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DIMENSIONAL REGULARIZATION, BOCHNER'S THEOREM AND PERTURBATIVE CALCULATIONS

by

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ABSTRACT

It is shown that when doing perturbative calculations with dimensional regularization the straightforward use of Bochner theorem gives an alternative method to the usual ones, without following Feynman or Bogolinbov prescriptions. Several examples are discussed. The method is equally applicable when working with analytic regularization.

Key-words: Field theory; Perturbative calculations; Renormalization.

1 INTRODUCTION

When studying perturbative quantum field theory by using Feynman diagrams techniques, the bare particles have as propagators the Feynman causal Green function Δ (with mass or without it). A second order loop leads then to the product $\Delta_1\ \Delta_2$ which is not well defined due to the singular behavior at the origin.

Working in momentum space with Fourier transforms we have the well known expression of the convolution theorem

$$(2\pi)^{\nu}\ F(\Delta_1\Delta_2) = F(\Delta_1) * F(\Delta_2) = P_1 * P_2 \tag{1}$$

where $P_1 = \int dx^{\nu}\Delta_1\ e^{ip.x}$
is the propagator in momentum space .

In (1) the singular behavior manifests itself as the famous ultraviolet divergences present in the convolution integration.

The usual procedure to deal with (1) consists in the use of Feynman parameters[1] or Bogoliubov[2] method to get an expression which is then regularized to obtain sensible results.

The singular character of (1) can have a different aspect if we leave the number of dimensions as a free regularizing parameter[3]. The Fourier antitransform of the propagator in momentum space is then an analytic function of ν

$$F^{-1}(P_1) = \Delta_1(\nu) \tag{2}$$

$$F^{-1}(P_1 * P_2) = (2\pi)^{\nu}\ \Delta_1(\nu)\ \Delta_2(\nu) \tag{3}$$

$$P_1 * P_2 = (2\pi)^{\nu}\ F(\Delta_1(\nu)\ \Delta_2(\nu)) \tag{4}$$

The ultraviolet divergences appear now as poles of the analytic function of ν defined by (4). The Feynman, (or Bogolinbov) trick is an elegant way to cast the convolution integration into a simpler expression, leaving the "complications" to a final integration over the extra auxiliary variables. We want to show that we can get the same results without the aid of extra parameters (except ν) by means of a systematic use of the following well known theorem:

Bochner Theorem[4]. If a function $f(x,x_2 \dots x_\nu)$ depends only on $x=(x_1^2+x_2^2+\dots+x_\nu^2)^{1/2}$ its Fourier transform:

$$I(p_1 \dots p_2) = \int dx^{\nu}\ f(x)\ e^{i\vec{p}\cdot\vec{x}} \tag{5}$$

depends only on

$$p = \left(p_1^2 + p_2^2 + \dots + p_\nu^2\right)^{1/2}$$

and:

$$I(p) = \frac{(2\pi)^{-\nu}}{p^{\frac{\nu}{2}-1}} \int_0^{\infty} f(x)\ x^{\frac{\nu}{2}}\ J_{(\frac{\nu}{2}-1)}(px)\ dx \tag{6}$$

We want to point out that the causal propagators are functions of $t^2-x^2+i\varepsilon$. So, its Fourier transforms are functions of $E^2-p^2-i\varepsilon$. We must also keep in mind that this is equivalent to perform all calculations in euclidean metric, and later on a dilatation of time, plus an analytic continuation in the coeficient of the dilatation (say K).

Finally make $K=i\pm\varepsilon$ (see ref. [5]).

Having decided to use euclidean metric we are forced to use Bochner theorem and the computation of (1) or (4) is done according to the following steps.

First we take the antiFourier transform of each factor of the convolution. This can be done with the aid of Bochner theorem or simply by the use of a table of Fourier transforms. For example, if we use ref. [6] we get the following results:

a) For a massless particle (note than when using Bochner formula (6) for F^{-1}, the factor $(2\pi)^{-\nu}$ has to be suppressed).

