observations, the q = 0 curve ν_{21} vs. d presents two branches: (a) in the interval $1 \leq d \leq 2$, ν_{21} presents a minimum at $d \simeq 1.5$ and $\nu_{21} \simeq 3.36$, and diverges in both $d \to 1+0$ and $d \to 2-0$ lim its; (b) for $d \geq 2$, ν_{21} monotonously decreases from infinity to one while d increases from 2 to infinity.

Points (i) and (ii) above mentioned have been verified for ν_{21}. Although we have not systematically checked, similar facts are expected for $\nu_{bb'}$. Summarizing, three regimes are observed, namely: (i) $0 \leq q < q_{min}$: except for a local maximum in the neighbourhood of $d = 2$, the general trend of $\nu_{bb'}$ is to decrease from infinity to one while d increases from one to infinity; (ii) $q_{min} \leq q \leq q_{max}$: $\nu_{bb'}$ monotonously decreases from infinity to one while d increases from one to infinity; (iii) $q > q_{max}$: $\nu_{bb'}$ presents a minimum while d increases from one to infinity ($\nu_{bb'}$ diverges in the $d \to 1$ limit, and goes to one in the $d \to \infty$ limit.

Another point which deserves to be commented is the $b \to \infty$ behaviour of $t^*_{bb'}$ and $\nu_{bb'}$. Our numerical results are consistent with the following behaviours:

(i) diamond lattices: $t^*_{b,1} \sim 1 - (d-1)\ell n b/b$ ($\forall q$), and $t^*_{b,b-1} \sim 1 - A(d,q)/b$, $A(d,q)$ being a pure number which satisfies $A(1,q) = 0$ (similar laws are obtained for the tress lattices);

(ii) diamond and tress lattices: $\nu_{b,1} \sim B(d)\ell n b/\ell n \ell n b$ ($\forall q$), $B(d)$ being a pure number which decreases for increasing d; $\nu_{b,b-1}$ almost independs from b, and practically coincides, in the $b \to \infty$ limit, with $\nu_{21}(d,q)$ (see Fig. 5)

The result obtained for $\nu_{b,1}$ is in variance with the behaviour expected for lattices with *finite* critical temperature (i.e., $0 < \lim_{\substack{b\to\infty \\ b' \leq b}} t^*_{bb'} < 1$): in such cases, finite size scaling arguments [7] usually suggest, in the $b \to \infty$ limit, a *logarithmic* approach to a *finite* value.

Let us now turn our attention onto a different type of limit, namely the differential one (i.e., $b'=1$ and $b=1+\mu$ with $\mu \to 0+$). We first notice that if we consider the hierarchical lattices generated by the b-sized d-dimensional generalized Wheatstone-bridge graphs (see [2] and references therein) with transmissivity noted G_w, we have, for all $(t;b,d,q)$,

$$G_D(t;b,d,q) \le G_W(t;b,d,q) \le G_T(t;b,d,q) \tag{7}$$

This is a trivial consequence of the fact that the transmissivity of *any* graph is a monotonously increasing function of the elementary transmissivity of any of its bonds, together with the fact that the breaking (collapsing) of all the "transverse" bonds of the Wheatstone-bridge graph precisely yields the diamond (tress) graph [6]. It is then straightforward to verify that, in the $b \to 1$ limit, the RG_{b1} recursive relation is *one and the same* for *both* diamond and tress cases (and *consequently for the Wheatstone-bridge case as well*, as it is between them), namely

$$t' \simeq t + \mu\left\{ t\ln t - (d-1)\,\frac{(1-t)\,[1+(q-1)t]}{q}\,\ln\left[\frac{1-t}{1+(q-1)t}\right]\right\} \tag{8}$$

The associated critical fixed point t^* satisfies

$$t^*\ln t^* = (d-1)\,\frac{(1-t^*)\,[1+(q-1)t^*]}{q}\,\ln\left[\frac{1-t^*}{1+(q-1)t^*}\right] \tag{9}$$

This equation yields the results presented in Fig. 6 as well as the following ones:

$$t^* \sim 1 - q^{-\frac{1}{d-1}} \qquad (d \to 1 + 0) \qquad (10)$$

which coincides with the asymptotically exact result for d-dimensional hypercubic lattice [8];

$$t^* = \frac{1}{\sqrt{q}+1} \qquad (d = 2) \qquad (11)$$

which coincides with the exact result for the square-lattice; and

$$t^* \sim e^{-(d-1)} \qquad (d \to \infty) \qquad (12)$$

which differs from the exact result for d-dimensional hypercubic lattice.

The fact that the $d \to 1$ result is asymptotically coincident with that of the d-dimensional hypercubic lattice comes from the fact that the linear chain has a special geometrical property, namely to simultaneously be scale invariant (hierarchical lattice) and translationally invariant (Bravais lattice). The fact that the $d = 2$ result exactly recovers that of the square-lattice comes from the confluence of the diamond and tress transmissivities on the *self-dual* Wheatstone-bridge transmissivity. This is a manner for understanding why the *differential* Migdal-Kadanoff approach preserves self-duality.

From Eq. (8) we also obtain the thermal critical exponent:

$$\nu^{-1} = 1 + \ell n t^* - \frac{d-1}{q} \left\{ \left[q - 2 - 2(q-1)t^* \right] \ell n \frac{1 - t^*}{1 + (q-1)t^*} - q \right\} \quad (13)$$

This equation yields the results presented in Fig. 7 as well as the following ones:

$$\nu \sim 1/(d-1) \qquad (d \to 1 + 0) \qquad (14)$$

which recovers the exact result for d-dimensional hypercubic lattice [8]; and

$$\nu^{-1} = 2\left[1 - \frac{1}{\sqrt{q}}\,\ell n(\sqrt{q} + 1)\right] \qquad (d = 2) \qquad (15)$$

and

$$\nu \to 1 \qquad (d \to \infty) \qquad (16)$$

which do not recover the exact results for the hypercubic lattice.

4 CONCLUSION

Let us summarize the main features of the present RG approach of the q-state Potts ferromagnet in hierarchical lattices. This approach is based on the renormalisation of b-sized two-rooted d-dimensional Migdal-Kadanoff-like graphs into b'-sized ones ($b'<b$). The results associated with $b' = 1$ are, as usual, exact for the corresponding hierarchical lattices.

Let us first stress an important point: transitions are, for all $d \geq 1$ and all $q \geq 0$, of the continuous type. This fact presents a remarkable discrepancy with Bravais lattices, which are known to yield first-order phase transitions for all $d > 1$ if q is high enough. In other words, the loss of the translational invariance of the system makes discontinuous phase transitions disappear.

Another interesting point is that, for fixed (b, b', d, q), the diamond and tress types present a different critical point but share *one and the same value of* ν. The d-dependence of ν, at a fixed value of q, presents three different shapes according to whether q is in the interval $[0, q_{min})$, $[q_{min}, q_{max}]$ or (q_{max}, ∞). In the first case ν presents a local minimum and a local maximum in the interval $1 < d \leq 2$, and monotonously decreases down to one for d increasing above 2. In the second case, ν monotonously decreases down to one for d increasing above one. In the third case, ν presents a minimum at a value of d which increases when q increases; also $\lim_{q \to \infty} \nu = 1/d$.

The $b \to \infty$ behaviours for t^* and ν are partially different from what is normally found for Bravais lattices. However, the reason for that might be not the loss of translational invariance but rather the fact that the critical temperature for the present cases is, in the $b \to \infty$ limit, not finite ($T_c = 0$ for diamonds, and $T_c \to \infty$ for tresses).

