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GROUP GEOMETRIC METHODS IN SUPERGRAVITY AND SUPERSTRING THEORIES*

by

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1. Introduction

The purpose of these notes is to give a brief and pedagogical account of the group-geometric approach to (super)gravity and superstring theories. Full details can be found in the book written in collaboration with R. D' Auria and P. Fré [1]; here we summarize the main ideas, and apply them to selected examples.

Group geometry provides a natural and unified formulation of gravity and gauge theories. The invariances of both are interpreted as diffeomorphisms on a suitable group manifold. This geometrical framework has a fruitful output, in that it provides a systematic algorithm for the gauging of Lie algebras, and the construction of (super)gravity or (super)string lagrangians.

The basic idea is to associate fundamental fields to the group generators. This is done by considering first a basis of tangent vectors on the group manifold. These vectors close on the same algebra as the abstract group generators. The dual basis, i.e. the vielbeins (cotangent basis of one-forms) is then identified with the set of fundamental fields. Thus, for example, the vielbein V^a and the spin connection ω^{ab} of ordinary Einstein-Cartan gravity are seen as the duals of the tangent vectors corresponding to translations and Lorentz rotations, respectively.

Dealing with forms is particularly appropriate when having in mind to construct integrands (lagrangians). Also, this formalism extends to p- forms ($p > 1$), and gives an algebraic "raison d' être" to antisymmetric tensor fields as well. The relevant structures are the so-called free differential algebras [2,3,1] and generalize the Cartan-Maurer equations of ordinary Lie algebras.

We set up the geometric framework in sections 2 and 3, and we apply it to the derivation of D=4 gravity and supergravity lagrangians in sections 4 and 5. Section 6 is devoted to free differential algebras. By adjoining a fermionic (central) charge to the Lie algebra G, we show in section 7 that BRST symmetry can be seen as a global coordinate change in the fermionic direction. This is generalized to free differential algebras, and provides a geometric rationale to the "russian" formula of Stora for the BRST transformations of antisymmetric fields. Finally, section 8 contains the geometric derivation of the type II superstring lagrangian in an arbitrary background.

Except for part of section 7, the material presented here is not new, and can be found in the original references quoted at the end. We have tried to write a self-consistent review and only elementary knowledge of differential geometry and group theory is assumed.

2. Group manifolds

Let us start from a Lie algebra Lie(G), with generators T_A satisfying the commutation relations

$$[T_A, T_B] = C^C{}_{AB} T_C \tag{2.1}$$

A generic group element $g \in G$ connected with the identity * can be expressed as

$$g = \exp(y^A T_A) \equiv y \tag{2.2}$$

where y^A are the (exponential) coordinates of the group manifold. Each element of G is labelled by the coordinates y^A, and for notational economy we denote it simply by y. Similarly yx stands for $\exp(y^A T_A)\exp(x^B T_B)$, the product of two group elements, and by $(yx)^M$ we denote the corresponding coordinates.

Consider now $(yx)^M$ as a function ** of x^A:

$$(yx)^M = y^M + e_A{}^M(y)x^A + e_{AB}{}^M(y)x^A x^B + ... \tag{2.3}$$

For infinitesimal x:

$$(yx)^M = y^M + (x^A \vec{t}_A) y^M = (1 + x^A \vec{t}_A) y^M, \quad \vec{t}_A \equiv e_A{}^N(y) \frac{\partial}{\partial y^N} \tag{2.4}$$

so that the $\vec{t}_A$ are a differential representation of the abstract generators T_A, and satisfy therefore the same algebra:

$$[\vec{t}_A, \vec{t}_B] = C^C{}_{AB} \vec{t}_C \tag{2.5}$$

The geometrical meaning of the components $e_A^N(y)$ in eq. (2.3) is clear: consider the infinitesimal displacement $\delta_A y^M$ due to the (right) action of $1 + \varepsilon T_A$ (ε = infinitesimal parameter). Then

$$\delta_A y^M = \varepsilon e_A{}^M(y) \tag{2.6}$$

and the dim$G$ vectors $e_A{}^M(y)$, A=1,...dim$G$ are simply the tangent vectors at y in the direction of the displacements $\delta_A y^M$ (see fig.1). It is customary to call tangent vector along the T_A direction the whole differential operator $\vec{t}_A \equiv e_A{}^N(y)\frac{\partial}{\partial y^N}$.

Note that $e_A{}^M$ is an invertible matrix, since the map $y \to yx$ is a diffeomorphism.

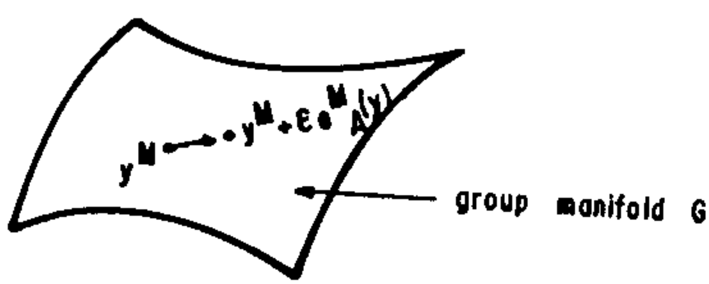

fig. 1

* Hereafter G indicates the part of the group connected with the identity.

** Since G is a Lie group, this function is smooth.

The $\vec{t}_A(y)$ span the tangent space of G at y: they form a contravariant basis. The "coordinate" basis given by the vectors $\frac{\partial}{\partial y^N}$ is related to the $\vec{t}_A$ (the intrinsic basis) via the nondegenerate matrix ${e_A}^N$. The indices A,B,... are tangent space indices ("flat" indices) and are inert under y coordinate transformations. The indices M,N,... are coordinate indices ("world" indices) and do transform under coordinate transformations in the usual way (see later). Next we define the one-forms $\sigma^A(y)$ as the duals of the $\vec{t}_A$:

$$\sigma^A(\vec{t}_B) = \delta^B_A \tag{2.8}$$

The σ^A are a covariant basis (the intrinsic vielbein basis) for the dual of the tangent space, called cotangent space (the space of 1-forms). The "coordinate" cotangent basis dual to the $\frac{\partial}{\partial y^N}$ vectors is given by the differentials dy^M $(dy^M(\frac{\partial}{\partial y^N}) = \delta^M_N)$. The components of $\sigma^A(y)$ on the coordinate basis are denoted ${e_M}^A(y)$: A/

$$\sigma^A(y) = {e_M}^A(y)\; dy^M \tag{2.9}$$

From the duality of the tangent and cotangent bases we have:

$$\begin{aligned} {e_M}^A\; {e_B}^M &= \delta^A_B \\ {e_A}^M\; {e_N}^A &= \delta^M_N \end{aligned} \tag{2.10}$$

Exercise 1: Substitute eq. (2.7) into the commutator (2.5) and find the differential condition on ${e_A}^M(y)$:

$$-2e_{[A}{}^N\; e_{B]}{}^M \partial_N {e_M}^C = {C^C}_{AB} \tag{2.11}$$

Exercise 2: compute the exterior derivative of σ^A using eq.s (2.9) and (2.11) and find

$$d\sigma^A + \frac{1}{2}{C^A}_{BC}\sigma^B \wedge \sigma^C = 0 \tag{2.12}$$

These are called the Cartan-Maurer equations, and provide a dual formulation of Lie algebras in terms of the one-forms σ^A. It is immediate to verify that the closure of the exterior derivative d ($d^2 = 0$) is equivalent to the Jacobi identities for the structure constants:

$$C^A{}_{B[C}C^B{}_{DE]} = 0 \tag{2.13}$$

(apply d to eq. (2.12)).

Note

Defining $\sigma(y) \equiv \sigma^A(y)T_A$ the Cartan-Maurer eq.s (2.12) take the form

$$d\sigma + \sigma \wedge \sigma = 0 \tag{2.14}$$

The Lie-valued one-form $\sigma(y)$ can also be constructed directly from the group element y:

$$\sigma(y) = y^{-1}dy \tag{2.15}$$

It is easy to verify that (2.15) satisfies the Cartan-Maurer equation (2 .14) (use $dy^{-1} = -y^{-1}dy\ y^{-1}$). Moreover, it takes the same value as $e_M{}^A dy^M\ T_A$ at the origin $y = 0$. Indeed from the definition of $e_A{}^M$ in eq. (2.3) one sees that $e_A{}^M(y=0) = \delta_A^M$, and therefore $e_M{}^A(0)dy^M\ T_A = dy^A\ T_A$. This value coincides with $y^{-1}dy|_{y=0}$ since $y^{-1}|_{y=0}$ =[group unit], and $dy|_{y=0} = dy^A T_A$ (from (2.2)). This observation suffices to conclude that $y^{-1}dy$ is equal to $e_M{}^A(y)dy^M T_A$.

3. One-forms as dynamical fields

Consider a smooth deformation $\tilde{G}$ of the group manifold G. Its vielbein field is given by the intrinsic cotangent basis, defined for any differentiable manifold:

$$\mu^A(y) = \mu_M{}^A(y)dy^M \tag{3.1}$$

In general μ^A does not satisfy the Cartan-Maurer equations any more, so that

$$d\mu^A + \frac{1}{2}C^A{}_{BC}\mu^B \wedge \mu^C \equiv R^A \neq 0 \tag{3.2}$$

The extent of the deformation $G \to \tilde{G}$ is measured by the curvature two-form R^A. $R^A = 0$ implies $\mu^A = \sigma^A$ and viceversa.

Applying the external derivative d to the definition (3.2), using $d^2 = 0$ and the Jacobi identities on $C^A{}_{BC}$, yields the Bianchi identities

$$(\nabla R)^A \equiv dR^A - C^A{}_{BC}R^B \wedge \mu^C = 0 \tag{3.3}$$

The main idea is to consider the one-forms $\mu^A(y)$ as the fundamental fields of our geometric theory. The deformation $G \to \tilde{G}$ is necessary in order to allow configurations with nonvanishing curvature.

As a first example, consider $\tilde{G}$= smooth deformation of the Poincaré group, whose structure constants are read off the corresponding Lie algebra :

$$\begin{aligned}
&[P_a, P_b] = 0 \\
&[M_{ab}, M_{cd}] = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} \\
&[M_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b
\end{aligned} \tag{3.4}$$

Denoting by V^a and ω^{ab} the vielbein μ^A when the index A runs on the translations and on the Lorentz rotations respectively, eq.s (3.2) take the form:

$$R^a = dV^a - \omega^{ab} \wedge V^c \eta_{bc}$$
$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{db} \eta_{cd} \tag{3.5}$$

The fundamental fields V^a and ω^{ab} are interpreted as the ordinary vierbein and the spin connection, respectively, and eq.s (3.5) define the torsion and the Riemann curvature. These satisfy the Bianchi identities*

$$dR^a + R^{ab}V^b - \omega^{ab}R^b \equiv \mathcal{D}R^a + R^{ab}V^b = 0$$
$$dR^{ab} + R^{ac}\omega^{cb} - \omega^{ac}R^{cb} \equiv \mathcal{D}R^{ab} = 0 \tag{3.6}$$

where $\mathcal{D}$ is the Lorentz covariant derivative, and repeated indices are contracted with the Minkowski metric η_{ab}.

How do we find the dynamics of $\mu^A(y)$? We want to obtain a geometric theory, i.e. invariant under diffeomorphisms of the soft group manifold $\tilde{G}$. We need therefore to construct an action invariant under diffeomorphisms, and this is simply achieved by using only diffeomorphic invariant operations as the exterior derivative and the wedge product. Our building blocks are the one-form μ^A and its curvature two-form R^A, and exterior products of them can make up a lagrangian D-form (where D is the dimension of space-time, see later).

An immediate problem presents itself: the fields $\mu^A(y)$ depend on all the soft group manifold coordinates y. In the Poincaré example, this means that the vierbein and the spin connection depend on the coordinates y^a associated to the translations (the ordinary space- time coordinates) *and* on the coordinates y^{ab} associated to the Lorentz rotations. Since we want to have *space-time* fields at the end of the game, we have to find a way to remove the y^{ab} dependence. This is achieved when the curvatures are *horizontal* in the y^{ab} directions, as we explain below.

First we discuss the variation under diffeomorphisms of the vielbein field $\mu^A(y)$:

$$\begin{aligned} \mu^A(y+\delta y) - \mu^A(y) &= \delta[\mu_M{}^A(y)dy^M] = \\ &= (\partial_N \mu_M{}^A)\delta y^N dy^M + \mu_M{}^A(\partial_N \delta y^M)dy^N = \\ &= dy^N[\partial_N \delta y^A + \delta y^M(\partial_M \mu_N{}^A - \partial_N \mu_M{}^A)] = \\ &= d\delta y^A - 2\mu^B \delta y^C (d\mu^A)_{BC} = d(i_{\vec{\delta y}}\mu^A) + i_{\vec{\delta y}} d\mu^A \end{aligned} \tag{3.7}$$

where

$$\delta y^A \equiv \delta y^M \mu_M{}^A, \quad \vec{\delta y} \equiv \delta y^M \partial_M, \quad d\mu^A \equiv (d\mu^A)_{BC}\mu^B \wedge \mu^C, \tag{3.8}$$

and the contraction $i_{\vec{t}}$ along a tangent vector $\vec{t}$ is defined on p-forms

* products between forms are understood to be exterior products. The wedge symbol $\wedge$ is omitted in the following

$$\omega_{(p)} = \omega_{B_1 \dots B_p} \mu^{B_1} \wedge \dots \wedge \mu^{B_p}$$

as

$$i_{\vec{t}}\, \omega_{(p)} = p\; t^A \omega_{AB_2 \dots B_p} \mu^{B_2} \wedge \dots \wedge \mu^{B_p} \tag{3.9}$$

Note that $i_{\vec{t}}$ maps p-forms into $(p-1)$-forms. The operator

$$l_{\vec{t}} \equiv d\; i_{\vec{t}} + i_{\vec{t}}\; d \tag{3.10}$$

is called the *Lie derivative* along the tangent vector $\vec{t}$ and maps p-forms into p-forms. As shown in eq. (3.7), the Lie derivative of the one-form μ^A along $\vec{\delta y}$ gives its variation under the diffeomorphism $y \to y + \delta y$. This holds true for any p-form.