$$P = p^{-2} \qquad \Delta(\nu) = F^{-1}\left(\frac{1}{p^2}\right) = 2^{\frac{\nu}{2}-2}\,\Gamma\left(\frac{\nu}{2}-1\right)\left(x^2\right)^{1-\frac{\nu}{2}} \tag{7}$$

or, more general (ref. [6] p.365)

$$P = p^{-2\alpha} \qquad \Delta(\nu) = 2^{\frac{\nu}{2}-2\alpha}\,\frac{\Gamma\left(\frac{\nu}{2}-\alpha\right)}{\Gamma(\alpha)}\left(x^2\right)^{\alpha-\frac{\nu}{2}} \tag{8}$$

b) For a massive particle, we have

$$p = \frac{1}{p^2+m^2} \qquad\qquad \Delta(\nu) = \frac{m^{\frac{\nu}{2}-1} K_{\frac{\nu}{2}-1}(mx)}{x^{\frac{\nu}{2}-1}} \tag{9}$$

or, for arbitrary powers (ref. [6] p.365)

$$p = \frac{1}{\left(p^2+m^2\right)^{\lambda}} \qquad\qquad \Delta(\nu) = \frac{2^{1-\lambda}\, m^{\frac{\nu}{2}-\lambda}}{\Gamma(\lambda)} \frac{K_{\frac{\nu}{2}-\lambda}(mx)}{x^{\frac{\nu}{2}-\lambda}} \tag{10}$$

now for the second step we take of course the product of the functions whose convolution (in p-space) we want to evaluate. In configuration space both functions of the product depend only on x and therefore we can use again Bochner theorem to get the final answer. For that purpose, all we need is a table of integrals containing the case in which the integrand is the product of a power of the variable times (at most, for the usual theories) three Bessel functions (see ref. [7] p. 694, form 6.578).

We shall illustrate how the method works in the common example.

2 CONVOLUTION OF MASSLESS PROPAGATORS

From (7) we get:

$$F^{-1}\left(\frac{1}{p^2} * \frac{1}{p^2}\right) = 2^{\nu-4}\, \Gamma^2\left(\frac{\nu}{2} - 1\right) \left(x^2\right)^{2-\nu} \tag{11}$$

Taking Fourier transform with (6) we have:

$$\frac{1}{p^2} * \frac{1}{p^2} = \frac{2^{\nu-4}(2\pi)^{-\nu}}{p^{\frac{\nu}{2}-1}} \Gamma^2\left(\frac{\nu}{2} - 1\right) \int_0^\infty dx \; x^{4-2\nu} \; x^{\frac{\nu}{2}} \; J_{\frac{\nu}{2}-1}(px) \tag{12}$$

From ref. [7] p.684. We get:

$$\int_0^\infty dx \; x^{\mu} \; J_{\rho}(ax) = 2^{\mu} \; a^{-\mu-1} \; \frac{\Gamma\left(\frac{1+\rho+\mu}{2}\right)}{\Gamma\left(\frac{1+\rho-\mu}{2}\right)} \tag{13}$$

So, with appropriate substitutions, we obtain:

$$\frac{1}{p^2} * \frac{1}{p^2} = 2^{-\frac{\nu}{2}} \left(2\pi\right)^{-\nu} \frac{\Gamma^2\left(\frac{\nu}{2}-1\right)\Gamma\left(2-\frac{\nu}{2}\right)}{p^{4-\nu} \; \Gamma(\nu-2)} \tag{14}$$

This formula can easily be generalized to arbitrary powers of the massless propagators (see eq. 8) (analytic regularization)

$$\frac{1}{p^{2\alpha}} * \frac{1}{p^{2\beta}} = \frac{2^{\nu-2(\alpha+\beta)}}{(2\pi)^{\nu} \; p^{\frac{\nu}{2}-1}} \Gamma\left(\frac{\nu}{2}-\alpha\right)\Gamma\left(\frac{\nu}{2}-\beta\right) \int_0^\infty dx \; x^{2(\alpha+\beta)-2\nu} \; x^{\frac{\nu}{2}} \; J_{\frac{\nu}{2}-1}(px)$$

and using (13)

$$\frac{1}{p^{2\alpha}} * \frac{1}{p^{2\beta}} = 2^{-\frac{\nu}{2}} \left(2\pi\right)^{-\nu} \frac{\Gamma\left(\frac{\nu}{2}-\alpha\right)\Gamma\left(\frac{\nu}{2}-\beta\right)\Gamma\left(\alpha+\beta-\frac{\nu}{2}\right)}{p^{2(\alpha+\beta)-\nu} \; \Gamma(\nu-\alpha-\beta)} \tag{15}$$