Finally, let us note that, in both $b' = b - 1$ with $b \to \infty$ and $b' = 1$ with $b \to 1$ cases, the linear expansion factor b/b' tends to unity. However, important differences are found for these two situations. For instance, in the former $t^* \to 0$ or 1, while in the latter, t^* becomes a finite value between 0 and 1. In some sense, this type of discrepancy reinforces the well known fact that the knowledge of the intrinsic fractal dimensionality of an hierarchical lattice is nothing but one (though important) of the many ingredients which determine their criticality.

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CAPTION FOR FIGURES

Fig. 1 - b-sized d-dimensional two-rooted graphs and corresponding hierarchical lattices (o and ● respective denote the roots and internal sites).

Fig. 2 - $q = d = 2$ critical point within the $RG_{bb'}$ approach (with $b' = 1$ and also $b' = b - 1$) for both diamond and tress types; —·—·— denotes the exact result for the Ising ferromagnet in square lattice.

Fig. 3 - q-and d-dependences of the critical point within the RG_{12} approach. (a) diamond (the exact result for square lattice has been included for comparison); (b) diamond (the results corresponding to the Ising ferromagnet in hypercubic lattice have been included for comparison; the dashed line is a guide to the eye); (c) and (d) the same for the tress type.

Fig. 4 - $q = d = 2$ critical exponent ν within the $RG_{bb'}$ approach (with $b' = 1$ and also $b' = b - 1$) for both diamond and tress types (one and the same); —·—·— denotes the exact result for the Ising ferromagnet in square lattice.

Fig. 5 - q - and d-dependances of the critical exponent ν within the RG_{12} approach (one and the same for both diamond and tress types) (a) for typical values of d (the exact result for square lattice has been included for comparison; (b) for typical values of q (the results corresponding to the Ising ferromagnet in hypercubic lattice have been included for comparison; the dashed line is a guide to the eye).

Fig. 6 - q - and d-dependances of the critical point within the differential RG ($b' = 1$ and $b \to 1$). (a) for typical values of d (the $d = 2$ curve coincides with the exact one for square lattice); (b) for typical values of q (the results corresponding to the Ising ferromagnet in hypercubic lattice have been included for comparison; the dashed line is a guide to the eye).

Fig. 7 - q - and d-dependances of the critical exponent ν within the differential RG ($b' = 1$ and $b \to 1$). (a) for typical values of d (the exact result for square lattice has been included for comparison); (b) for typical values of q (the results corresponding to the Ising ferromagnet in hypercubic lattice have been included for comparison; the dashed line is a guide to the eye).

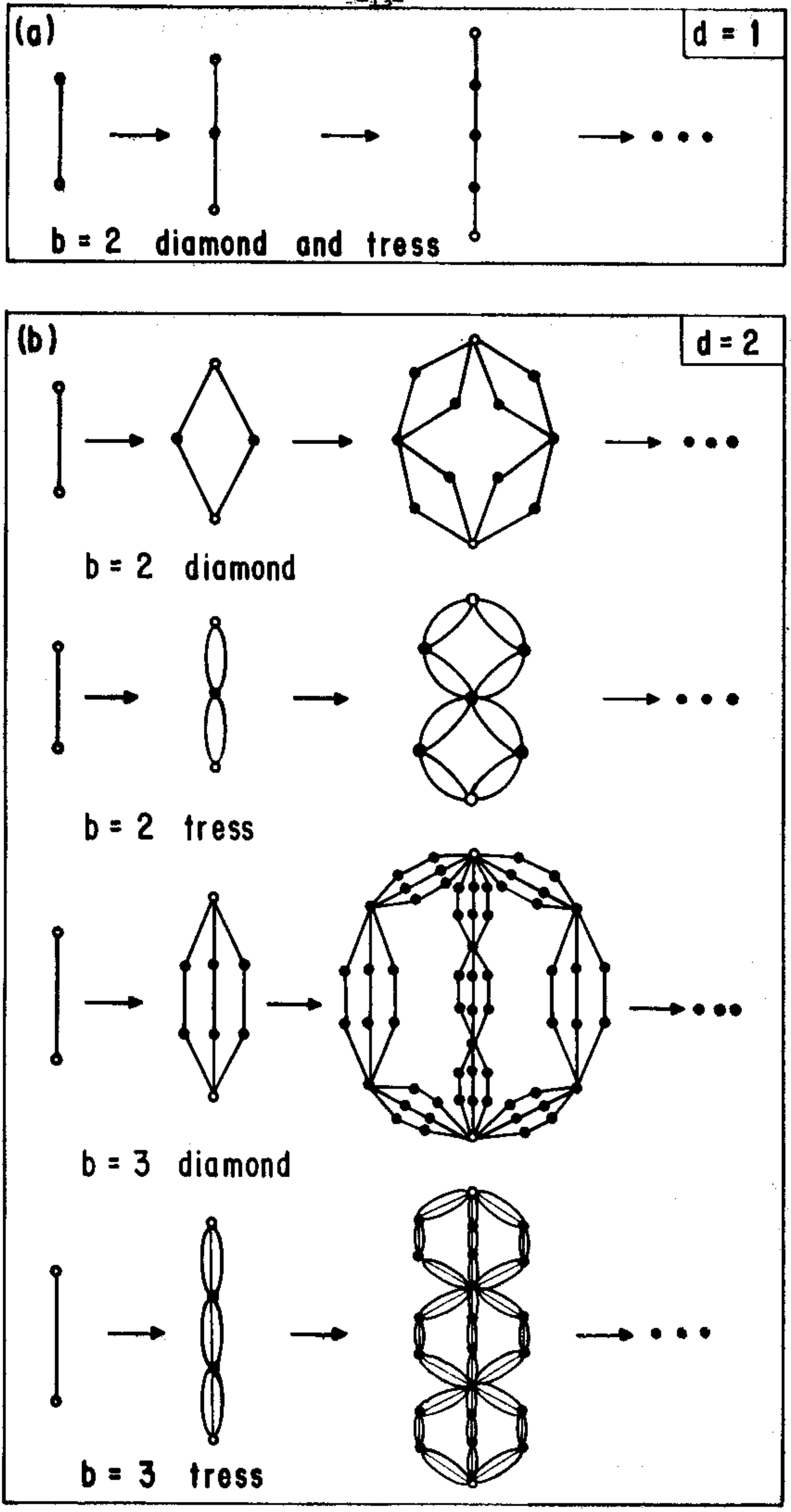

(a)
d = 1
b = 2 diamond and tress
(b)
d = 2
b = 2 diamond
b = 2 tress
b = 3 diamond
b = 3 tress