We now rewrite the variation $\delta\mu^A$ of eq. (3.7) in a suggestive way, by adding and subtracting $C^A{}_{BC}\mu^B \delta y^C$:

$$\begin{aligned} \delta\mu^A &= d\delta y^A + C^A{}_{BC}\mu^B \delta y^C - 2\mu^B \delta y^C (d\mu^A)_{BC} - C^A{}_{BC}\mu^B \delta y^C \\ &= (\nabla \delta y)^A + i_{\vec{\delta y}} R^A \end{aligned} \tag{3.11}$$

where we have used the definition (3.2) for the curvature, and the G-covariant derivative ∇ acts on δy^A as

$$(\nabla \delta y)^A \equiv d\mu^A + C^A{}_{BC}\mu^B \delta y^A \tag{3.12}$$

All the invariances of our geometric theory are contained in eq. (3.11). In particular, suppose that the two-form $R^A = R^A{}_{BC}\mu^B \wedge \mu^C$ has vanishing components along the directions of a subgroup H of G:

$$R^A{}_{BH} = 0 \qquad \begin{matrix} \text{A runs on G} \\ \text{H runs on H} \end{matrix} \tag{3.13}$$

Then we say that R^A is *horizontal* on H, and the diffeomorphisms along the H-directions reduce to *gauge transformations*:

$$\delta\mu^A(y) = (\nabla \delta y)^A \tag{3.14}$$

Moreover, the dependence on the y^H coordinates becomes inessential, in the sense that it factorizes after a finite gauge transformation. Indeed, let us examine eq. (3.14) in more detail: separating the H-indices and the K-indices (those along the directions of the coset space G/H) we have *

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* we recall that for semisimple Lie algebras (or direct products of semisimple Lie algebras with U(1) factors) it is always possible to find a tensor transformation on the generators $T^A \to S^A_B T^B$ such that the Killing metric $g_{AB} = C^C{}_{AD}\, C^D{}_{BC}$ becomes diagonal. On this basis G/H (for any H) is *reductive*, i.e. the structure constants $C^H{}_{H'K}$ vanish. Indeed $C^H{}_{H'K}$ is proportional to $C_{HH'K} = C_{KH'H} = 0$ (indices are lowered with the Killing metric, and C_{ABC} is totally antisymmetric because of Jacobi identities).

$$\begin{aligned} \delta\mu^H &= d\delta y^H + C^H{}_{H'H''}\mu^{H'}\delta y^{H''} \\ \delta\mu^K &= C^K{}_{K'H}\mu^{K'}\delta y^H \end{aligned} \tag{3.15}$$

for diffeomorphisms along y^H. These equations have the typical form of gauge variations of the H-gauge field μ^H and of a field μ^K transforming in a representation $(T_H)^K{}_{K'} = C^K{}_{K'H}$ of the subgroup H. It is clear that invariance of the theory under (3.5) requires the field μ^H to appear in the action only through the curvatures R^A, whereas "naked" μ^K can appear since they do transform homogeneously. A finite H-coordinate transformation can be used to remove the y^H dependence in the objects appearing in the action: for example by integrating the second equation in (3.15) we find

$$\mu^K(y^K, y^H) = D_{K'}{}^K(y^H)\mu^{K'}(y^K, y^H = 0) \tag{3.16}$$

where $D_A{}^B$ is the adjoint representation of G, so that the whole dependence on y^H is contained in the D matrices. If invariant H tensors are used to contract indices in the lagrangian, the adjoint D matrices cancel out, and the fields really live on the coset space $\tilde{G}/H$. The theory "remembers" the invariance under y^H-diffeomorphisms by retaining the gauge invariance under H (eq.(3.14)), with δy^H interpreted now as a gauge parameter. This mechanism is illustrated in the examples of next sections.

In Poincaré gravity, we have horizontality of the curvatures along the Lorentz directions (see next Section): then the fields V^a and ω^{ab} live on the coset space

$$\frac{\mathrm{G}}{\mathrm{H}} = \frac{\mathrm{Poincare'}}{\mathrm{Lorentz}} \tag{3.17}$$

i.e. on ordinary spacetime. The lagrangian is integrated on a D- volume (D-dimensional spacetime), and is therefore a D-form. The resulting theory is invariant under D-spacetime diffeomorphisms, and under local Lorentz rotations.

4. Poincaré gravity in D=4

From the discussion in the preceding section, we know that the lagrangian must be a 4-form, and therefore at most quadratic in the curvatures:

$$\mathcal{L} = \Lambda^{(4)} + R^A \nu_A^{(2)} + \frac{1}{2} R^A R^B \nu_{AB}^{(0)} + \text{total differential} \tag{4.1}$$

with the group index A splitting into $A = ab, a$. The cosmological term Λ and the ν terms are exterior polynomials in the $\mu^A = \omega^{ab}, V^a$ group vielbeins:

$$\begin{aligned} \Lambda^{(4)} &= C_{ABCD}\; \mu^A\mu^B\mu^C\mu^D \\ \nu_A^{(2)} &= C_{APQ}\; \mu^P\mu^Q \\ \nu_{AB}^{(0)} &= C_{AB} \end{aligned} \tag{4.2}$$

The constant coefficients C in (4.2) are Lorentz invariant tensors (since we want a Lorentz invariant theory, see preceding section) .

The terms quadratic in the curvatures can be dropped: indeed they are equivalent to total differentials+terms linear in R^A. To prove this, we observe that the only Lorentz invariant tensors of the type ν_{AB} are:

$$\begin{aligned} &C_{ab,cd} = \varepsilon_{abcd} \\ &C'_{ab,cd} = \eta_{ae}\eta_{bf}\ \delta^{ef}_{cd} \equiv \eta_{ae}\eta_{bf}\frac{1}{2}(\delta^e_c\delta^f_d - \delta^f_c\delta^e_d) \\ &C_{a,b} = \eta_{ac}\delta^c_b \end{aligned} \tag{4.3}$$

and therefore

$$R^A R^B \nu^{(2)}_{AB} = c_1 R^{ab} R^{cd}\varepsilon_{abcd} + c_2 R^{ab} R^{ab} + c_3 R^a R^a \tag{4.4}$$

The first two terms are closed forms:

$$\begin{aligned} &d(R^{ab}R^{cd}\varepsilon_{abcd}) = \mathcal{D}(R^{ab}R^{cd}\varepsilon_{abcd}) = 0 \\ &d(R^{ab}R^{ab}) = \mathcal{D}(R^{ab}R^{ab}) = 0 \end{aligned} \tag{4.5}$$

because of Lorentz invariance and of the second Bianchi identity in (3.6). The two terms are then locally exact, indeed:

$$\begin{aligned} &R^{ab}R^{cd}\varepsilon_{abcd} = d(\varepsilon_{abcd}\ \omega^{ab}R^{cd} - \varepsilon_{abcd}\ \omega^{al}\omega^{lb}\omega^{cd}) \\ &R^{ab}R^{ab} = d(\omega^{ab}R^{ab} - \frac{1}{3}\omega^{la}\omega^{lm}\omega^{ma}) \end{aligned} \tag{4.6}$$

Their spacetime integrals give topological numbers, respectively the first Pontriagyn number and the Euler characteristic of the 4- dimensional spacetime manifold. For the last term in (4.4) we have:

$$\begin{aligned} R^a R^a = \mathcal{D}V^a\ \mathcal{D}V^a =&\mathcal{D}(V^a\mathcal{D}V^a) + V^a\mathcal{D}R^a = \\ =&d(V^a\mathcal{D}V^a) + V^a(-R^{ab}V^b) \end{aligned} \tag{4.7}$$

and therefore can be reduced to a term linear in R^{ab} plus a total derivative. Thus the most general lagrangian is at most linear in the curvatures:

$$\mathcal{L} = \alpha\varepsilon_{abcd}\ V^aV^bV^cV^d + \beta\varepsilon_{abcd}R^{ab}V^cV^d + \gamma R^{ab}V^aV^b \tag{4.8}$$

Note that ω^{ab} can appear only through the $SO(1,3)$- covariant curvature R^{ab} (local Lorentz invariance).

A simple scaling consideration allows us to discard the first term (cosmological term) in (4.8): the curvature definitions (3.5) and the Bianchi identities (3.6) are invariant under the rescaling:

$$V^a \rightarrow \lambda V^a; R^a \rightarrow \lambda R^a \tag{4.9}$$

A theory based on the algebraic structure encoded in (3.5-6) must have the same rescaling invariance. In other words, the lagrangian must scale homogeneously under (4.9). The β and γ terms scale as λ^2, whereas the cosmological term scales as λ^4 and has therefore to be dropped (alone it would lead to the drastically simple field equations $V^a = 0$). Note that the same scaling argument could have been used for the first two terms in (4.4).

This argument does not mean that we cannot describe gravity with a cosmological constant. This can be done starting from a different Lie group, namely the de Sitter $SO(1,4)$ or anti de-Sitter $SO(2,3)$ group.

Another criterion we may use in building lagrangians is the requirement that the vacuum (defined by $R^A = 0$) be a solution of the variational equations.* This again rules out the cosmological term in Poincaré gravity (but not in (anti) de-Sitter gravity).

The last two terms in (4.8) have opposite parity and cannot coexist in the same lagrangian. The field equations from the γ term, obtained by varying in V^a and ω^{ab}, read:

$$\begin{aligned} &R^{ab}V^a = 0 \\ &R^aV^b - R^bV^a = 0 \end{aligned} \tag{4.10}$$

and are identically satisfied by $R^a = 0$. The curvature R^{ab} remains free, since the first equation in (4.10) is simply equivalent to the first Bianchi identity in (3.6) when $R^a = 0$. This choice does not lead to any dynamics.

We are thus left with the Cartan-Einstein action:

$$\mathcal{A} = \int_{M^4} R^{ab}V^cV^d\varepsilon_{abcd} \tag{4.11}$$

where the integration is on Minkowski spacetime $M^4 \subset$ [smooth deformation of the Poincaré group]. By excluding bare ω^{ab} in the lagrangian we have ensured the horizon-

* We can justify this as follows. Suppose that the vacuum is not a solution. Then we have two cases: either there are no solutions at all or there is a solution of the type $R^A = \frac{1}{2}F^A_{BC}\mu^B\mu^C$ where F^A_{BC} are constants. Indeed the field equations are algebraic equations for R^A_{BC} (with constant coefficients: cfr. for example the field equations $\frac{\delta\mathcal{L}}{\delta\mu^A}$ derived from (4.1,2)) and therefore either have no solution or have also constant solutions. Bringing $\frac{1}{2}F^A_{BC}\mu^B\mu^C$ to the left hand side of the previous equation we see that the constant curvature solution is given by the zero curvature of a new group whose structure constants are $C^A_{BC} - F^A_{BC}$. These satisfy Jacobi identities (to see this substitute in the Bianchi identity (3.3) for R^A its value $\frac{1}{2}F^A_{BC}\mu^B\mu^C$, and use the definition (3.2) to eliminate the derivative $d\mu^A$) and are therefore "bona fide" structure constants. Then we could consider the lagrangian based on this new group, whose field equations would now admit the zero curvature solution. Hence there is no loss of generality in requiring $R^A = 0$ to be a solution of the variational equations.

tality of the curvatures in the Lorentz directions. This will be verified in the variational equations for the action (4.11).

Contact with the usual Einstein action is made as follows:

$$\begin{aligned}\mathcal{L} &= R^{ab}(\omega)V^cV^d\varepsilon_{abcd} = R^{ab}{}_{ef}V^eV^fV^cV^d\varepsilon_{abcd} = \\ &= R^{ab}{}_{ef}V_\mu{}^eV_\nu{}^fV_\rho{}^cV_\sigma{}^d\varepsilon_{abcd}\,dx^\mu dx^\nu dx^\rho dx^\sigma = R^{ab}{}_{ef}V_\mu{}^eV_\nu{}^fV_\rho{}^cV_\sigma{}^d\varepsilon_{abcd}\,\varepsilon^{\mu\nu\rho\sigma}d^4x = \\ &= R^{ab}{}_{ef}\,\varepsilon^{efcd}\,\varepsilon_{abcd}\,detV d^4x = -4R^{ab}{}_{ab}\sqrt{-g}d^4x\end{aligned} \tag{4.12}$$

where we have used the horizontality of the Lorentz curvature.