3 CONVOLUTION OF A MASSLESS PROPAGATOR WITH A MASSIVE ONE

Here we have to use the product of (7) and (9) and the Bochner theorem

$$\frac{1}{p^2} * \frac{1}{p^2+m^2} = \frac{2^{\frac{\nu}{2}-2}}{(2\pi)^\nu} \frac{m^{\frac{\nu}{2}-1}\,\Gamma\left(\frac{\nu}{2}-1\right)}{p^{\frac{\nu}{2}-1}} \int_0^\infty dx\; x^{3-\frac{3\nu}{2}}\; x^{\frac{\nu}{2}}\; J_{\frac{\nu}{2}-1}(px)\; K_{\frac{\nu}{2}-1}(mx)$$

We now use ref. [6] p. 693 (6.576-3)

$$\int_0^\infty dx\; x^{-\lambda}\; K_\mu(ax)\; J_\rho(bx) = \frac{b^\rho \Gamma\left(\frac{\rho-\lambda+\mu+1}{2}\right)\Gamma\left(\frac{\rho-\lambda-\mu+1}{2}\right)}{2^{\lambda+1}\; a^{\rho-\lambda+1}\; \Gamma(1+\rho)}\; F\left(\frac{\rho-\lambda+\mu+1}{2},\frac{\rho-\lambda-\mu+1}{2};\rho+1;-\frac{b^2}{a^2}\right)$$

and we finally get (Cf. ref. [3]):

$$\frac{1}{p^2} * \frac{1}{p^2+m^2} = \frac{2^{-\frac{\nu}{2}}\; m^{\nu-4}}{(2\pi)^\nu} \frac{\Gamma\left(\frac{\nu}{2}-1\right)\Gamma\left(2-\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\; F\left(1,2-\frac{\nu}{2};\frac{\nu}{2};\frac{-p^2}{m^2}\right) \tag{16}$$

It is also easy to evaluate the vertex for zero momentum transfer. For this case, the convolution to be done is:

$$\frac{1}{p^2} * \frac{1}{\left(p^2+m^2\right)^2}$$

The calculation is almost the same as that for (16). The only difference is that now we have to use (10) with $\lambda=2$. We get

$$\frac{1}{p^2} * \frac{1}{\left(p^2+m^2\right)^2} = \frac{2^{-\frac{\nu}{2}} m^{\nu-6}}{(2\pi)^{\nu}} \frac{\Gamma\left(\frac{\nu}{2}-1\right)\Gamma\left(3-\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} F\left(1,3-\frac{\nu}{2};\frac{\nu}{2};\frac{-p^2}{m^2}\right) \tag{17}$$

4 CONVOLUTION OF MASSIVE PROPAGATORS

For massive propagators we can use (10). Bochner theorem (eq.(6)) leads us to an integral that contains a power of the integration variable and the product of a Bessel function of the first kind $J_{\frac{\nu}{2}}(px)$, times two Bessel functions of the third kind $K_{\rho}(m_1x)$ and $K_{\sigma}(m_2x)$. However a Bessel K_{λ} function can be written as a combination of Bessel J_{λ}-functions (see ref. [7] p.951) so that the integration is reduced to a combination of integrals containing a power of x and a product of three Bessel functions of the first kind. The answer, then, can be obtained by means of ref. [7] form 6.578-1 p.694 or 6.578-2.

5 DISCUSSION

In a way, the method just explained is the most natural one, as it is based on the canonical application of the convolution theorem and the generalized Fourier transform of causal distributions. This together with the use of Bochner theorem and the dimensional regularization techniques, completes the scheme.

The procedure can be expressed more synthetically in the following alternative way:

Let us take two spherically symmetric functions $f_1(p)$ and $f_2(p)$ (two propagators, for example). Then we can use Bochner theorem to write

$$g_1(x) = F^{-1}(f_1(p)) = x^{1-\frac{\nu}{2}} \int_0^\infty dp\; p^{\frac{\nu}{2}} f_1(p)\; J_{\frac{\nu}{2}-1}(px) \tag{18}$$

Now, we multiply together g_1 and g_2

$$g_1(x)g_2(x) = x^{2-\nu} \int_0^\infty dp_1\; f_1(p_1)p_1^{\frac{\nu}{2}}\; J_{\frac{\nu}{2}-1}(p_1x) \int_0^\infty dp_2\; f_2(p_2)\; p_2^{\frac{\nu}{2}}\; J_{\frac{\nu}{2}-1}(p_2x) \tag{19}$$