FIG.1

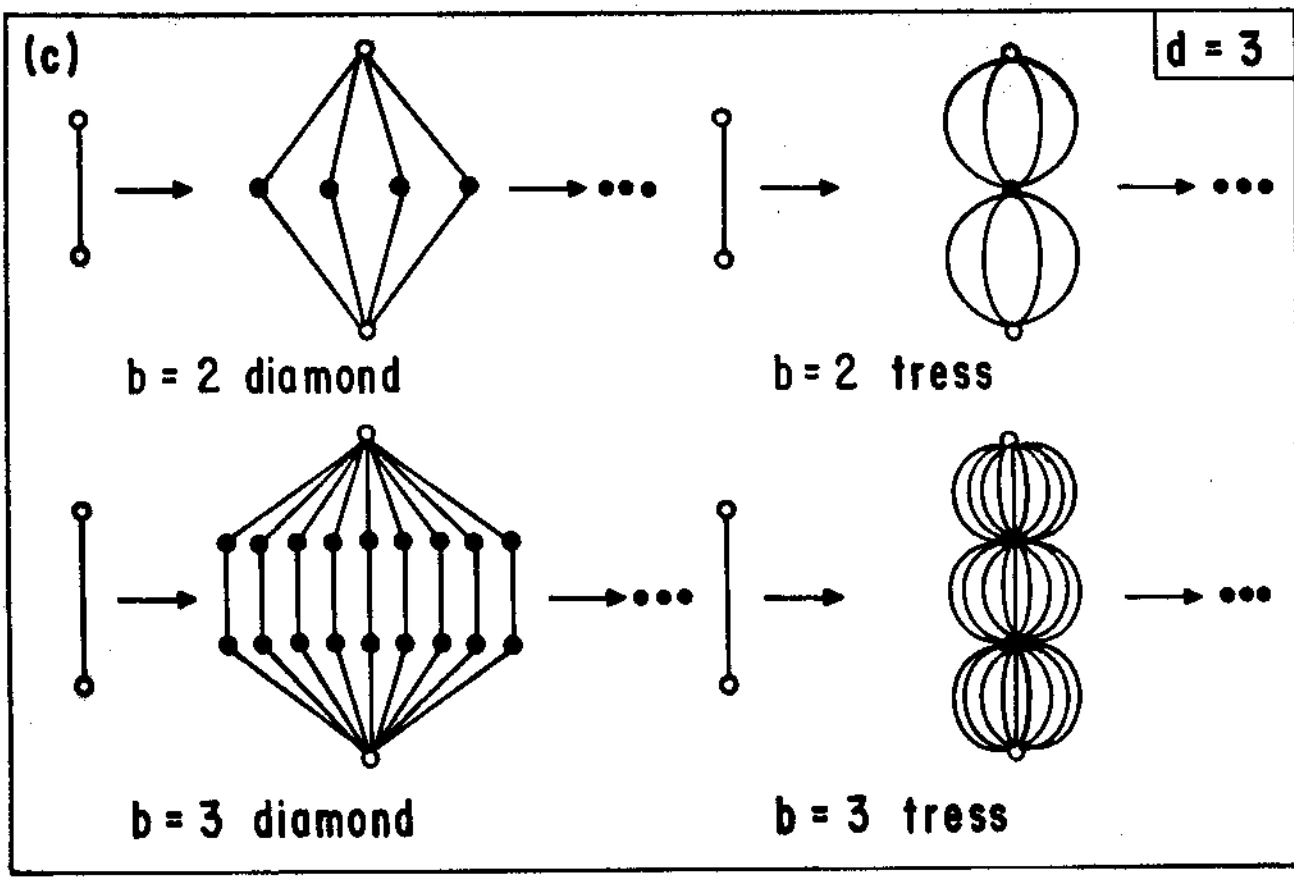


FIG.1

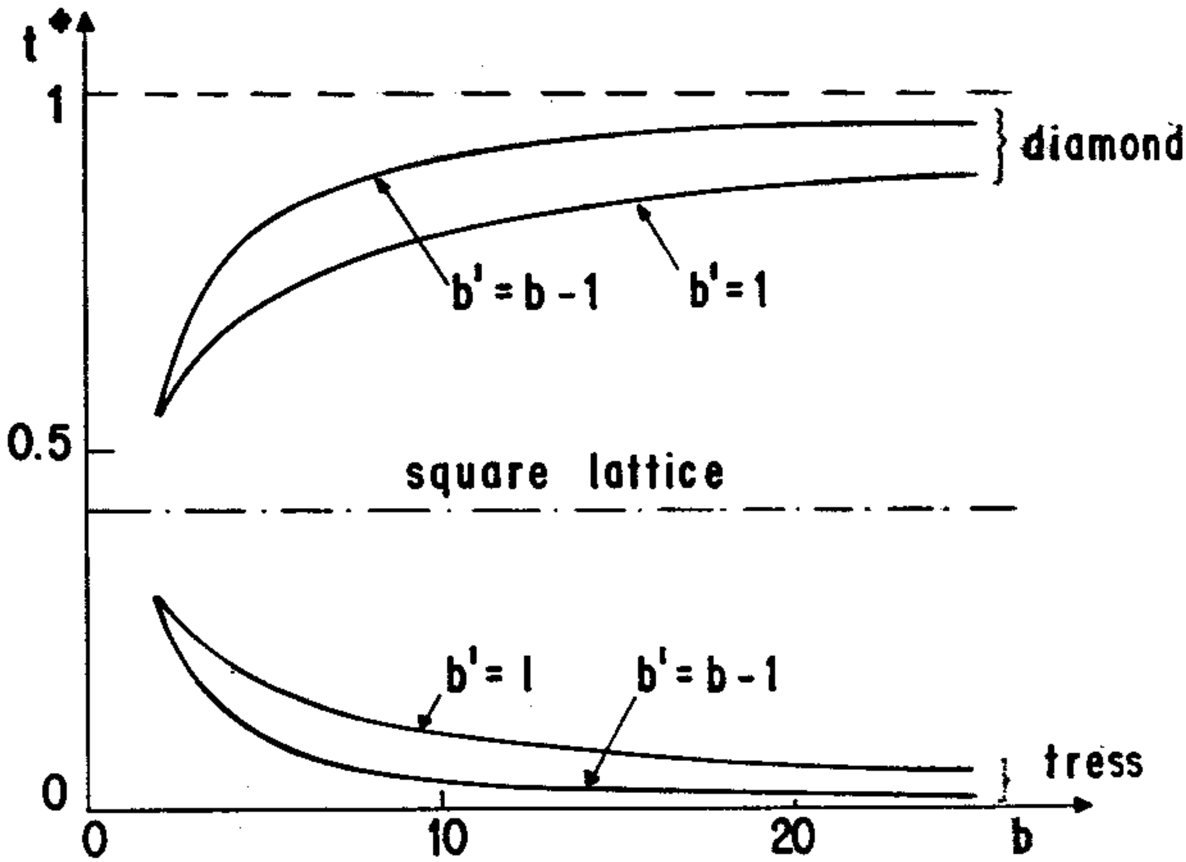


FIG.2

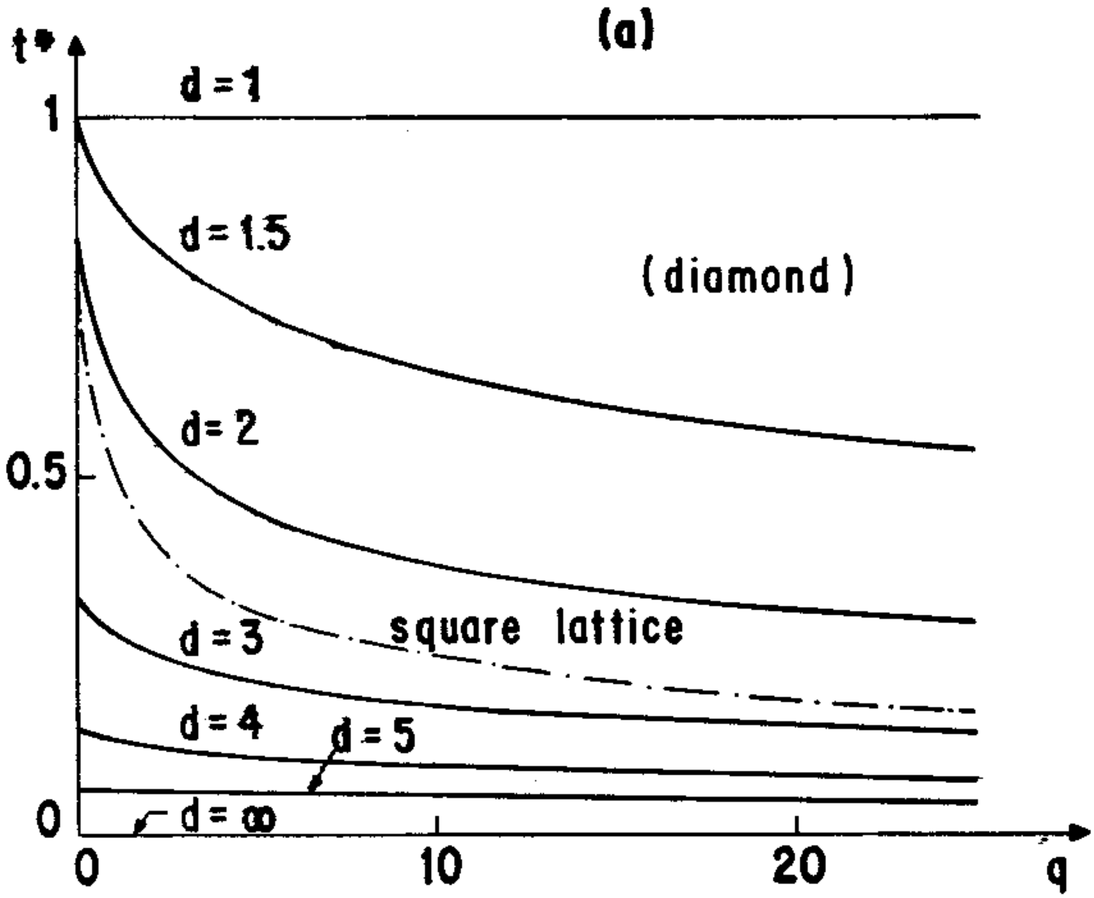