Variational equations

$$\frac{\delta\mathcal{A}}{\delta V^a} = 0 \Rightarrow R^{ab}V^c\varepsilon_{abcd} = 0 \tag{4.13a}$$

$$\frac{\delta\mathcal{A}}{\delta\omega^{ab}} = 0 \Rightarrow R^aV^b\varepsilon_{abcd} = 0 \tag{4.13b}$$

To examine their content, first expand the curvatures $R^A = (R^a, R^{ab})$ on the complete basis of 2-forms $\mu^A \wedge \mu^B$:

$$R^A = R^A{}_{BC}\mu^B\mu^C = R^A{}_{ab}V^aV^b + R^A{}_{a,bc}V^a\omega^{bc} + R^A{}_{ab,cd}\,\omega^{ab}\omega^{cd} \tag{4.14}$$

Projecting the 3-form equations (4.13a) on independent components of the complete basis $\mu^A \wedge \mu^B \wedge \mu^C$ we find the three equations:

$$R^{ab}{}_{ef}\;V^eV^fV^c\varepsilon_{abcd} = 0 \tag{4.15a}$$

$$R^{ab}{}_{e,fg}\;V^e\omega^{fg}V^c\varepsilon_{abcd} = 0 \tag{4.15b}$$

$$R^{ab}{}_{ef,gh}\;\omega^{ef}\omega^{gh}V^c\varepsilon_{abcd} = 0 \tag{4.15c}$$

From the first equation, after setting

$$V^eV^fV^c \equiv \varepsilon^{efcg}\Omega_g \tag{4.16}$$

we retrieve the Einstein equations:

$$R^{ab}{}_{ef}\varepsilon^{efcg}\varepsilon_{abcd} = -3!\;\delta^{efg}_{abd}R^{ab}{}_{ef} = 0 \Rightarrow \quad R^{ac}{}_{bc} - \frac{1}{2}\delta^a_bR^{cd}{}_{cd} = 0 \tag{4.17}$$

The other two equations (4.15b,c) imply the horizontality of R^{ab}, as anticipated:

$$R^{ab}{}_{e,fg} = R^{ab}{}_{cd,ef} = 0 \tag{4.18}$$

With similar considerations we find that the torsion equation (4.13b) yields horizontality conditions on R^a, and $R^a{}_{bc} = 0$. These constraints, arising as field equations, are summarized by the zero-torsion condition:

$$R^a = 0 \tag{4.19}$$

i.e. the torsion vanishes as a two-form on the whole soft group manifold. Eq. (4.19) can be solved for the spin connection in terms of the vielbein, its first derivatives and its inverse:

$$\omega_\mu{}^{ab} = (q_{\lambda|\mu\nu} + q_{\nu|\lambda\mu} - q_{\mu|\nu\lambda}) V_c{}^\lambda V_d{}^\nu \eta^{ac} \eta^{bd} \tag{4.20}$$

where

$$q_{\lambda|\mu\nu} = V_\lambda{}^a \partial_{[\mu} V_{\nu]}{}^b \eta_{ab} \tag{4.21}$$

(expand $R^a = dV^a - \omega^{ab} V^b = 0$ in the coordinate basis $dx^\mu \wedge dx^\nu$, multiply by a vielbein $\eta_{ac} V_\rho{}^c$ and sum cyclic permutations in the curved indices with signs $++-$).

Inserting ω^{ab} as given in (4.20) into the Einstein equation (4.17) which is of first order in derivatives of ω^{ab}, we obtain a second order equation for the vielbein field. The conclusion is that starting from the Cartan-Einstein action (4.11), the propagation of the vielbein field is obtained via the torsion mechanism $R^a = 0$, allowing the elimination of the spin connection in terms of $V_\mu{}^a$, the only physical (propagating) field.

Symmetries

The symmetries of Poincaré gravity are given by the diffeomorphisms on the Poincaré (soft) group manifold. Applying the general formula (3.11) with

$$\vec{\delta y} = \varepsilon^a \partial_a + \varepsilon^{ab} \partial_{ab}$$

we find

$$\delta_\varepsilon V^a = (\nabla\varepsilon)^a + i_\varepsilon R^a = \mathcal{D}\varepsilon^a + \varepsilon^{ab} V^b \tag{4.22a}$$

$$\delta_\varepsilon \omega^{ab} = (\nabla\varepsilon)^{ab} + i_\varepsilon R^{ab} = \mathcal{D}\varepsilon^{ab} + 2\varepsilon^c V^d R^{ab}{}_{cd} \tag{4.22b}$$

We have used $R^a = 0$ in the first equation. The reader may verify as an exercise that the variation of V^a in (4.22a) induces precisely the variation (4.22b) for the spin connection given by eq. (4.20) in terms of the vierbein field.

Diffeomorphisms along Lorentz directions ($\varepsilon^a = 0$) become local Lorentz rotations gauged by the spin connection. Diffeomorphisms along the translations ($\varepsilon^{ab} = 0$) give the usual variations of V^a and ω^{ab} under general coordinate transformations. Actually for the vierbein field this is true modulo a field dependent Lorentz rotation. Indeed from eq. (4.22a) we find the variation of the vierbein components:

$$\delta_\varepsilon V_\mu{}^a = \mathcal{D}_\mu(V_\nu{}^a\varepsilon^\nu) = (\delta^a_b\partial_\mu + \omega_\mu{}^{ab})(V_\nu{}^b\varepsilon^\nu) =$$
$$= (\mathcal{D}_\mu V_\nu{}^a)\varepsilon^\nu + V_\nu{}^a\partial_\mu\varepsilon_\nu = (\mathcal{D}_\mu V_\nu{}^a - \mathcal{D}_\nu V_\mu{}^a)\varepsilon^\nu + \mathcal{D}_\nu V_\mu{}^a\varepsilon^\nu + V_\nu{}^a\partial_\mu\varepsilon^\nu = \quad (4.23)$$
$$= (\mathcal{D}_\nu V_\mu{}^a)\varepsilon^\nu + V_\nu{}^a\partial_\mu\varepsilon^\nu = (\partial_\nu V_\mu{}^a)\varepsilon^\nu + V_\nu{}^a\partial_\mu\varepsilon^\nu + \omega_\nu{}^{ab}V_\mu{}^b\varepsilon^\nu$$

(use $\mathcal{D}_\mu V_\nu{}^a - \mathcal{D}_\nu V_\mu{}^a = 0$ because of the zero-torsion condition $R^a = \mathcal{D}V^a = 0$. Note that the final expression for $\delta_\varepsilon V_\mu{}^a$ does not change when $R^a \neq 0$: then the covariant curl of $V_\mu{}^a$ cancels with the R^a term in (4.22a)). This reproduces the usual transformation law plus a Lorentz rotation with field dependent parameter $\varepsilon^\nu\omega_\nu{}^{ab}$. Since the theory is separately invariant under local Lorentz rotations, the usual transformation of $V_\mu{}^a$ is a symmetry, as it should.

5. D=4, N=1 supergravity

This section supersymmetrizes the previous one. $D = 4, N = 1$ supergravity is based on the superPoincaré Lie algebra:

$$[P_a, P_b] = 0 \quad (5.1)$$
$$[M_{ab}, M_{cd}] = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} \quad (5.2)$$
$$[M_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b \quad (5.3)$$
$$[M_{ab}, \bar{Q}_\beta] = \frac{1}{4}\bar{Q}_\alpha(\gamma_{ab})_{\alpha\beta} \quad (5.4)$$
$$\{\bar{Q}_\alpha, \bar{Q}_\beta\} = i(C\gamma^a)_{\alpha\beta}P_a \quad (5.5)$$

where the supersymmetry charge $\bar{Q}_\alpha \equiv Q^\dagger_\beta(\gamma_0)_{\beta\alpha}$ is a Majorana fermion:

$$\bar{Q} = Q^T C, \quad C = \text{charge conjugation matrix} \quad (5.6)$$

Greek indices are spinor indices.

We write the Lie-valued one-form of eq. (2.14) as

$$\sigma(y) = \sigma^A T_A = \frac{1}{2}\omega^{ab}M_{ab} + V^a P_a + \bar{Q}\psi \quad (5.7)$$

and we use the notation (V^a=vierbein, ω^{ab}=spin connection, ψ^α=gravitino) also for the vielbein μ^A of the *soft* superPoincaré manifold. Then the curvature definitions (3.2) take the form

$$R^a = dV^a - \omega^{ab}V^b - \frac{i}{2}\bar{\psi}\gamma^a\psi \quad (5.8a)$$
$$R^{ab} = d\omega^{ab} - \omega^{ac}\omega^{ca} \quad (5.8b)$$
$$\rho = d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi \equiv \mathcal{D}\psi \quad (5.8c)$$

and the corresponding Bianchi identities are

$$\mathcal{D}R^a + R^{ab}V^b - i\bar{\psi}\gamma^a\rho = 0 \tag{5.9a}$$
$$\mathcal{D}R^{ab} = 0 \tag{5.9b}$$
$$\mathcal{D}\rho + \frac{1}{4}R^{ab}\gamma_{ab}\psi = 0 \tag{5.9c}$$

Eq.s (5.8,9) are invariant under the rescaling:

$$V^a \to \lambda V^a;\ \psi \to \sqrt{\lambda}\psi\ \ (\Rightarrow R^a \to \lambda R^a;\ \rho \to \sqrt{\lambda}\rho) \tag{5.10}$$

The fields depend in principle on the coordinates x^a, x^{ab} and θ^α (this last being a fermionic coordinate corresponding to the supersymmetry generator Q_α) of the soft superPoincaré group manifold. Horizontality of the curvatures in the Lorentz directions will remove the x^{ab} dependence, as in Poincaré gravity, and ensure local Lorentz invariance. How about the dependence on θ ? This translates into new degrees of freedom when expanding $\mu^A(x,\theta)$ in series of the anticommuting θ^α.

Notice that here we do not need *superfields*, i.e. fields living in the superspace (x^a, θ^α), whose expansion in the anticommuting θ yields a supersymmetric multiplet (for ex. the vielbein and the gravitino). Indeed we already have both the vielbein and the gravitino as part of the same superPoincaré vielbein ! The dependence on θ is therefore redundant in our framework and we must find a way to dispose of it.

This is done by a mechanism which resembles horizontality, but is a weaker requirement on the curvatures, called "rheonomy". It simply consists in having curvatures with outer components (i.e. components in the θ directions) expressible as linear combinations of inner (or spacetime) components:

$$R^A_{\ \alpha\Lambda} = C^A_{\ \alpha\Lambda}|^{\mu\nu}_B R^B_{\ \mu\nu} \tag{5.11}$$

If this happens, the purely spacetime configuration $(\mu_\mu{}^A(x,0),\ \partial_\mu\mu_\nu{}^A(x,0))$ *determines* the extension of $\mu_\nu{}^A(x,0)dx^\nu$ to the whole superspace:

$$\mu^A(x,0) \equiv \mu_\nu{}^A(x,0)dx^\nu \longrightarrow \mu_\nu{}^A(x,\theta)dx^\nu + \mu_\alpha{}^A(x,\theta)d\theta^\alpha \tag{5.12}$$

Proof: consider formula (3.11) for a diffeomorphism $\theta \to \theta + \varepsilon(x,\theta)$:

$$\begin{aligned}\mu^A(x,\theta=\varepsilon) &= \mu^A(x,0) + (\nabla\varepsilon)^A + 2\bar{\varepsilon}^\alpha R^A_{\ \alpha\Lambda}(x,0)dy^\Lambda = \\ &= \mu^A(x,0) + \delta^A_\alpha\partial_\Lambda\varepsilon^\alpha dy^\Lambda + C^A_{\ B\alpha}\mu^B(x,0)\varepsilon^\alpha + 2\bar{\varepsilon}^\alpha C^A_{\ \alpha\Lambda}|^{\mu\nu}_B R^B_{\ \mu\nu}(x,0)dy^\Lambda\end{aligned} \tag{5.13}$$

where we have used the rheonomy condition (5.10), and $dy^\Lambda = (dx^\mu, d\theta^\alpha)$. Since the spacetime components of the curvature are given by

$$R^A_{\mu\nu} \equiv \partial_{[\mu}\mu_{\nu]}{}^A + \frac{1}{2}C^A{}_{BC}\mu_\mu{}^B\mu_\nu{}^C \tag{5.14}$$

we see that , by integrating eq. (5.13), we can reconstruct the whole superspace vielbein from the knowledge of its spacetime restriction $\mu_\mu{}^A(x,0)$ and the spacetime derivatives $\partial_\mu\mu_\nu{}^A(x,0)$ appearing in (5.14).

This is the solution to our problem: the unwanted extra degrees of freedom in μ^A due to its θ dependence (i.e. the fields appearing in the θ and $d\theta$ expansion of $\mu^A(x,\theta)$) are not really independent and all the physical information resides in the spacetime field $\mu_\mu{}^A(x,0)$. If eq. (5.10) (rheonomy) holds, we can rewrite the transformation (5.13) as

$$\delta_\varepsilon\mu^A(x,0) = \delta^A_\alpha\partial_\mu\varepsilon^\alpha dx^\mu + C^A_{B\alpha}\mu^B(x,0)\varepsilon^\alpha + 2\bar{\varepsilon}^\alpha C^A_{\alpha D}|^{\mu\nu}_B R^B{}_{\mu\nu}(x,0)\mu^D(x,0) \tag{5.15a}$$

or in components:

$$\delta_\varepsilon\mu_\rho{}^A(x,0) = \delta^A_\alpha\partial_\rho\varepsilon^\alpha + C^A_{B\alpha}\mu_\rho{}^B(x,0)\varepsilon^\alpha + 2\bar{\varepsilon}^\alpha C^A_{\alpha D}|^{\mu\nu}_B R^B_{\mu\nu}(x,0)\mu_\rho{}^D(x,0) \tag{5.15b}$$

and consider it as a symmetry between purely spacetime fields $\mu_\mu{}^A(x,0)$ (supersymmetry), with $\varepsilon(x)^\alpha$ as fermionic infinitesimal parameter. By construction the soft group manifold action is invariant under the supersymmetry (5.15), since it is invariant under the superspace diffeomorphisms (5.13) reducing to (5.15) when the curvatures are rheonomic. Then, if we restrict this action on spacetime (setting $\theta = 0$, $d\theta = 0$ in the lagrangian D-form, and integrating on spacetime), the supersymmetry variation (5.15) is still a symmetry, transforming spacetime fields into spacetime fields. This is how we arrive at spacetime supersymmetric actions.