Using Bochner theorem again

$$f_1(p) * f_2(p) = \frac{p^{1-\frac{\nu}{2}}}{(2\pi)^{\nu}} \int_0^\infty dx\; x^{2-\nu}\; x^{\frac{\nu}{2}}\; J_{\frac{\nu}{2}-1}(px) \left\{\int_0^\infty dp_1\; f_1(p_1)p_1^{\frac{\nu}{2}}\; J_{\frac{\nu}{2}-1}(p_1x).\right.$$
$$\left. .\int_0^\infty dp_2\; f(p_2)\; p_2^{\frac{\nu}{2}}\; J_{\frac{\nu}{2}-1}(p_2x)\right. \tag{20}$$

But we find in ref. [7] p.696; 6.57-9

$$\int_0^\infty dx\; x^{1-\alpha}\; J_\alpha(ax)\; J_\alpha(bx)\; J_\alpha(cx) = \frac{2^{\alpha-1}\; \Delta^{2\alpha-1}}{(abc)^\alpha\; \Gamma(\alpha+\frac{1}{2})\; \Gamma(\frac{1}{2})}$$

where Δ is the area of the triangle whose sides are a, b and c. When a b c cannot form a triangle the integral is zero. This "amusing" formula allows us to write

$$f_1(p)*f_2(p) = \frac{2^{\frac{\nu}{2}-2}}{(2\pi)^\nu}\frac{\pi^{-\frac{1}{2}}}{\Gamma(\frac{\nu-1}{2})}p^{1-\frac{\nu}{2}}\int_0^\infty dp_1\int_0^\infty dp_2\; p_1^{\frac{\nu}{2}}\; p_2^{\frac{\nu}{2}}\; f_1(p_1)\;\frac{f_2(p_2)\Delta^{\nu-3}}{(pp_1p_2)^{\frac{\nu}{2}-1}}$$

$$f_1(p)*f_2(p) = \frac{2^{\frac{\nu}{2}-2}\,\pi^{-\frac{1}{2}}\,p^{2-\nu}}{(2\pi\;)^\nu\;\Gamma(\frac{\nu-1}{2})}\int_0^\infty dp_1p_1\int_0^\infty dp_2p_2\; f_1(p_1)f_2(p_2)\Delta^{\nu-3} \tag{21}$$

It is not difficult to see that

$$\Delta = \frac{1}{4}\left(2p^2p_1^2+2p_1^2p_2^2+2p_2^2p^2-p_1^4-p_2^4-p^4\right)^{1/2} \tag{22}$$

Eq. (21) can be considered to be an extension of Bochner theorem to the convolution of two spherically symmetric functions.

If we choose to integrate first with respect to p_2, we can write.

$$\Delta = \frac{1}{4}\left(p_2^2-(p-p_1)^2\right)^{1/2}\left((p+p_1)^2-p_2^2\right)^{1/2} \tag{23}$$

So that

$$f_1(p)*f_2(p) = \frac{2^{\frac{\nu}{2}-2}\;\pi^{-\frac{1}{2}}\;p^{2-\nu}}{(2\pi)^\nu\;\Gamma\left(\frac{\nu-1}{2}\right)}\int_0^\infty dp_1p_1f(p_1)\int_{(p-p_1)}^{(p+p_1)}dqf_2(q)q\Big[q^2\; -$$

$$-(p-p_1)^2\Big]^{\frac{\nu-3}{2}} \Big[(p+p_1)^2-q^2\Big]^{\frac{\nu-3}{2}} \frac{1}{4^{\nu-3}}$$

where we took into account that $\Delta=0$ when $p_2 \leq p-p$, or $p_2 \geq p+p$.

If we take, for example, $f_2(p)$ equal to an arbitrary power of the massless propagators, we have to evaluate:

$$I = \int_{a^2}^{b^2} dq^2 \, \frac{1}{q^{2\lambda}} \left(q^2-a^2\right)^{\frac{\nu-3}{2}} \left(b^2-q^2\right)^{\frac{\nu-3}{2}} \tag{25}$$

or changing variable to $x=q^2-a^2$

$$I = \int_0^{b^2-a^2} dx \quad x^{\frac{\nu-3}{2}} \, (x+a^2)^{-\lambda} \left(b^2-a^2-x\right)^{\frac{\nu-3}{2}} \tag{26}$$

whose value can be found in ref. [7] p.287, 3.19-8. And so on.

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