FIG.3

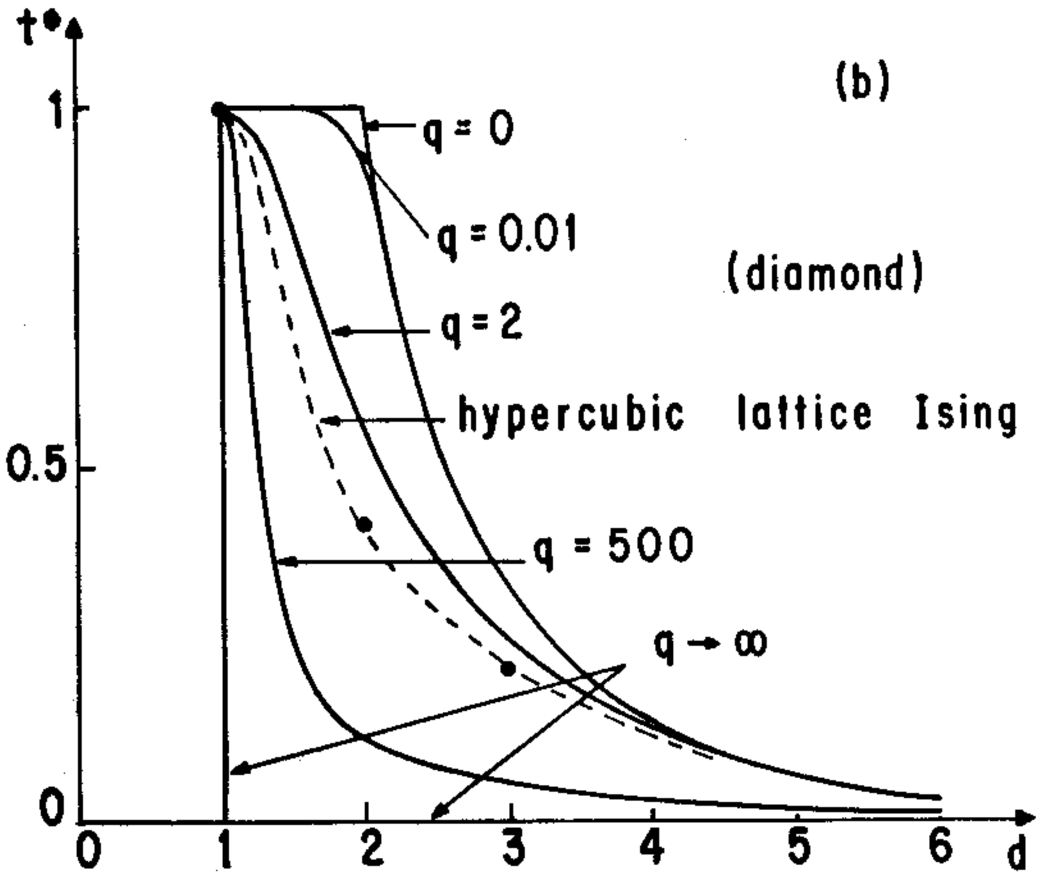

t*
(b)
1
q = 0
q = 0.01
(diamond)
q = 2
hypercubic lattice Ising
0.5
q = 500
q → ∞
0
0
1
2
3
4
5
6
d