Note 1: the rheonomy property (5.10) does not depend on the particular basis chosen for the 1-forms. The indices α, Λ in (5.10) are curved indices, but an analogous relation holds for the flat components (i.e. the components along the vielbein basis μ^A).

Note 2: there is an interesting analogy between analiticity and rheonomy:

$$\text{vielbein} \leftrightarrow \text{analytic function}$$

$$\text{superspace} \leftrightarrow \text{complex plane}$$

$$\text{spacetime} \leftrightarrow \text{real line}$$

$$\text{Cauchy} - \text{Riemann conditions} \leftrightarrow \text{rheonomy conditions}$$

so that rheonomy can be seen as a kind of analiticity in superspace.

Let us now return to our specific example of N=1 supergravity in D=4. The most general lagrangian 4-form is of the type (4.1). We have five possible terms quadratic in the curvatures:

$$R^A R^B \nu^{(2)}_{AB} = c_1 R^{ab} R^{cd} \varepsilon_{abcd} + c_2 R^{ab} R^{ab} + c_3 R^a R^a + c_4 \bar{\rho}\rho + c_5 \bar{\rho}\gamma_5\rho \tag{5.16}$$

The first two are total derivatives as in the gravity case (4.6), since the definition of R^{ab} is the same. The last three can be reduced to linear terms in the curvatures plus total derivatives. Actually the scaling invariance of (5.10) eliminates all the terms in (5.16) except $R^a R^a$, since the Einstein term (4.12) scales as λ^2. The torsion-squared term can be reduced to a linear term since

$$R^a R^a = (\mathcal{D}V^a - \frac{i}{2}\bar{\psi}\gamma^a\psi)R^a = d(V^a R^a) + V^a(-R^{ab}V^b + i\bar{\psi}\gamma^a\rho) - \frac{i}{2}\bar{\psi}\gamma^a\psi R^a \tag{5.17}$$

in virtue of the Bianchi identity (5.9a). This leaves us with a lagrangian of the form:

$$\mathcal{L} = \Lambda + \nu_{ab} R^{ab} + \nu_a R^a + \bar{\nu}\rho \tag{5.18}$$

where

$$\begin{aligned}
&\Lambda = \alpha_1 \varepsilon_{abcd} V^a V^b V^c V^d + i\alpha_2 \varepsilon_{abcd} \bar{\psi}\gamma^{ab}\psi V^c V^d + i\alpha_3 \bar{\psi}\gamma^{ab}\psi V^a V^b \\
&\nu_{ab} = \beta_1 \varepsilon_{abcd} V^c V^d + \beta_2 V^a V^b + i\beta_3 \bar{\psi}\gamma_{ab}\psi + i\beta_4 \varepsilon_{abcd}\bar{\psi}\gamma^{cd}\psi \\
&\nu_a = i\eta_1 \bar{\psi}\gamma_a\psi \\
&\nu = \delta_1 \gamma_5\gamma_a\psi V^a + i\delta_2 \gamma_a \psi V^a
\end{aligned} \tag{5.19}$$

are the most general Lorentz covariant terms. Notice that the only nonvanishing $\psi\psi$ currents are $\bar{\psi}\gamma^a\psi$ and $\bar{\psi}\gamma^{ab}\psi$ (ψ^t and ψ commute since they are fermionic *one-forms*). Correct λ^2 scaling of $\mathcal{L}$ under (5.10) drastically reduces the possible terms: $\alpha_1 = \alpha_2 = \alpha_3 = \beta_3 = \beta_4 = 0$. Moreover parity conservation implies $\beta_2 = \eta_1 = \delta_2 = 0$ (all terms must have the same parity as the Einstein term $R^{ab}V^cV^d\varepsilon_{abcd}$, i.e. must be pseudoscalars). Thus we finally have:

$$\mathcal{L} = \beta_1 \varepsilon_{abcd} R^{ab} V^c V^d + \delta_1 \bar{\psi}\gamma_5\gamma_a \rho V^a \tag{5.20}$$

The requirement that the vacuum be a solution of the field equations fixes the last parameter $a = \delta_1/\beta_1$. Indeed the field equations obtained by varying (5.20) in the fields V^a, ω^{ab}, ψ are respectively:

$$2R^{ab}V^c\varepsilon_{abcd} + a\bar{\psi}\gamma_5\gamma_d\rho = 0 \tag{5.21a}$$

$$2\mathcal{D}V^cV^d\varepsilon_{abcd} + \frac{1}{4}a\bar{\psi}\gamma_5\gamma_d\gamma_{ab}\psi V^d = 0 \tag{5.21b}$$

$$2a\gamma_5\gamma_a\rho V^a - a\gamma_5\gamma_a\psi R^a = 0 \tag{5.21c}$$

To find the first is immediate; for the second we only have to recall that varying ω^{ab} in R^{ab} yields $\delta R^{ab} = \mathcal{D}(\delta\omega^{ab})$, and that by integrating by parts the Lorentz covariant derivative $\mathcal{D}$ can be transferred on V^a. Finally for the gravitino variation we have

$$\begin{aligned}\frac{1}{a}\delta\mathcal{L} &= a(\delta\bar{\psi})\gamma_5\gamma_a\mathcal{D}\psi V^a + a\bar{\psi}\gamma_5\gamma_a\mathcal{D}(\delta\psi)V^a = \\ &= (\delta\bar{\psi})\gamma_5\gamma_a\mathcal{D}\psi V^a + \bar{\psi}\gamma_5\gamma_a\delta\psi\mathcal{D}V^a + \delta\bar{\psi}\gamma_5\gamma_a\mathcal{D}\psi V^a = \\ &= (\delta\bar{\psi})\gamma_5\gamma_a\mathcal{D}\psi V^a - \delta\bar{\psi}\gamma_5\gamma_a\psi(R^a + \frac{i}{2}\bar{\psi}\gamma^a\psi) = \\ &= (\delta\bar{\psi})(2\gamma_5\gamma_a\mathcal{D}\psi V^a - \gamma_5\gamma_a\psi R^a)\end{aligned} \tag{5.22}$$

in virtue of $\bar{\psi}\gamma_5\gamma_a(\delta\psi) = (\delta\bar{\psi})\gamma_5\gamma_a\psi$ and the Fierz identity

$$\gamma_a\psi\bar{\psi}\gamma^a\psi = 0 \tag{5.23}$$

Note that using the gamma-algebra identity:

$$\gamma_5\gamma_d\gamma_{ab} = 2\gamma_5\delta_{d[a}\gamma_{b]} - i\varepsilon_{abcd}\gamma^c \tag{5.24}$$

the variational equation (5.21b) can be recast in the form:

$$2R^cV^d\varepsilon_{abcd} + \frac{1}{4}(a-4)\bar{\psi}\gamma_5\gamma_d\gamma_{ab}\psi V^d = 0 \tag{5.25}$$

so that the vacuum, defined by vanishing curvatures, is a solution of the field equations (5.21) only if $a = 4$.

Analysis of the field equations

Let us find what are the constraints on the curvatures due to eq.s (5.21) with $a = 4$. For short, we refer to these equations as a), b) and c), respectively. We expand the curvatures on a complete basis of 2-forms:

$$R^a = R^a{}_{bc}V^bV^c + \bar{\theta}^a{}_c\psi V^c + \bar{\psi}K^a\psi \tag{5.26a}$$
$$R^{ab} = R^{ab}{}_{cd}V^cV^d + \bar{\theta}^{ab}{}_c\psi V^c + \bar{\psi}K^{ab}\psi \tag{5.26b}$$
$$\rho = \rho_{ab}V^aV^b + H_c\psi V^c + \Omega_{\alpha\beta}\psi^\alpha\psi^\beta \tag{5.26c}$$

where $\theta^{ab}{}_c$, $\theta^a{}_c$ are spinor-tensors, $K^{ab} = -K^{ba}$, K^a and H_c are 4×4 matrices in spinor space, and the 4×4 matrix $\Omega_{\alpha\beta}$ is a Majorana spinor. Inserting the parametrizations (5.26) for the curvatures into the field equations a),b) and c) yields in the various sectors:

$\psi\psi\psi$ sector:
a) $\Rightarrow$ $\Omega_{\alpha\beta} = 0$
c) $\Rightarrow$ $K^a = 0$

$\psi\psi V$ sector:
a) $\Rightarrow\ 2\bar{\psi}K^{ab}\psi V^c\varepsilon_{abcd} + 4\bar{\psi}\gamma_5\gamma_d H_c\psi V^c = 0$
b) $\Rightarrow\ 0 = 0$
c) $\Rightarrow\ \bar{\theta}^a{}_c = 0$

ψVV sector:
a) $\Rightarrow\ 2\bar{\theta}^{ab}{}_e\psi V^e V^c\varepsilon_{abcd} + 4\bar{\psi}\gamma_5\gamma_d\rho_{ab}V^a V^b = 0$
b) $\Rightarrow\ 0 = 0$
c) $\Rightarrow\ \gamma_5\gamma_a H_b\psi V^b V^a - 4\gamma_5\gamma_c\psi R^c{}_{ab}V^a V^b = 0$

VVV sector:
a) $\Rightarrow\ R^a{}_{bc} = 0$
b) $\Rightarrow\ R^{ac}{}_{bc} - \frac{1}{2}\delta^a_b R^{cd}{}_{cd} = 0$
c) $\Rightarrow\ \gamma_5\gamma_{ab}\psi_c\varepsilon^{abcd} = 0$

The last VVV sector contains the propagation equations (Einstein and Rarita-Schwinger equations). The other sectors determine the "outer" components of the curvatures. Indeed using $R^a{}_{bc} = 0$ into the last of ψVV eq.s yields $H_c = 0$, which inserted into the first of the $\psi\psi V$ eq.s implies $K^{ab} = 0$. The only nontrivial relation is the first of the ψVV sector:

$$(\bar{\theta}^{ab}{}_e\varepsilon^{ecgh}\varepsilon_{abcd} + 2\bar{\rho}_{ab}\gamma_5\gamma_d\varepsilon^{abgh})\psi\Omega_{gh} = 0 \tag{5.27}$$

where we used for the area element $V^a V^b = \varepsilon^{abcd}\Omega_{cd}$. Carrying out the ε contractions leads to:

$$\bar{\theta}^{ab}{}_d + 2\delta^{[a}_d\bar{\theta}^{b]c}{}_c + \varepsilon^{abce}\bar{\rho}_{ce}\gamma_5\gamma_d = 0 \tag{5.28}$$

Contracting in the indices b and d yields $\theta^{ac}{}_c = \frac{1}{2}\varepsilon^{abcd}\bar{\rho}_{cd}\gamma_5\gamma_b$, which substituted into (5.28) finally gives:

$$\bar{\theta}^{ab}{}_e = -\varepsilon^{abcd}\bar{\rho}_{cd}\gamma_5\gamma_e - \delta^{[a}_e\varepsilon^{b]cdf}\bar{\rho}_{df}\gamma_5\gamma_c \tag{5.29}$$

In summary, the field equations deduced from the supergravity lagrangian

$$\mathcal{L} = \varepsilon_{abcd}R^{ab}V^c V^d + 4\bar{\psi}\gamma_5\gamma_a\rho V^a \tag{5.30}$$

have determined the curvatures:

$$R^{ab} = R^{ab}{}_{cd}V^c V^d + (-\varepsilon^{abcd}\bar{\rho}_{cd}\gamma_5\gamma_e - \delta^{[a}_e\varepsilon^{b]cdf}\bar{\rho}_{df}\gamma_5\gamma_c)\psi V^e \tag{5.31a}$$

$$R^a = 0 \tag{5.31b}$$

$$\rho = \rho_{ab}V^{ab} \tag{5.31c}$$

where the spacetime components $R^{ab}{}_{cd}$ and ρ_{ab} satisfy the propagation equations found in the VVV sector. As we see from eq.s (5.31) the curvatures are rheonomic, i.e. their outer components (the only nonvanishing outer components are those of R^{ab}) are expressed in terms of inner ones (here in terms of ρ_{ab}). Then the θ diffeomorphism invariance of the superspace lagrangian (5.30) become spacetime supersymmetry of its spacetime restriction, as explained in the previous paragraphs. Inserting the curvatures given in eq.s (5.31) into the general formula (3.11), with $\vec{\delta y} = \varepsilon$ a generic tangent vector of the soft superPoincaré group manifold

$$\varepsilon = \bar{\varepsilon}^{\alpha}\partial_{\alpha} + \varepsilon^{a}\partial_{a} + \varepsilon^{ab}\partial_{ab} \tag{5.32}$$

we can deduce the transformation laws:

$$\delta_{\varepsilon}V^{a} = (\nabla\varepsilon)^{a} = \mathcal{D}\varepsilon^{a} + \varepsilon^{ab}V^{b} + i\bar{\varepsilon}\gamma^{a}\psi \tag{5.33a}$$

$$\delta_{\varepsilon}\omega^{ab} = (\nabla\varepsilon)^{ab} + i_{\varepsilon}R^{ab} = \mathcal{D}\varepsilon^{ab} + \varepsilon^{ac}\omega^{cb} + \varepsilon^{bc}\omega^{ac} + 2\varepsilon^{c}V^{d}R^{ab}{}_{cd} + 2\bar{\theta}^{ab}{}_{c}\psi\varepsilon^{c} + 2\bar{\theta}^{ab}{}_{c}\varepsilon V^{c} \tag{5.33b}$$

$$\delta_{\varepsilon}\psi = \nabla\psi + i_{\varepsilon}\rho = \mathcal{D}\varepsilon + \frac{1}{4}\varepsilon^{ab}\gamma_{ab}\psi + 2\varepsilon^{a}\rho_{ab}V^{b} \tag{5.33c}$$

with $\bar{\theta}^{ab}{}_{c}$ given by (5.29). On the spacetime restriction of the D=4 N=1 surgravity lagrangian, the above transformations are interpreted as ordinary D=4 diffeomorphisms with parameter ε^{a}, supersymmetry variations with parameter ε and local Lorentz rotations with parameter ε^{ab}.