FIG. 3

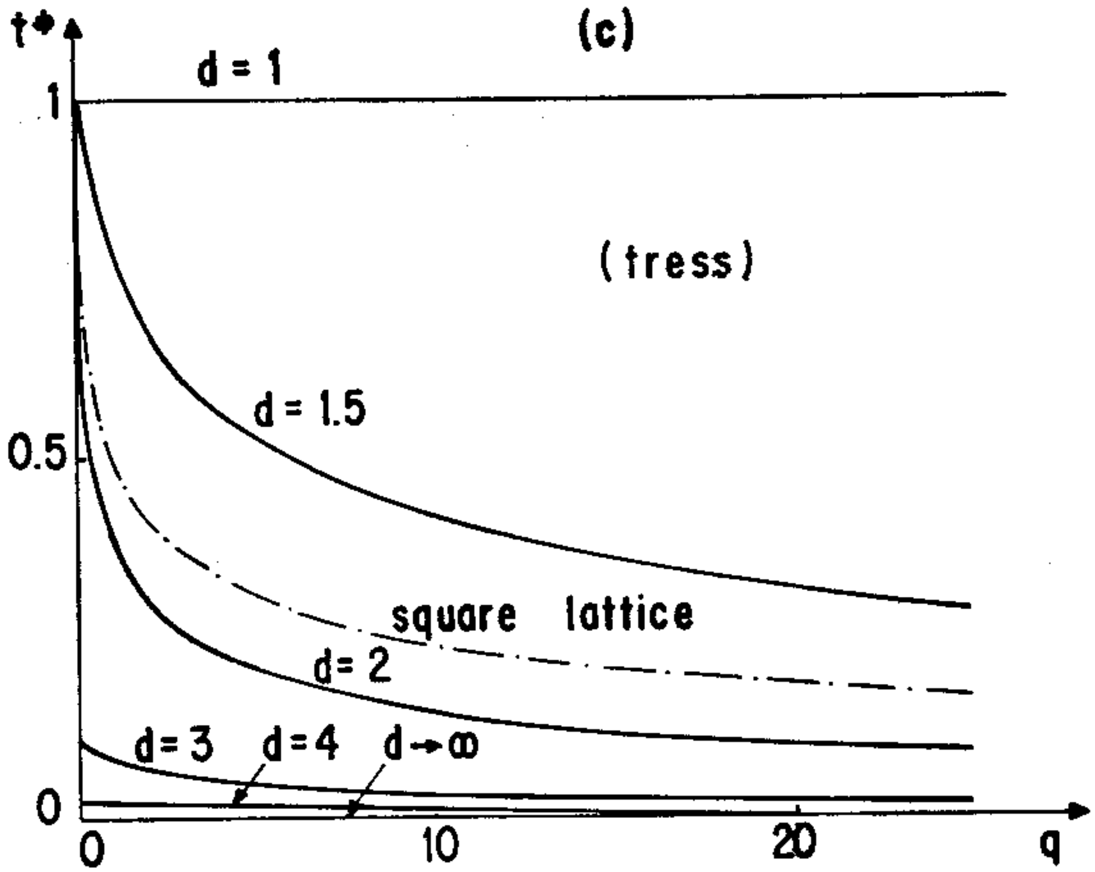


FIG.3

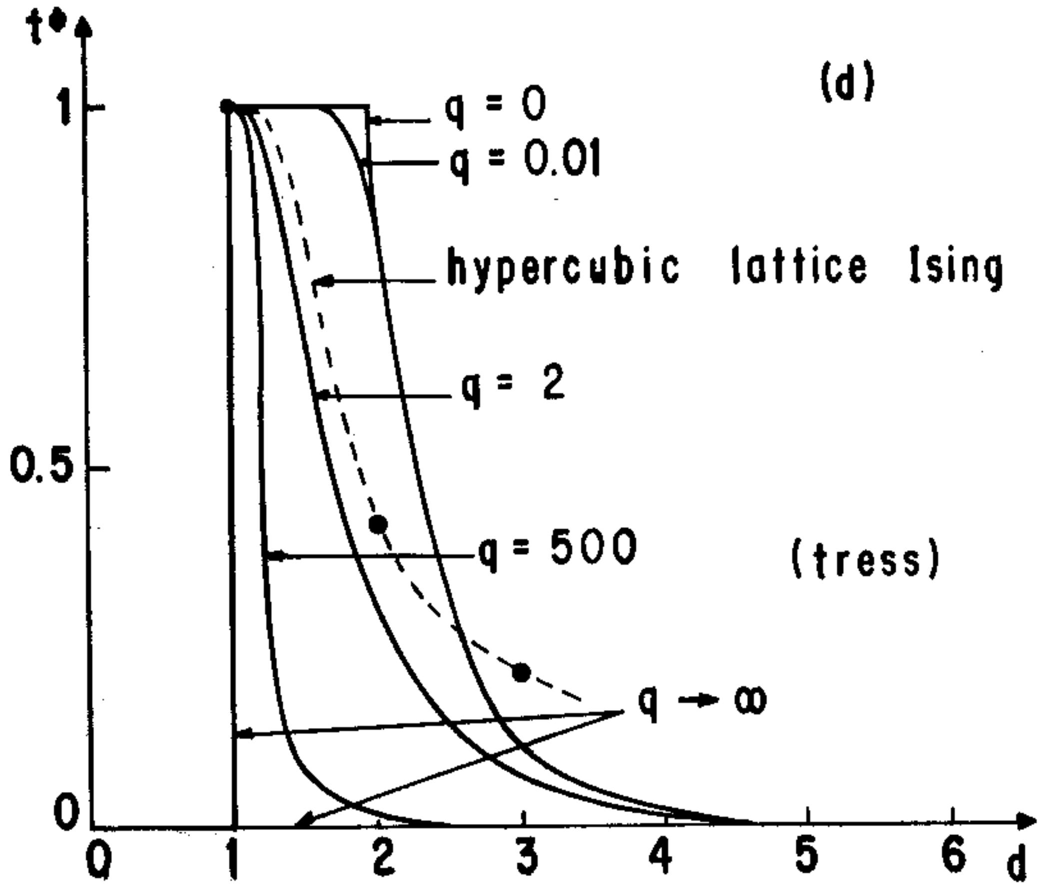

t*
1
0.5
0
0
1
2
3
4
5
6
d
(d)
q = 0
q = 0.01
hypercubic lattice Ising
q = 2
q = 500
(tress)
q → ∞

FIG.3

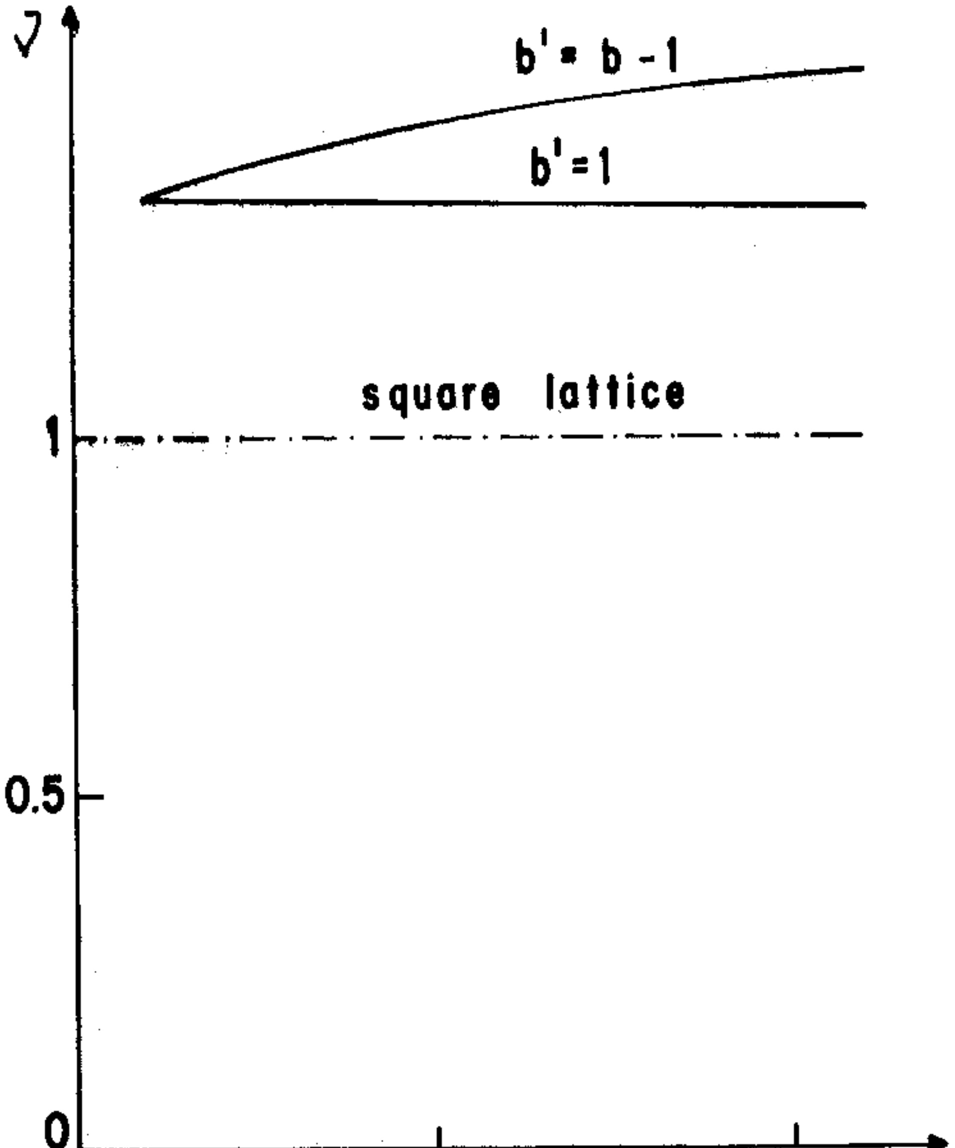


FIG. 4

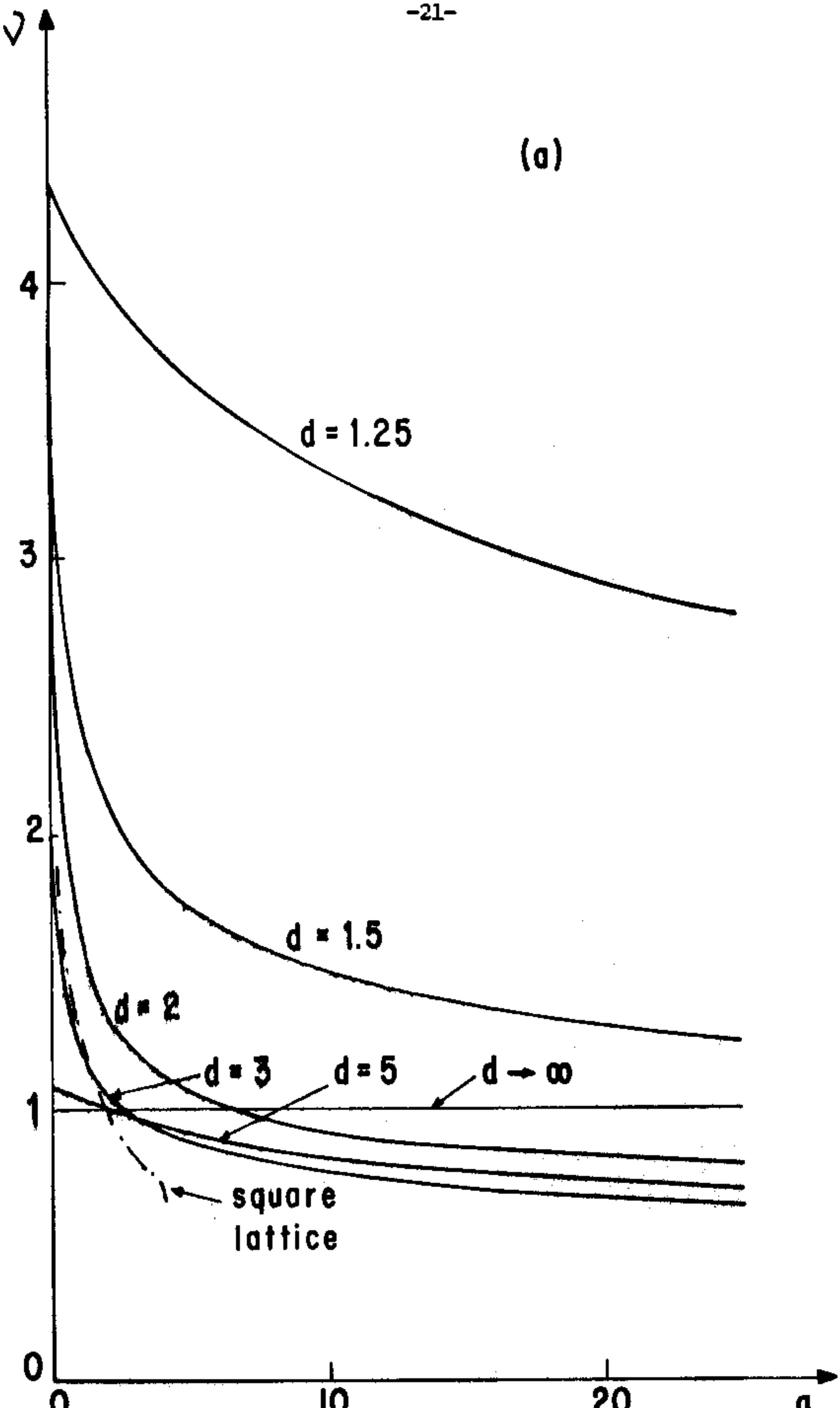
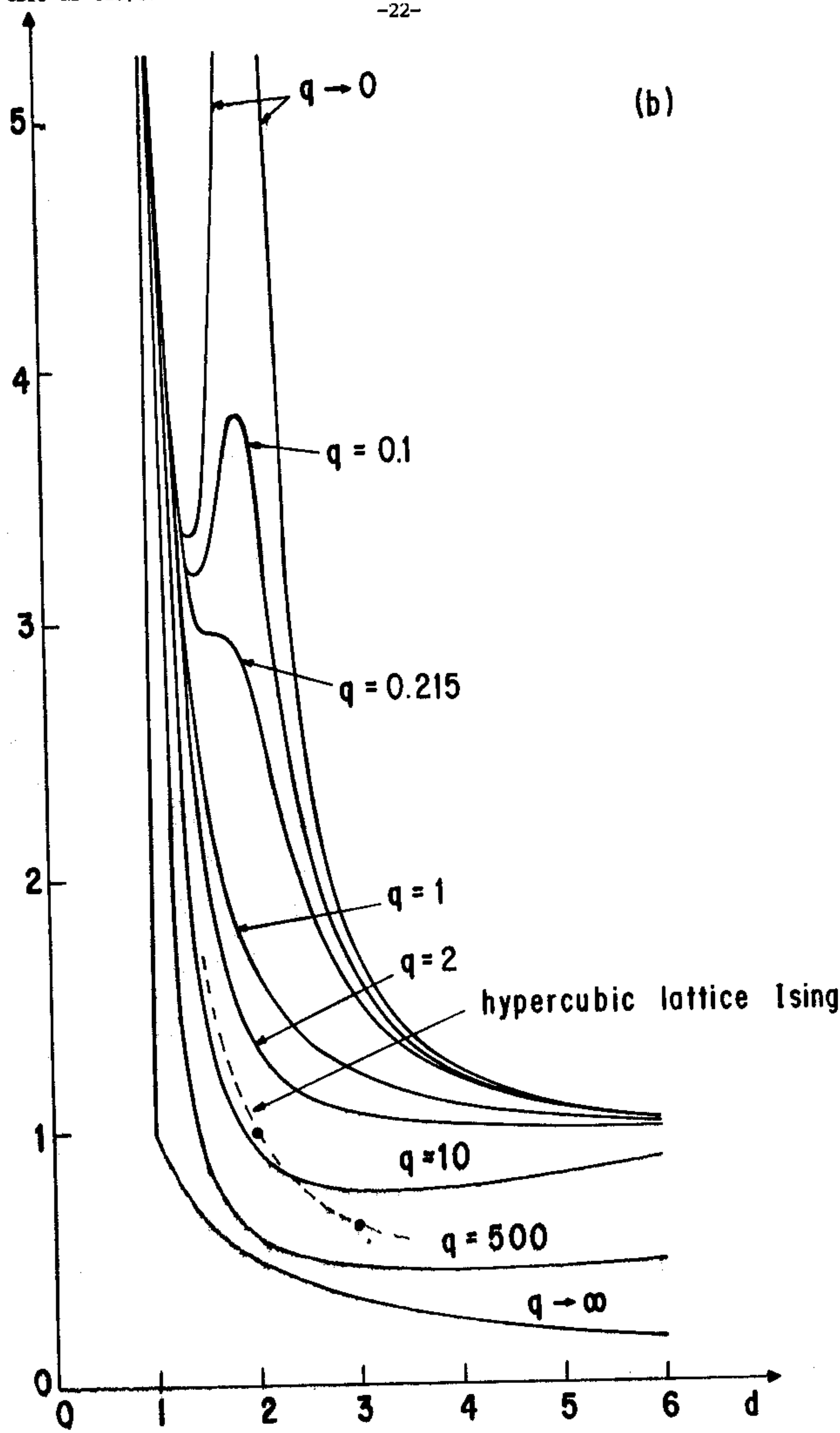


FIG.5



(b)
q → 0
q = 0.1
q = 0.215
q = 1
q = 2
hypercubic lattice Ising
q = 10
q = 500
q → ∞
5
4
3
2
1
0
0
1
2
3
4
5
6
d