These symmetries are really *on shell symmetries* of the supergravity action, since in deducing the transformations (5.33) we used the the curvatures given in eq.s (5.30), obtained via the field equations. Of course, one could use the general parametrization of the curvatures (5.26) instead, and the resulting variations obtained through formula (3.11) would then be totally off-shell, the price being the introduction of extra auxiliary fields (the outer components of the curvatures).

Note: The Einstein equation

$$R^{ac}{}_{bc}(\omega) - \frac{1}{2}\delta^{a}_{b}R^{cd}{}_{cd}(\omega) = 0, \tag{5.34}$$

deduced from (5.25) with $a = 4$, is formally the same as in ordinary gravity. However, the spin connection ω is different. To find it, we solve the zero-torsion constraint (5.31b) as we did for ordinary gravity, and find:

$$\omega_{\mu}{}^{ab} = \overset{\circ}{\omega}_{\mu}{}^{ab} + \Delta\omega_{\mu}{}^{ab} \tag{5.35}$$

where $\overset{\circ}{\omega}{}^{ab}$ is the usual Riemannian spin connection (4.20) and

$$\Delta\omega_{\mu}{}^{ab} = \frac{i}{4}(\bar{\psi}_{\mu}\gamma_{\lambda}\psi_{\nu} + \bar{\psi}_{\lambda}\gamma_{\nu}\psi_{\mu} - \bar{\psi}_{\nu}\gamma_{\mu}\psi_{\lambda} - (\lambda \leftrightarrow \nu))V_{c}{}^{\lambda}V_{d}{}^{\nu}\eta^{ac}\eta^{bd} \tag{5.36}$$

Substituting (5.35) in the supergravity curvatures:

$$R^{ab}(\omega) = R^{ab}(\mathring{\omega} + \Delta\omega) = R^{ab}(\mathring{\omega}) + \mathcal{D}(\mathring{\omega})\Delta\omega^{ab} - \Delta\omega^{ac}\Delta\omega^{cd} \qquad (5.37a)$$

$$\rho = \mathcal{D}(\omega)\psi = \mathcal{D}(\mathring{\omega} + \Delta\omega) = \mathcal{D}(\mathring{\omega})\psi - \frac{1}{4}\Delta\omega^{ab}\gamma_{ab}\psi \qquad (5.37b)$$

the spacetime field equations read:

$$R^{ac}{}_{bc}(\mathring{\omega}) - \frac{1}{2}\delta^a_b R^{cd}{}_{cd}(\mathring{\omega}) + T^a_b(\psi) \qquad (5.38a)$$

$$8\gamma_5\gamma_a\mathcal{D}_b(\mathring{\omega})\psi_c\varepsilon^{abcd} = 2\gamma_5\gamma_a\Delta\omega^{ef}{}_b\gamma_{ef}\psi_c\varepsilon^{abcd} \qquad (5.38b)$$

where

$$T^a_b(\psi) \equiv \mathcal{D}_{[c}\Delta\omega^{cd}{}_{b]} - \Delta\omega^{cd}{}_{[c}\Delta\omega^{ca}{}_{b]} - \frac{1}{2}[\delta^a_b\mathcal{D}_c(\mathring{\omega})\Delta\omega^{cd}{}_d - \Delta\omega^{cd}{}_{]c}\Delta\omega^{ce}{}_{e[}] \qquad (5.39)$$

is the energy momentum tensor of the gravitino field.

6. Free differential algebras

The dual formulation of Lie algebras provided by the Cartan-Maurer equations (2.12) can be naturally extended to p-forms ($p > 1$):

$$d\theta^i_{(p)} + \sum \frac{1}{n} C^i{}_{i_1 \ldots i_n}\theta^{i_1}_{(p_1)} \wedge \ldots \wedge \theta^{i_n}_{(p_n)} = 0, \qquad p+1 = p_1 + \ldots + p_n \qquad (6.1)$$

$p, p_1, \ldots p_n$ are, respectively, the degrees of the forms $\theta^i, \theta^{i_1}, \ldots, \theta^{i_n}$; the indices $i, i_1, \ldots, i_n$ run on irreps of a group G, and $C^i{}_{i_1 \ldots i_n}$ are generalized structure constants satisfying generalized Jacobi identities due to $d^2 = 0$. When $p = p_1 = p_2 = 1$ and i, i_1, i_2 belong to the adjoint representation of G, eq.s (6.1) reduce to the ordinary Cartan-Maurer equations. The (anti)symmetry properties of the indices $i_1, \ldots i_n$ depend on the bosonic or fermionic character of the forms $\theta^{i_1}, \ldots \theta^{i_n}$

If the generalized Jacobi identities hold, eq.s (6.1) define a *free differential algebra* [2](FDA). The possible FDA extensions G' of a Lie algebra G have been studied in ref.s [2,3], and rely on the existence of Chevalley cohomology classes in G [5]. Suppose that, given an ordinary Lie algebra G, there exists a p-form:

$$\Omega^i{}_{(p)}(\sigma) = \Omega^i{}_{A_1 \ldots A_p}\sigma^{A_1} \wedge \ldots \wedge \sigma^{A_p}, \quad \Omega^i{}_{A_1 \ldots A_p} = \text{constants, i runs on a G} - \text{irrep} \qquad (6.2)$$

which is covariantly closed but not covariantly exact, i.e.

$$\nabla\Omega^i{}_{(p)} \equiv d\Omega^i{}_{(p)} + \sigma^A \wedge D(T_A)^i{}_j \Omega^j{}_{(p)} = 0, \quad \Omega^i{}_{(p)} \neq \nabla\Phi^i{}_{(p-1)} \tag{6.3}$$

Then $\Omega^i{}_{(p)}$ is said to be a representative of a Chevalley cohomology class in the $D^i{}_j$ irrep of G. ∇ is the boundary operator satisfying $\nabla^2 = 0$ (it would be proportional to the curvature 2-form on the *soft* group manifold). The existence of $\Omega^i{}_{(p)}$ allows the extension of the original Lie algebra G to the FDA G':

$$d\sigma^A + \frac{1}{2}C^A{}_{BC}\sigma^B \wedge \sigma^C = 0 \tag{6.4a}$$

$$\nabla\Sigma^i{}_{(p-1)} + \Omega^i_{(p)}(\sigma) = 0 \tag{6.4b}$$

where $\Sigma^i_{(p-1)}$ is a new $p-1$-form, not contained in G. Closure of eq.s (6.4) is ensured because $\nabla\Omega^i_{(p)} = 0$.

It is clear that $\Omega^i_{(p)}$ differing by exact pieces $\nabla\Phi^i_{(p-1)}$ lead to equivalent FDA's, via the redefinition $\Sigma^i_{(p-1)} \to \Sigma^i_{(p-1)} + \Phi^i_{(p-1)}$. What we are interested in are really nontrivial cohomology classes satisfying eq.s (6.3).

The whole game can be repeated on the free differential algebra G' which now contains σ^A, $\Sigma^i_{(p-1)}$. One looks for the existence of polynomials in σ^A, $\Sigma^i_{(p-1)}$

$$\Omega^i_{(q)}(\sigma, \Sigma) = \Omega^i_{A_1 \ldots A_r i_1 \ldots i_s}\sigma^{A_1} \wedge \ldots \wedge \sigma^{A_r} \wedge \Sigma^{i_1}_{(p-1)} \wedge \ldots \wedge \Sigma^{i_s}_{(p-1)} \tag{6.5}$$

satisfying the cohomology conditions (6.3). If such a polynomial exists, the FDA of eq.s (6.4) can be further extended to G'', and so on.

In constructing D-dimensional supergravity theories we usually choose as starting point the superPoincaré Lie algebra, whose dual formulation is given in eq.s (3.4). The possible G' extensions to FDA's depend on the spacetime dimension D. For example in $D = 11$ there is a cohomology class of the superPoincaré algebra in the identity representation:

$$\Omega(V, \omega, \psi) = \frac{1}{2}\bar{\psi}\Gamma^{ab}\psi V^a V^b \tag{6.6}$$

$d\Omega = 0$ holds because of the $D = 11$ Fierz identity

$$\bar{\psi}\Gamma^{ab}\psi \; \bar{\psi}\Gamma^a\psi V^b = 0 \tag{6.7}$$

This allows the extension of the algebra (3.4) by means of a three -form A:

$$dA - \Omega(V, \omega, \psi) = 0 \tag{6.8}$$

Note Only nonsemisimple algebras can have FDA extensions in nontrivial G-irreps. Indeed a theorem by Chevalley and Eilenberg [5] states that there is no nontrivial cohomology class of G in nontrivial G-irreps when G is semisimple.

As we have done in the case of ordinary Lie algebras, we find a dynamical theory based on FDA's by allowing nonvanishing curvatures. This means, for example, that $D = 11$ supergravity is based on a deformation of the fields V, ω, ψ, A such that the superPoincaré curvatures and the A-curvature of (6.8) are different from zero. The construction of the action proceeds along the same lines outlined in section 4, and we refer the reader to refs. [1,3]. Other theories containing higher forms (i.e. antisymmetric tensors) have been interpreted as gaugings of free differential algebras: we refer the reader to ref.s [1,3] for a detailed study of these theories.

7. BRST geometry

In this section we provide a geometric interpretation of BRST symmetry [6]. The basic idea is to enlarge with an extra grassmann coordinate θ the group manifold G of the original theory, be it a gauge, a (super)gravity or a (super)string theory. Adding θ to the spacetime coordinates was already considered in refs. [7-8], and indeed in this way one achieves a *superspace* formulation of BRST invariant theories. Here we want to take a step further, and consider the theory as living on the enlarged *group manifold* G + Q, obtained by adding a fermionic central charge Q to the original group generators T_A, satisfying $Q^2 = 0$. This is, in our opinion, the natural geometric arena of BRST-invariant theories. Also, our formulation is easily extended to describe the BRST structure of theories containing antisymmetric tensors. In this case, the relevant geometry is that of an enlarged free differential algebra FDA + Q.

BRST symmetry is a global fermionic symmetry. Global symmetries are described in our formalism by rigid translations along some group manifold coordinates y^B. Also, the would be gauge potentials associated to the corresponding generators T_B become pure gauge, and thus effectively disappear from the theory, if one imposes $R^B = 0$.

In the case of BRST symmetry, we assume therefore that the Q-curvature R[Q] vanishes. Moreover, in order to remove the θ-dependence in the fields of the theory, we impose the horizontality constraints:

$$R^A{}_{\theta C} = R^A{}_{C\theta} = R^A{}_{\theta\theta} = 0 \tag{7.1}$$

The enlarged Lie algebra we start from is given by the (anti)commutations:

$$\begin{aligned} &[T_A, T_B] = C^C{}_{AB} T_C \\ &[T_A, Q] = 0 \\ &\{Q, Q\} = 0 \end{aligned} \tag{7.2}$$

Using the structure constants of (7.2) in eq. (3.2), we find the curvature definitions:

$$\begin{aligned} &R^A \equiv d\mu^A + \frac{1}{2} C^A{}_{BC} \mu^B \wedge \mu^C \\ &R[Q] \equiv d\mu[Q] \end{aligned} \tag{7.3}$$

where $\mu[Q]$ is the potential corresponding to Q. The Bianchi identities are:

$$\begin{aligned} &dR^A - C^A{}_{BC}\, R^B \wedge \mu^C = 0 \\ &dR[Q] = 0 \end{aligned} \tag{7.4}$$

Note that the horizontality constraints in eq. (7.1) and the "rigidity" constraint

$$R[Q] = 0 \tag{7.5}$$

are consistent with the Bianchi identities (7.4).

Our claim is that the gauging of the extended Lie algebra (7.2), supplemented with the constraints (7.1) and (7.5), yields a BRST-invariant theory. The proof is simple. First we expand the vielbein one-form μ on the basis of differentials $(dy^\alpha, d\theta)$:

$$\mu^A(y,\theta) = \mu^A_\alpha\, dy^\alpha + \mu^A_\theta\, d\theta \equiv \mu^A_\alpha\, dy^\alpha + g^A\, d\theta \tag{7.6a}$$

$$\mu[Q](y,\theta) = \mu[Q]_\alpha\, dy^\alpha + \mu[Q]_\theta\, d\theta \tag{7.6b}$$

$\mu[Q]$ being a pure gauge because of eq. (7.5), we will concentrate on the transformation laws for μ^A. Note that in eq. (7.6a) we have renamed g^A the $d\theta$ component of μ^A. The reason is that the fermionic zero-form g^A will play the role of the ghost field associated to the gauge potential μ^A_α. Thus, gauge fields and ghost fields are parts of the same fundamental field μ^A.