FIG. 5

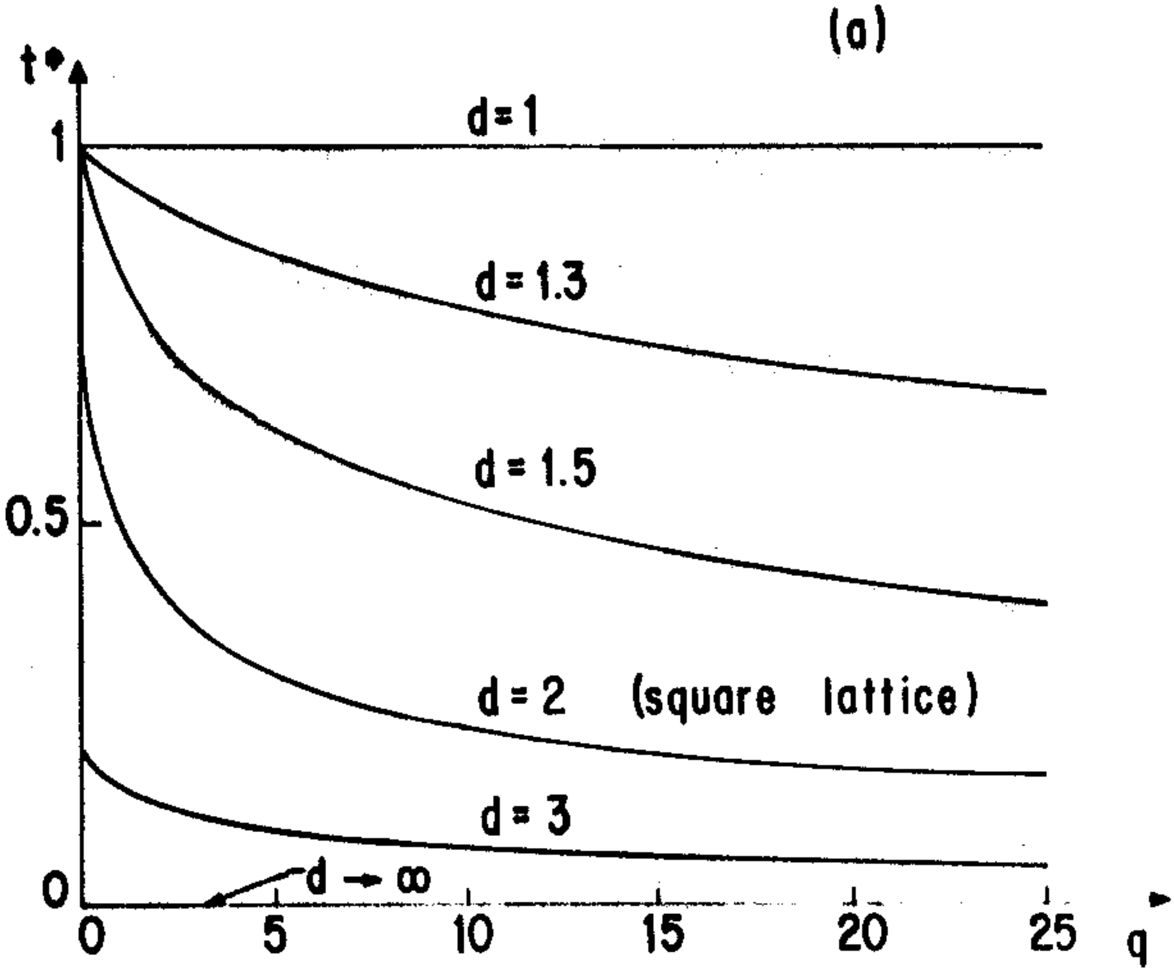


FIG.6

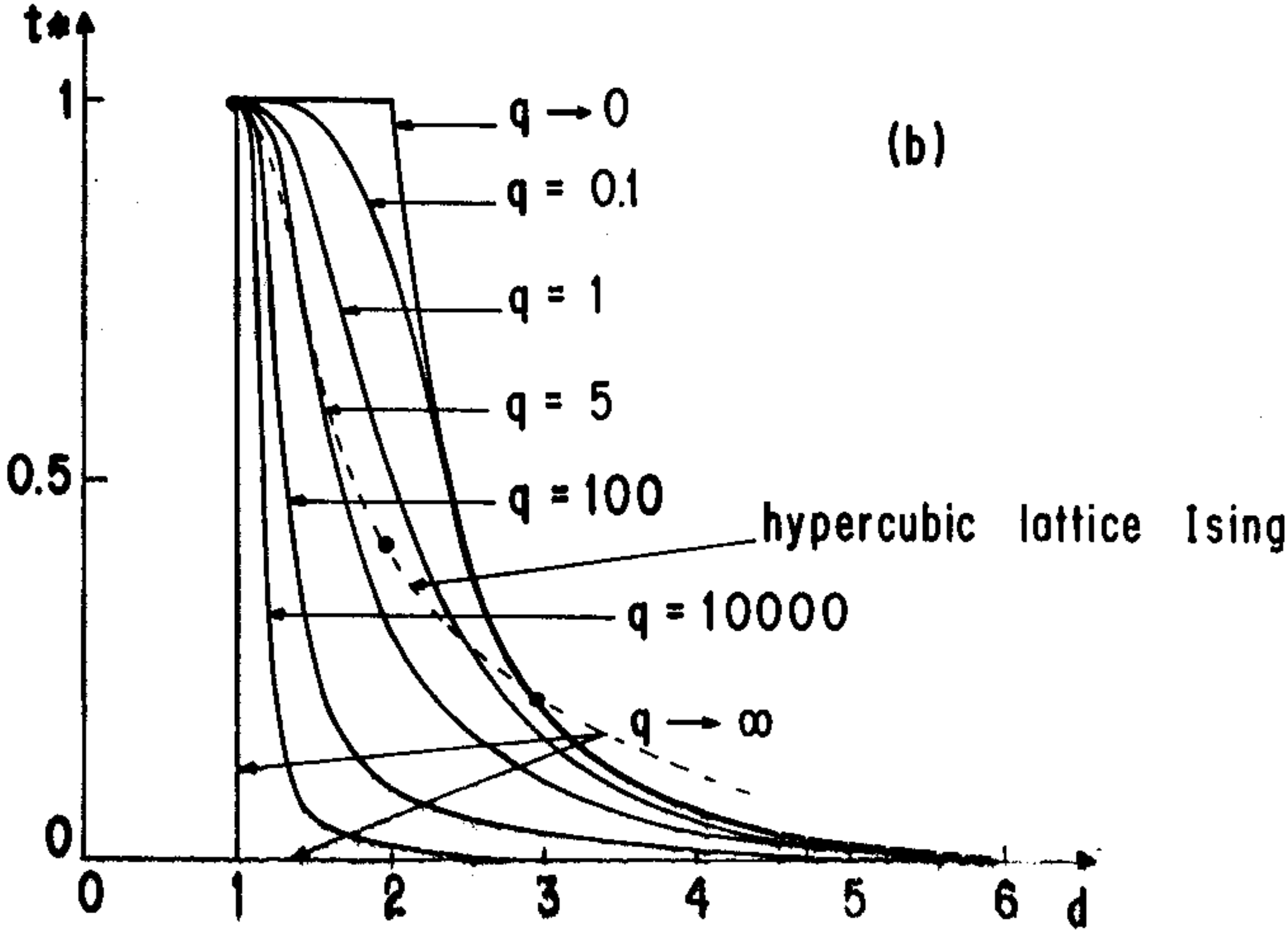

t*
1
0.5
0
q → 0
(b)
q = 0.1
q = 1
q = 5
q = 100
hypercubic lattice Ising
q = 10000
q → ∞
0
1
2
3
4
5
6
d