Consider now the general formula (3.11) for all the symmetry transformations of the theory. The coordinate variation δy^A has a flat (adjoint) index A, and can be expressed in terms of coordinate variations with curved indices as:

$$\delta y^A = \delta y^\alpha \mu^A_\alpha + \delta\theta\ \mu^A_\theta \tag{7.7}$$

Let us specialize the variation $(\delta y^\alpha, \delta\theta)$ to describe a rigid translation in the θ direction. Then $\delta y^\alpha = 0$, $\delta\theta =$ constant and eq. (3.11) takes the form:

$$\begin{aligned} \delta\mu^A &= \nabla(\delta\theta\ g^A) = d(\delta\theta\ g^A) + C^A{}_{BC}\mu^B(\delta\theta\ g^C) \\ &= (-dg^A - C^A{}_{BC}\mu^B g^C)\delta\theta \end{aligned} \tag{7.8}$$

where the curvature term drops because of horizontality. Projecting on the differentials dy^α, $d\theta$ yields the BRST transformation laws of the gauge fields μ^A_α and ghost fields g^A:

$$\delta\mu^A_\alpha = -(\nabla_\alpha g^A)\delta\theta \tag{7.9}$$

$$\delta g^A = (-\partial_\theta g^A - C^A{}_{BC} g^B g^C)\delta\theta = -\frac{1}{2} C^A{}_{BC} g^B g^C \delta\theta \tag{7.10}$$

In the last equation we have used the curvature definition (3.2) and the horizontality condition $R^A_{\theta\theta} = 0$ to express $\partial_\theta g^A$ as:

$$\partial_\theta g^A = -\frac{1}{2} C^A{}_{BC} g^B g^C \tag{7.11}$$

This concludes the proof. The theory is BRST invariant, this invariance being on the same conceptual footing as the other invariances of the theory. All of them have the same geometric origin, i.e. are relics of diffeomorphism invariance on the enlarged group manifold $\tilde{G}$+Q, i.e. the soft group manifold associated to the algebra (7.2).

The whole discussion can be straightforwardly extended to the case of free differential algebras. It suffices to enlarge the FDA to FDA+Q. Let us see how this works in a particular case.

The FDA we consider is the simplest extension of a Lie algebra (in the following denoted by FDA1):

$$\begin{aligned} &d\sigma^A + \frac{1}{2} C^A{}_{BC}\, \sigma^B \sigma^C = 0 \\ &dB^i + C^i{}_{Aj} \sigma^A B^j + \frac{1}{6} C^i{}_{ABC} \sigma^A \sigma^B \sigma^C \equiv \nabla B^i + \frac{1}{6} C^i{}_{ABC} \sigma^A \sigma^B \sigma^C = 0 \end{aligned} \tag{7.12}$$

where B^i is a two-form in a representation $D^i{}_j$ of G. The generalized Jacobi identities ($d^2 = 0$), besides the usual ones for $C^A{}_{BC}$, are

$$C^i{}_{Aj} C^j{}_{Bk} - C^i{}_{Bj} C^j{}_{Ak} = C^C{}_{AB} C^i{}_{Ck}, \quad \text{representation condition} \tag{7.13a}$$

$$-\frac{1}{6} C^j{}_{[ABC} C^i{}_{D]j} + \frac{1}{4} C^E{}_{[DA} C^i{}_{BC]E} = 0, \quad 3-\text{cocycle condition} \tag{7.13b}$$

Eq. (7.2a) implies that $(C_A)^i{}_j \equiv C^i{}_{Aj}$ is a matrix representation of G, while eq. (7.2b) is just the statement that $C^i \equiv C^i{}_{ABC} \sigma^A \sigma^B \sigma^C$ is a 3-cocycle, i.e. $\nabla C^i = 0$.

To this algebra we adjoin the central fermionic charge Q and we allow the left hand sides of eq.s (7.12) to be nonvanishing curvatures R^A, R^i respectively, satisfying generalized Bianchi identities:

$$dR^A - C^A{}_{BC}\, R^B \mu^C = 0 \tag{7.14a}$$

$$dR^i - C^i{}_{Aj} R^A B^j + C^i{}_{Aj} \mu^A R^j - \frac{1}{2} C^i{}_{ABC} R^A \mu^B \mu^C = 0 \tag{7.14b}$$

$$dR[Q] = 0 \tag{7.14c}$$

The ghost fields are contained in the expansion of μ^A and B^i along $d\theta$ differentials:

$$\mu^A(y,\theta) = \mu^A_\alpha \, dy^\alpha + g^A \, d\theta \tag{7.15a}$$
$$B^i = B^i{}_{\alpha\beta} dy^\alpha dy^\beta + b^i{}_\alpha dy^\alpha d\theta + g^i d\theta d\theta \tag{7.15b}$$

As discussed in ref.s [8,9], in the case of FDA's we cannot require complete horizontality of the higher form curvatures. We will comment on this later. In general, if $R_{(p)}$ is a curvature p-form, one needs to consider all the components $(p,0),(p-1,1),...(2,p-2)$, where (r,s) denotes the components with r differentials dy^α and s differentials $d\theta$. In our specific case the three-form R^i is expanded as

$$R^i = R^i{}_{\alpha\beta\gamma} dy^\alpha dy^\beta dy^\gamma + r^i_{\alpha\beta} dy^\alpha dy^\beta d\theta \tag{7.16}$$

We want to prove again that coordinate transformations in the θ direction reproduce the correct BRST transformations on the FDA1 fields. Applying the Lie derivative (3.10) to B^i yields:

$$\delta B^i \equiv (i_{\delta y} d + d i_{\delta y}) B^i = i_{\delta y}(R^i - C^i{}_{Aj}\mu^A B^j - \frac{1}{6} C^i{}_{ABC}\mu^A\mu^B\mu^C) + d(i_{\delta y} B^i) \tag{7.17}$$

Specializing the tangent vector δy to point in the θ direction, i.e:

$$\delta y = \delta\theta \frac{\partial}{\partial\theta} \tag{7.18}$$

and projecting eq. (7.17) respectively onto the complete basis of 2-forms ($dy^\alpha dy^\beta$, $dy^\alpha d\theta$, $d\theta d\theta$), one arrives at the transformation rules:

$$\frac{\delta}{\delta\theta} B^i{}_{\alpha\beta} = -r^i{}_{\alpha\beta} + C^i{}_{Aj} g^A B^j{}_{\alpha\beta} + C^i{}_{Aj}\mu^A_{[\alpha} b^j_{\beta]} + \frac{1}{2} C^i{}_{ABC} g^A \mu^B_\alpha \mu^C_\beta + \partial_{[\alpha} b^i_{\beta]} \tag{7.19a}$$
$$\frac{\delta}{\delta\theta} b^i_\alpha = 2\, C^i{}_{Aj} g^A b^j_\alpha + 2\, C^i{}_{Aj}\mu^A_\alpha g^j - C^i{}_{ABC}\mu^A_\alpha g^B g^C + 2\partial_\alpha g^i - \partial_\theta b^i_\alpha \tag{7.19b}$$
$$\frac{\delta}{\delta\theta} g^i = 3\, C^i{}_{Aj} g^A g^j - \frac{1}{2} C^i{}_{ABC} g^A g^B g^C + \partial_\theta(2g^i) \tag{7.19c}$$

where care has to be taken of the (anti)symmetrization properties of the various quantities (for ex. g^A and $d\theta$ anticommute etc.) As in the case for ordinary Lie algebras, we make use now of the horizontality conditions to get rid of the $\frac{\partial}{\partial\theta}$ derivatives in eq.s (7.19b) and (7.19c). From

$$R^i{}_{\alpha\theta\theta} = R^i{}_{\theta\theta\theta} = 0 \tag{7.20}$$

($R^i{}_{\alpha\beta\theta} = r^i{}_{\alpha\beta}$ from eq. (7.16)) we deduce

$$\partial_\alpha g^i - \partial_\theta b^i_\alpha = -C^i{}_{Aj}\mu^A_\alpha g^j - C^i{}_{Aj} g^A b^j + \frac{1}{2} C^i{}_{ABC}\mu^A_\alpha g^B g^C \tag{7.21a}$$
$$\partial_\theta g^i = -C^i{}_{Aj} g^A g^j + \frac{1}{6} C^i{}_{ABC} g^A g^B g^C \tag{7.21b}$$

so that the expressions for θ-diffeomorphisms (7.19) reduce to:

$$\frac{\delta}{\delta\theta}B^i{}_{\alpha\beta} = -r^i{}_{\alpha\beta} + C^i{}_{Aj}g^A B^j{}_{\alpha\beta} + C^i{}_{Aj}\mu^A_{[\alpha}b^j_{\beta]} + \frac{1}{2}C^i{}_{ABC}g^A\mu^B_\alpha\mu^C_\beta + \partial_{[\alpha}b^i_{\beta]} \quad (7.22a)$$

$$\frac{\delta}{\delta\theta}b^i_\alpha = C^i{}_{Aj}g^A b^j_\alpha + C^i{}_{Aj}\mu^A_\alpha g^j - \frac{1}{2}C^i{}_{ABC}\mu^A_\alpha g^B g^C + \partial_\alpha g^i \quad (7.22b)$$

$$\frac{\delta}{\delta\theta}g^i = C^i{}_{Aj}g^A g^j - \frac{1}{6}C^i{}_{ABC}g^A g^B g^C \quad (7.22c)$$

These are the usual BRST transformation rules for FDA1 as given for example in ref.s [8,9]. In these references the BRST transformations are obtained via the "Russian formula" algorithm due to Stora [4]. To compare them with eq.s (7.22) we recall that in our language the BRST operator s is really $d\theta\frac{\partial}{\partial\theta}$, and that our ghosts have to be multiplied by the $d\theta$ differentials before comparing. This is because in the algorithm of [4] the ghosts are defined by

$$\mu^A = \mu^A_\alpha dy^\alpha + g^A \quad (7.23a)$$

$$B^i = B^i{}_{\alpha\beta}dy^\alpha dy^\beta + b^i{}_\alpha dy^\alpha + g^i \quad (7.23b)$$

Let us comment now on the "almost horizontality" conditions (7.20). To see the necessity of $R^i{}_{\alpha\beta\theta} = r^i_{\alpha\beta} \neq 0$ consider the Bianchi identity (7.14b), and project it on $dy^\alpha dy^\beta d\theta d\theta$:

$$\partial_\theta r^i{}_{\alpha\beta} - C^i{}_{Aj}R^A{}_{\alpha\beta}g^j + C^i{}_{Aj}g^A r^j{}_{\alpha\beta} + \frac{1}{2}C^i{}_{ABC}R^A{}_{\alpha\beta}g^B g^C = 0 \quad (7.24)$$

If we had insisted on total horizontality of R^i in the θ direction, i.e. $r^i{}_{\alpha\beta} = 0$, eq. (7.24) would have implied an algebraic relation between the fields and the curvatures, so that the basic fields of the theory would not have been independent any more.

Can the fields still be considered to be independent of the fermionic coordinate θ ? After all, in the case of ordinary Lie algebras, this independence was due to horizontality. Here, however, horizontality is not complete, and we may wonder whether it is possible to remove the θ dependence. It turns out that the "almost horizontality" conditions (7.20) are enough to do the trick: indeed with their help we have removed the ∂_θ terms in the right hand sides of (7.19). We have therefore a sort of *global* rheonomy: the knowledge of the values of the fields at $\theta = 0$ allows to find their θ dependence just by integrating eq.s (7.22). The physical information resides then in the $\theta = 0$ restriction of the fields. In other words, the dependence on θ of the physical fields can be removed via a finite global θ-diffeomorphism (by integrating eq.s (7.22)). The resulting theory is invariant under the global BRST transformation (7.22), where now $\delta\theta$ is interpreted as the fermionic BRST transformation parameter.

In conclusion, we have proved that BRST transformations can be interpreted as diffeomorphisms in θ directions. In our framework, BRST invariant lagrangians can be obtained by "gauging" the enlarged free differential algebra FDA+Q, using the systematic algorithm discussed in sections 3 and 4.

Anti-BRST transformations are easily included in the game, just by considering another nilpotent central charge $\bar{Q}$, and gauging the augmented algebra $G+Q+\bar{Q}$. This of course introduces another grassmann coordinate $\bar{\theta}$, and the corresponding antighosts $\bar{g}^A$ etc.

8. The sigma model of type II superstrings

Type II superstring theories, although deemed to be inadequate for realistic phenomenology [10] (see however [11]), are been considered with renewed attention. This is due to the existence of a canonical map [12,13], called the *h-map* in refs. [14,15], that relates two different modular invariant heterotic models to every consistent modular invariant type II superstring theory. As emphasized in [13], the h-map is the analogue, at the level of 2D-conformal field theories, of the spin connection embedding into the gauge connection coupled to the heterotic fermions.

A classification of (2,2) heterotic superstring vacua has been given in [16], where the h-map is applied on the type II superstring compactifications on $SU(2)^3$ groupfolds (= twisted group manifolds) [17,18].

In this section, based on ref. [19], we provide the geometric construction of the relevant σ-model for type II superstrings propagating on an arbitrary target space $\mathcal{M}_{target}$. From the action of this σ-model, one can deduce the world-sheet supercurrent whose structure turns out to be different from the one considered in the free-fermion constructions of ref.s [20]. This supercurrent yields a more restricted set of solutions to the problem of constructing four-dimensional modular invariant theories that preserve world-sheet supersymmetry.

In subsection 8.1 we discuss the geometry of (1,1)-superspace, i.e. the space underlying the N=2 superconformal algebra in two dimensions. In subsection 8.2 we match this geometry together with the target space geometry, and in subsection 8.3 we derive the geometric action of the (1,1) σ-model.

8.1 World-sheet geometry: (1,1)-superspace

In this subsection we derive the geometry of (1,1)-superspace from its underlying algebraic structure, i.e. the two-dimensional N=2 superconformal algebra.