FIG. 6

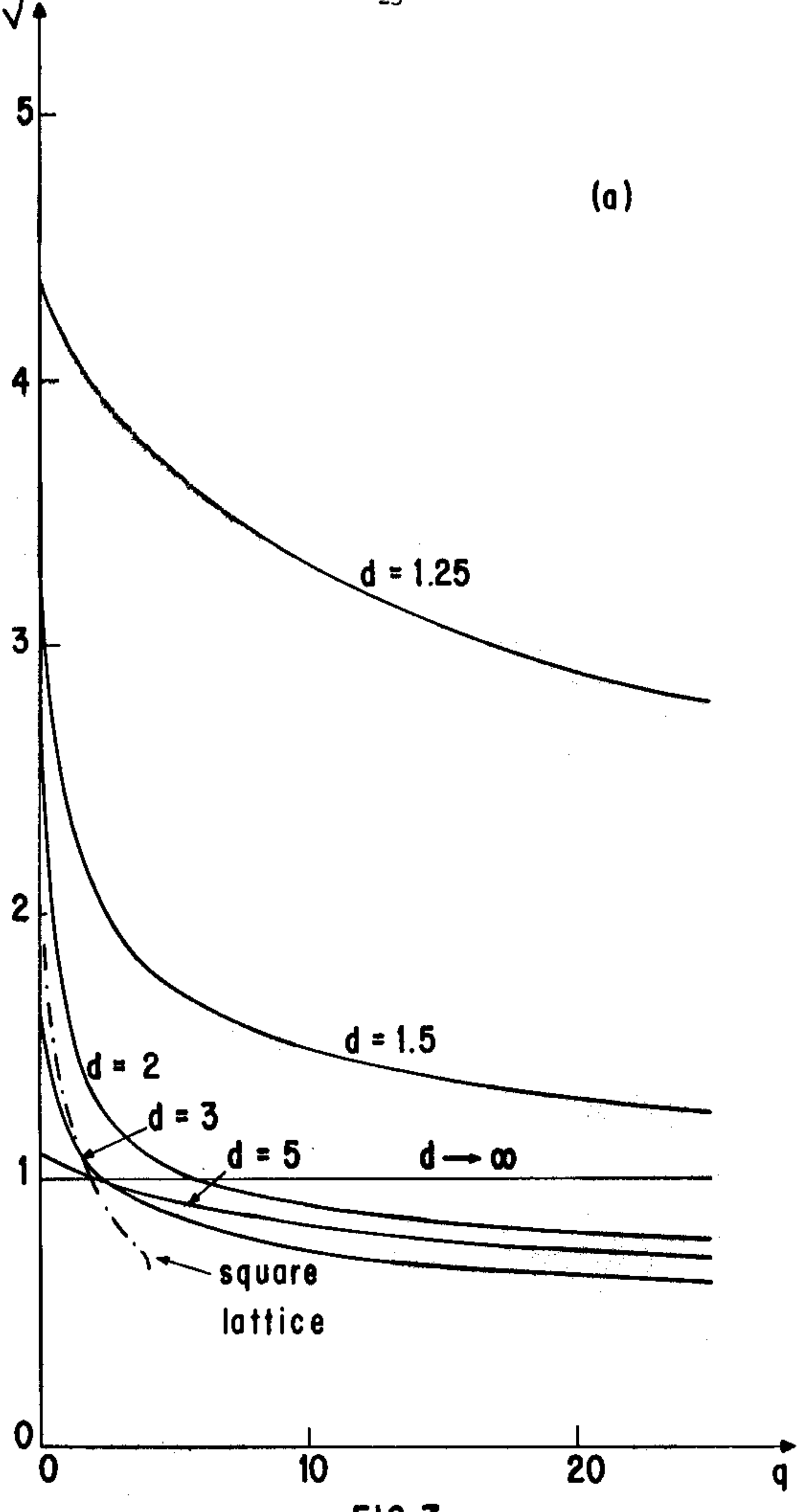


FIG. 7

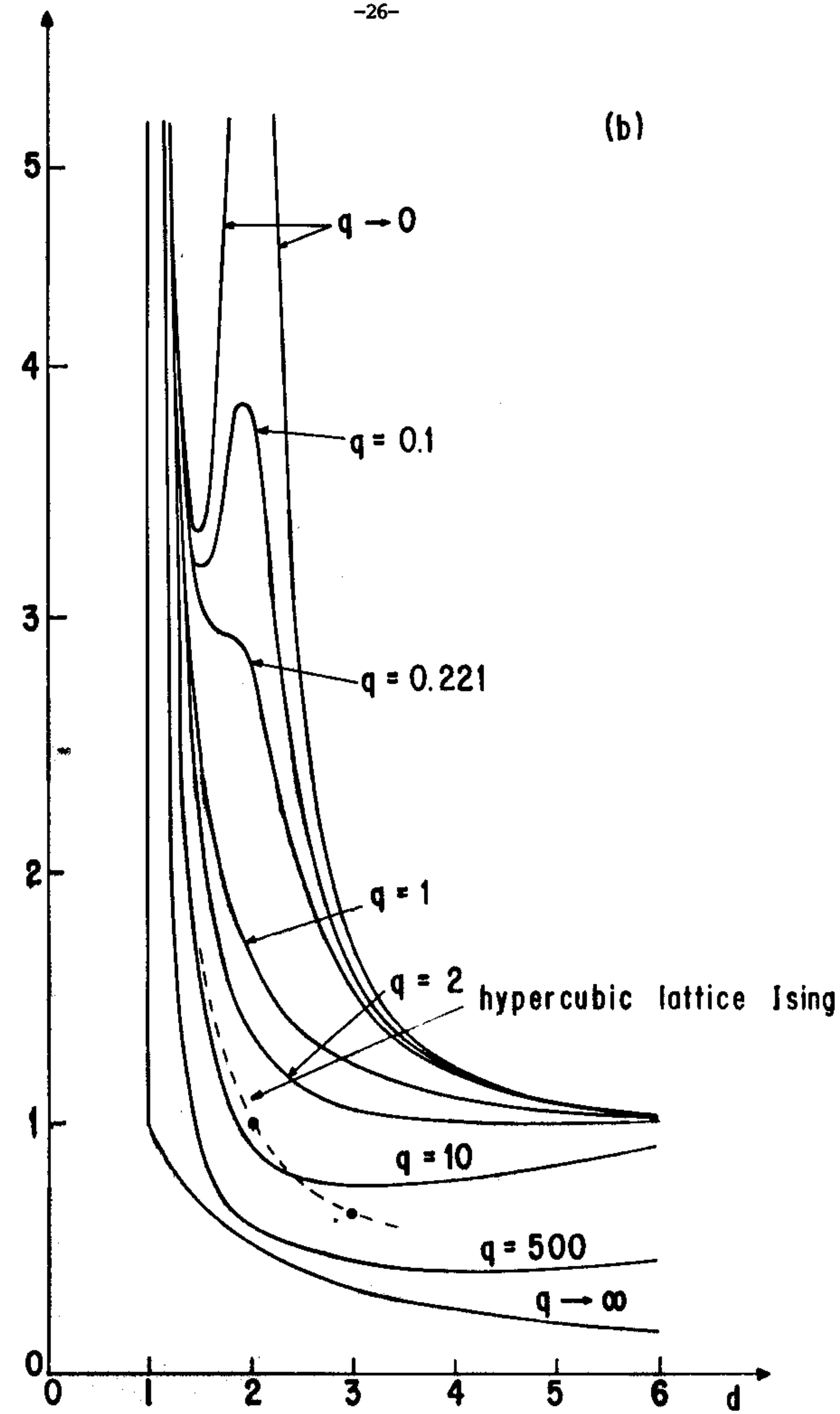

(b)
5
4
3
2
1
0
q → 0
q = 0.1
q = 0.221
q = 1
q = 2
hypercubic lattice Ising
q = 10
q = 500
q → ∞
0
1
2
3
4
5
6
d

FIG.7

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