This superalgebra contains

$$\begin{aligned}\text{translations} &\longrightarrow V^a\\ \text{conformal boosts} &\longrightarrow K^a\\ \text{Q}-\text{supersymmetry} &\longrightarrow \psi\\ \text{S}-\text{supersymmetry} &\longrightarrow \phi\\ \text{Lorentz rotations} &\longrightarrow \omega^{ab}\\ \text{dilatations} &\longrightarrow W\end{aligned} \tag{8.1}$$

and we have indicated on right-hand side of the arrow the corresponding gauge field one-forms. In the dual language of Cartan-Maurer equations the N=2 superconformal algebra reads:

$$\mathcal{D}V^a + W\wedge V^a - \frac{i}{2}\bar{\psi}\wedge\gamma^a\psi = 0 \tag{8.2a}$$

$$\mathcal{D}K^a - W\wedge K^a - \frac{i}{2}\bar{\phi}\wedge\gamma^a\phi = 0 \tag{8.2b}$$

$$\mathcal{D}\psi + \frac{1}{2}W\wedge\psi - i\gamma^a\phi\wedge V_a = 0 \tag{8.2c}$$

$$\mathcal{D}\phi - \frac{1}{2}W\wedge\phi - i\gamma^a\psi\wedge K = 0 \tag{8.2d}$$

$$d\omega^{ab} + \bar{\psi}\wedge\gamma^{ab}\phi - 4V^{[a}\wedge K^{b]} = 0 \tag{8.2e}$$

$$dW - \bar{\psi}\wedge\phi + 2V^a\wedge K^a = 0 \tag{8.2f}$$

The 2 gravitini ψ and ϕ are respectively Majorana-Weyl and Majorana anti-Weyl spinors, i.e.:

$$\bar{\psi}\equiv\psi^\dagger\gamma^0 = \psi^T C; \qquad \gamma_3\psi = \psi \tag{8.3a}$$

$$\bar{\phi}\equiv\phi^\dagger\gamma^0 = \phi^T C; \qquad \gamma_3\phi = -\phi \tag{8.3b}$$

Our conventions for the two dimensional gamma matrices and the charge conjugation matrix C are as in ref. [17]. Eqs. (1.3) are uniquely solved by setting

$$\psi = e^{-i\frac{\pi}{4}}\begin{pmatrix}\zeta\\0\end{pmatrix},\ \zeta^* = \zeta \tag{8.4a}$$

$$\phi = e^{-i\frac{\pi}{4}}\begin{pmatrix}0\\\chi\end{pmatrix},\ \chi^* = \chi \tag{8.4b}$$

Now we allow the left-hand sides of eqs.(8.2) to be nonvanishing, and define them to be the corresponding curvatures of the superconformal N=2 algebra. Moreover, we introduce the convenient basis:

$$e^{\pm} = \frac{1}{2}(V^0 \pm V^1),\ k^{\pm} = \frac{1}{2}(K^0 \pm K^1) \tag{8.5a, b}$$

$$\omega^{ab} = \epsilon^{ab}\omega,\ \omega^{\pm} = W \pm \omega \tag{8.5c, d}$$

In this basis the algebra (8.2) is rewritten as *:

$$\mathcal{T}^+ = de^+ + \omega^+ e^+ - \frac{i}{2}\zeta\zeta \tag{8.6a}$$

$$\mathcal{T}^- = de^- + \omega^- e^- \tag{8.6b}$$

$$\Sigma^+ = dk^+ - \omega^- k^+ \tag{8.6c}$$

$$\Sigma^- = dk^- - \omega^+ k^- + \frac{i}{2}\chi\chi \tag{8.6d}$$

$$\rho = d\zeta + \frac{1}{2}\omega^+\zeta - 2\chi e^+ \tag{8.6e}$$

$$\sigma = d\chi - \frac{i}{2}\omega^+\chi - 2\zeta k^- \tag{8.6f}$$

$$R^+ = d\omega^+ + 2i\zeta\chi + 8e^+k^- \tag{8.6g}$$

$$R^- = d\omega^- + 8e^-k^+ \tag{8.6h}$$

The associated superconformal transformations are nonlinearly realized in a suitable superspace, called (1,1)-superspace, described by two bosonic coordinates $z = x^0 + x^1, \bar{z} = x^0 - x^1$ and two fermionic coordinates θ and $\bar{\theta}$. Let us see how.

We consider the theory, whose basic fields are the 1-forms in (1.1), as living in the $(z, \bar{z}, \theta, \bar{\theta})$ superspace. A complete basis of 1-forms for this superspace is given by its supervielbein, which we choose to *identify* with (e^+, e^-, ζ, χ), i.e. with a subset of the 1-forms associated to the N=2 superconformal algebra in two dimensions.

The torsion and the curvature of (1,1)-superspace are therefore defined by:

$$T^+ = de^+ + \omega e^+ \tag{8.7a}$$

* wedge symbols are omitted.

$$T^{-} = de^{-} - \omega e^{-} \tag{8.7b}$$

$$T^{\bullet} = d\zeta + \frac{1}{2}\omega\zeta \tag{8.7c}$$

$$T^{\circ} = d\chi - \frac{1}{2}\omega\chi \tag{8.7d}$$

$$R = d\omega \tag{8.7e}$$

where ω is the superspace spin connection ($\omega = \omega^{10}$).

The remaining 1-forms $\omega^{\pm}, k^{\pm}$ in the algebra (8.6) live on (1,1)-superspace, and can be expanded on the $(e^{+}, e^{-}, \zeta, \chi)$ basis.

Now the question is: what is the geometry of (1,1) superspace ? In other words, which constraints arise on the torsion and curvature in (8.7) because of the underlying N=2 superconformal algebra ?

The answer is found by analysing the Bianchi identities of the superspace torsion and curvature, and of the superconformal curvatures (8.6).

These identities are immediately obtained by exterior differentiation of the definitions (8.6) and (8.7), and are satisfied by the following torsions and curvatures:

superconformal curvatures:

$$\mathcal{T}^{+} = 0 \tag{8.8a}$$

$$\mathcal{T}^{-} = -\frac{i}{2}\chi\chi \tag{8.8b}$$

$$\Sigma^{+} = \frac{i}{2}\zeta\zeta \tag{8.8c}$$

$$\Sigma^{-} = 0 \tag{8.8d}$$

$$\rho = \tau^{\bullet}e^{+}e^{-} + P\chi e^{+} \tag{8.8e}$$

$$\sigma = \tau^{\circ}e^{+}e^{-} + P\zeta e^{-} \tag{8.8f}$$

$$R^{+} = (8 + \mathcal{R})e^{+}e^{-} + i\tau^{\circ}\chi e^{+} - i\tau^{\bullet}\zeta e^{-} - iP\zeta\chi \tag{8.8g}$$

$$R^{-} = -\mathcal{R}e^{+}e^{-} - i\tau^{\circ}\chi e^{+} + i\tau^{\bullet}\zeta e^{-} + i(P + 2)\zeta\chi \tag{8.8h}$$

superspace curvatures:

$$T^+ = \frac{i}{2}\zeta\zeta \quad (8.9a)$$

$$T^- = -\frac{i}{2}\chi\chi \quad (8.9b)$$

$$T^\bullet = \tau^\bullet e^+ e^- + (P+2)\chi e^+ \quad (8.9c)$$

$$T^\circ = \tau^\circ e^+ e^- + (P+2)\zeta e^- \quad (8.9d)$$

$$R = -R^- \quad (8.9e)$$

$$\omega = \omega^+ = -\omega^- \quad (8.9f)$$

where $\mathcal{R}, \tau^\bullet$ and τ° are the only independent fields, and P is determined by

$$D_\bullet \tau^\bullet - (P+2)^2 = \frac{1}{2}\mathcal{R} \quad (8.10)$$

Let us comment what we have done so far. We have taken as initial algebraic structure G the N=2 superconformal algebra in two dimensions. We have then chosen the particular subgroup $H \subset G$ spanned by the Lorentz rotations, conformal boosts and dilatations and considered the "H-horizontal" fiber bundle $G = (G/H, H)$ where the fiber is H. The bundle is "H-horizontal" in the sense that all curvatures vanish in the H-directions of G. This allows to consider the basic left-invariant one-forms of G as living on G/H . This G/H is nothing else but the (1,1) superspace we have been discussing.

A geometrical theory based on G is invariant under G-differomorphisms. Therefore, to find the transformation rules of the basic fields in eq. (8.1) we apply the Lie-derivative rule and find for the (1,1) supervielbein:

$$\delta e^+ = D\xi^+ + (\phi + \Lambda)e^+ + i\epsilon\zeta \quad (8.10a)$$

$$\delta e^- = D\xi^- + (\phi - \Lambda)e^- - i\eta\chi \quad (8.10b)$$

$$\begin{aligned}\delta\zeta = D\epsilon + \frac{1}{2}(\phi + \Lambda)\zeta + (P+2)\eta e^+ + P\chi\xi^+ + \\ + \tau^\bullet \xi^+ e^- + \tau^\bullet e^+ \xi^-\end{aligned} \quad (8.10c)$$

$$\begin{aligned}\delta\chi = D\eta + \frac{1}{2}(-\phi + \Lambda)\chi + (P+2)\epsilon e^- + P\zeta\xi^- + \\ + \tau^\circ \xi^+ e^- + \tau^\circ e^+ \xi^-\end{aligned} \quad (8.10d)$$

where $\xi^+, \xi^-, \epsilon, \eta, \Lambda, \phi$ are the parameters of the left translations, right translations, Q-supersymmetry, S-supersymmetry, Lorentz rotations and dilatations respectively.

These transformations preserve the constraints (8.9) and allow the choice of a special superconformal gauge where:

$$e^+ = dz + \frac{i}{2}\theta d\theta$$

$$e^- = d\bar{z} + \frac{i}{2}\bar{\theta} d\bar{\theta}$$

$$\zeta = d\theta$$

$$\chi = d\bar{\theta} \tag{8.11}$$

and:

$$\begin{gathered} \tau^\bullet = \tau^\circ = \mathcal{R} = 0 \\ (\Rightarrow P = -2) \end{gathered} \tag{8.12}$$

8.2 Field equations of the target space supervielbein

In this subsection we introduce an embedding function $X^\mu(z, \bar{z}, \theta, \bar{\theta})$, mapping (1,1)-superspace into an arbitrary target manifold $\mathcal{M}_{target}$ and describing the superstring propagation on $\mathcal{M}_{target}$. From the Bianchi identities of the $\mathcal{M}_{target}$-curvatures we determine the field equations for the $\mathcal{M}_{target}$-vielbein $V^a(z, \bar{z}, \theta, \bar{\theta})$, which can be viewed as a 2-D superfield, i.e. a function of the (1,1)-superspace coordinates. The same holds for the target spin connection $\omega^{ab}(z, \bar{z}, \theta, \bar{\theta})$. Thus the Bianchi identities of $\mathcal{M}_{target}$, which is per se a purely bosonic manifold, become actually differential equations for the 2D superfield $V^a(z, \bar{z}, \theta, \bar{\theta})$ in (1,1)- superspace. As usual in supergravity theories without auxiliary fields, Bianchi identities determine the equations of motion of the physical fields, in our case the superfield $V^a(z, \bar{z}, \theta, \bar{\theta})$ (or its component fields along the complete basis of 1-forms for (1,1)-superspace, see. eq. (8.14)).

The torsion and curvature of $\mathcal{M}_{target}$ are defined as usual by

$$T^a \equiv dV^a + \omega^{ab} V^b = T^a{}_{bc} V^b V^c \tag{8.13a}$$

$$R^{ab} \equiv d\omega^{ab} + \omega^{ac}\omega^{cb} = R^{ab}{}_{cd} V^c V^d \tag{8.13b}$$

where the $\mathcal{M}_{target}$-vielbein can be expanded on the basis e^+, e^-, ζ, χ of (1,1)-superspace:

$$V^a = V^a_+ e^+ + V^a_- e^- + \lambda^a \zeta + \mu^a \chi \tag{8.14}$$

Thus, the superbielbein V^a contains two bosonic fields V_+^a, V_-^a and two 2-dimensional spinors λ^a, μ^a.

Inserting the decomposition (8.14) into the torsion definition (8.13a) yields, in the various sectors of (1,1)-superspace:

$$e^+e^- : -\nabla_- V_+^a + \nabla_+ V_-^a - 2T^a{}_{bc} V_+^b V_-^c = 0 \tag{8.15a}$$

$$e^+\zeta : -\nabla_\bullet V_+^0 + \nabla_+ \lambda^a - 2T^a{}_{bc} V_+^b \lambda^c = 0 \tag{8.15b}$$

$$e^+\chi : -\nabla_\circ V_+^a + \nabla_+ \mu^a - 2T^a{}_{bc} V_+^b \mu^c = 0 \tag{8.15c}$$

$$e^-\zeta : -\nabla_\bullet V_-^a + \nabla_- \lambda^a - 2T^a{}_{bc} V_-^b \lambda^c = 0 \tag{8.15d}$$

$$e^-\chi : -\nabla_\circ V_-^a + \nabla_- \mu^a - 2T^a{}_{bc} V_-^b \mu^c = 0 \tag{8.15e}$$

$$\zeta\zeta : \frac{i}{2} V_+^a + \nabla_\bullet \lambda^a + T^a{}_{bc} \lambda^b \lambda^c = 0 \tag{8.15f}$$

$$\zeta\chi : \nabla_\circ \lambda^a + \nabla_\bullet \mu^a + 2T^a{}_{bc} \lambda^b \mu^c = 0 \tag{8.15g}$$

$$\chi\chi : -\frac{i}{2} V_-^a + \nabla_\circ \mu^a + T^a{}_{bc} \mu^b \mu^c = 0 \tag{8.15h}$$

Consider now the Bianchi identity for the $\mathcal{M}_{target}$ torsion (8.13a):

$$\nabla T^a = \nabla^2 V^a = R^{ab} V^b \tag{8.16}$$

or:

$$\nabla^2 V_+^a = R^{ab} V_+^b \tag{8.17a}$$

$$\nabla^2 V_-^a = R^{ab} V_-^b \tag{8.17b}$$

$$\nabla^2 \lambda^a = R^{ab} \lambda^b \tag{8.17c}$$

$$\nabla^2 \mu^a = R^{ab} \mu^b \tag{8.17d}$$

As we discuss later (subsection 8.3), for a generic (1,1) action including a topological WZ term $\int_{M^3} H$, the components of the torsion T_{abc} will be ultimately identified with the components H_{abc} of the closed 3-form $H \equiv H_{abc}V^aV^bV^c$. This justifies taking T_{abc} completely antisymmetric and

$$d(T_{abc}V^aV^bV^c) = 0 \tag{8.18a}$$

or

$$\nabla_{[a}T_{bcd]} + 3T^{\ell}{}_{[da}T^{\ell}{}_{bc]} = 0 \tag{8.18b}$$

This ansatz is compatible with the Bianchi identities solved in this Section. Together with the torsion Bianchi identity (8.16):

$$\nabla_{[m}T^a{}_{bc]} + 2T^{a\ell}{}_{[m}T^{\ell}{}_{bc]} = R^a{}_{[mbc]} \tag{8.19}$$

the ansatz (8.18) implies the following identities:

$$\nabla_a T_{bcm} = -6T^{\ell}{}_{a[b}T^{\ell}{}_{cm]} + 3R^a{}_{[bcm]} \tag{8.20a}$$

$$R^a{}_{[bcm]} = T^{\ell}{}_{a[b}T^{\ell}{}_{cm]} \tag{8.20b}$$

Next from eq. (8.17c) we find

$$\begin{aligned} &\nabla\left(\nabla_+\lambda^a e^+ + \nabla_-\lambda^a e^- + \left(-\frac{i}{2}\right)V^a_+\zeta - T^{abc}\lambda^b\lambda^c\zeta + \nabla_\circ\lambda^a\chi\right) = \\ &= R^{ab}{}_{cd}(V^c_+c^+ + V^c_-e^- + \lambda^c\zeta + \mu^c\chi)\ (V^d_+e^+ + V^d_-e^- + \lambda^d\zeta + \mu^d\chi) \end{aligned} \tag{8.21}$$

For simplicity, we set

$$\nabla_\circ\lambda^a = 0 \tag{8.22}$$

in the following. This constraint is compatible with the Bianchi identities, as we see in this Section, and really amounts to a suitable redefinition of the spin connection ω^{ab}.

From the $\chi\chi$ sector of eq. (8.21) we find

$$-\frac{i}{2}\nabla_-\lambda^a = -R^{ab}{}_{cd}\ \lambda^b\mu^c\mu^d \tag{8.23}$$

which is the λ^a field equation.

Similarly from the $\zeta\zeta$ projection of eq. (8.17d) we have

$$\frac{i}{2}\nabla_+\mu^a = -iT^{abc}\mu^bV^c_+ - R^{ab}{}_{cd}\mu^b\lambda^c\lambda^d \tag{8.24}$$

i.e. the μ^a field equation. We have used

$$\nabla_\bullet\mu^a = -2T^a{}_{bc}\lambda^b\mu^c$$

from eq. (8.15g) and

$$\nabla_\bullet(T^a{}_{bc}\lambda^b\mu^c) = -\frac{i}{2}T^a{}_{bc}V^b_+\mu^c$$

where the $\nabla T + TT$ terms cancel after use of eqs. (8.20).

The $V^a_\pm$ field equations can be found from the $\chi\chi$ projection of (8.17a) and from the $\zeta\zeta$ projection of (8.17b):

$$\begin{aligned}\nabla_- V^a_+ &= 2iR^{ab}{}_{cd}V^b\lambda^c\lambda^d - 2iR^{ab}{}_{cd}V^b_+\mu^c\mu^d \\ &- 2\nabla_a R^{bc}{}_{de}\lambda^b\lambda^c\mu^d\mu^e + 8R^{de}{}_{\ell b}T^\ell{}_{ac}\lambda^d\lambda^e\mu^b\mu^c\end{aligned} \tag{8.25}$$

$$\begin{aligned}\nabla_+ V^a_- &= 2T^{abc}V^b_+V^c_- + 2iR^{ab}{}_{cd}V^b_-\lambda^c\lambda^d \\ &- 2iR^{ab}{}_{cd}V^b_+\mu^c\mu^d \\ &- 2\nabla_a R^{bc}{}_{de}\lambda^b\lambda^c\mu^d\mu^e + 8R^{de}{}_{\ell b}T^\ell{}_{ac}\lambda^d\lambda^e\mu^b\mu^c\end{aligned} \tag{8.26}$$

again after use of the identities (8.20).

No further information can be gleaned from the torsion Bianchi identity (8.16). As usual, the Bianchi identity for the $\mathcal{M}_{target}$ curvature (8.13b) does not give new conditions when the other Bianchi identities are satisfied.

8.3 Geometric action of the (1,1) σ-model

In this subsection we construct the action that yields the parametrizations (8.15) and the field equations (8.23-8.26) together with the constraint (8.18).

We begin be writing a natural extension of the (1,0) lagrangian (in our notations, this lagrangian is given in Part. 6 of ref. [1]) where now many more terms are possible, because of the presence of the second bidimensional spinor μ^a:

$$\begin{aligned}S = \int_{\partial M} &[(V^a - \lambda^a\zeta - \mu^a\chi)(\Pi^a_+e^+ - \Pi^a_-e^-) + 2i\lambda^a\nabla\lambda^a e^+ + \alpha 2i\mu^a\nabla\mu^a e^- \\ &+ \lambda^a V^a\zeta + \beta\mu^a V^a\chi + \Pi^a_+\Pi^a_+e^+e^- + a\lambda^a\mu^a\zeta\chi \\ &+ t_1T^{abc}\lambda^a\lambda^b\lambda^c\zeta e^+ + t_2T^{abc}\mu^a\mu^b\mu^c\chi e^- + t_3T^{abc}\lambda^a\lambda^b\mu^c\chi e^+ + t_4T^{abc}\mu^a\mu^b\lambda^c\zeta e^- \\ &+ t_5T^{abc}\lambda^a\lambda^bV^ce^+ + t_6T^{abc}\mu^a\mu^bV^ce^- \\ &+ (r_1R^{ab}{}_{cd} + q_1T^{fab}T^{fcd})\lambda^a\lambda^b\mu^c\mu^de^+e^- + (r_2R^{ab}{}_{cd} + q_2T^{fab}T^{fcd})\lambda^a\mu^b\lambda^c\mu^de^+e^- \\ &+ (r_3R^a{}_b + q_3T^{acd}T^{bcd})\lambda^a\lambda^e\mu^b\mu^ee^+e^- + (r_4R + q_4T^{abc}T^{abc})\lambda^e\lambda^f\mu^e\mu^f] + \\ &+ \int_M H\end{aligned} \tag{8.27}$$

M is a three-dimensional manifold bounded by ∂M, and the three-form H is closed: $dH = 0$. Note that all the terms in S must have vanishing ω-weight, where the ω-weight is given by the coefficient of the ω-connection term in eqs. (8.7):

$$\begin{aligned} &[e^{\pm}] = \pm 1 \qquad && [\lambda^a] = -\frac{1}{2} \\ &[\zeta] = \frac{1}{2} && [\mu^a] = \frac{1}{2} \\ &[\chi] = -\frac{1}{2} \end{aligned} \tag{8.28}$$

since the action must be invariant under Weyl rescalings and two-dimensional Lorentz rotations, generated by $J^{12} \pm D$. The ω-weights of λ^a and μ^a can be deduced from eq. (8.14) (V^a has ω-weight= 0). Moreover, the action S is invariant also under the global rescalings

$$\begin{aligned} &V^a \to wV^a \qquad && \zeta \to \frac{1}{2}w\zeta \\ &\lambda^a \to \frac{1}{2}w\lambda^a && \chi \to \frac{1}{2}w\chi \\ &\mu^a \to \frac{1}{2}w\mu^a \\ \Rightarrow &T^{abc} \to w^{-1}T^{abc} \\ &R^{ab}{}_{cd} \to w^{-2}R^{ab}{}_{cd} \end{aligned} \tag{8.29}$$

which preserve the parametrizations (8.15) and the field equations (8.23-8.26).

The undetermined coefficients in the action (8.27) are computed by requiring that the variation δS reproduces the parametrizations (8.15) and the field eqs. (8.23-8.26).

Variations in $\Pi^a_{\pm}$ yield immediately

$$\Pi^a_{\pm} = V^a_{\pm} \tag{8.30}$$

namely the auxiliary 0-forms $\Pi^a_{\pm}$ are identified with the bosonic projections of the target vielbein.

The variation in the spinor λ^a leads to the following equations, in the various sectors of (1,1)-superspace:

$$\zeta\zeta : 0 = 0 \tag{8.31a}$$
$$\chi\chi : 0 = 0 \tag{8.31b}$$
$$\zeta\chi : (1+a)\mu^a = 0 \tag{8.31c}$$
$$\zeta e^+ : -2V^a_+ + 4i\nabla_\bullet\lambda^a + 3t_1T^{abc}\lambda^b\lambda^c + 2t_5T^{abc}\lambda^b\lambda^c = 0 \tag{8.31d}$$
$$\zeta e^- : t_4T^{abc}\mu^b\mu^c = 0 \tag{8.31e}$$
$$\chi e^+ : 4_i\nabla_\circ\lambda^a + 2t_3T^{abc}\lambda^b\mu^c + 2t_5T^{abc}\lambda^b\mu^c = 0 \tag{8.31f}$$

$$\chi e^- : 0 = 0 \tag{8.31g}$$

$$\begin{aligned} e^+e^- : & -4i\nabla_-\lambda^a - 2t_5 T^{abc}\lambda^b V_-^c + 2(r_1 R^{ab}{}_{cd} + q_1 T^{fab}T^{fcd})\lambda^b\mu^c\mu^d + \\ & + 2(r_2 R_{[a}{}^b{}_{c]}{}^d + q_2 T^f{}_{[a}{}^b T^f{}_{c]}{}^d)\mu^b\lambda^c\mu^d + (r_3 R^a{}_b + q_3 T^{fga}T^{fgb})\lambda^q\mu^b\mu^q - \\ & - (r_3 R^p{}_q + q_3 T^{fgp}T^{fgq})\lambda^p\mu^q\mu^a + 2(r_4 R + q_4 T^{bcd}T^{bcd})\lambda^e\mu^a\mu^e = 0 \end{aligned} \tag{8.31h}$$

leading immediately to

$$a = -1, \quad t_4 = 0, \quad t_3 = -t_5 \tag{8.32}$$

(the last equality being due two the choice $\nabla_\circ\lambda^a = 0$). Imposing further

$$3t_1 + 2t_5 = 4i \tag{8.33}$$

we see that (8.15f) is correctly reproduced. Also, comparing (8.31h) with the field equation (8.23) we see that

$$t_5 = 0 \left(\Rightarrow t_3 = 0, \quad t_1 = \frac{4}{3}i \right) \tag{8.34}$$

is necessary since the corresponding term $T^{abc}\lambda^b V_-^c$ cannot be otherwise eliminated. Moreover, the last four terms in the action (8.27), i.e. the terms with $r_1, q_1, \cdots r_4, q_4$ must sum up to $4R^{ab}{}_{cd}\, \lambda^a\lambda^b\mu^c\mu^d$, so that eq. (8.31h) indeed reduces to the field equation (8.23).

Varying now S in $\delta\mu^a$ we find:

$$\zeta\zeta : 0 = 0 \tag{8.35a}$$

$$\chi\chi : (\alpha + \beta)\mu^a = 0 \tag{8.35b}$$

$$\zeta\chi : (\beta + 1)\lambda^a = 0 \tag{8.35c}$$

$$\zeta e^+ : 0 = 0 \tag{8.35d}$$

$$\zeta e^- : 4i\alpha\nabla_\bullet\mu^a + 2t_6 T^{abc}\mu^b\lambda^c = 0 \tag{8.35e}$$

$$\chi e^+ : -(1+\beta)V_+^a = 0 \tag{8.35f}$$

$$\chi e^- : 4i\nabla_\circ\mu^a + 2V_-^a + (3t_2 + 2t_6)T^{abc}\mu^b\mu^c = 0 \tag{8.35g}$$

$$e^+e^- : 4i\nabla_+\mu^a + 2t_6 T^{abc}\mu^b V_+^c + 8R^{ab}{}_{cd}\mu^b\lambda^c\lambda^d = 0 \tag{8.35h}$$

These equations imply

$$\beta = -1, \alpha = 1, t_6 = 4i, t_2 = -\frac{4}{3}i \tag{8.36}$$

and the eqs. (8.15g,h) and (8.24) are satisfied if

$$t_6 = 4i, t_2 = -\frac{4}{3}i \tag{8.37}$$

All the coefficients in the action (8.27) have been determined, and S takes the form:

$$\begin{aligned} S = \int_{\partial M} & [V^a - \lambda^a\zeta - \mu^a\chi)(\Pi^a_+ e^+ - \Pi^a_- e^-) + 2i\lambda^a\nabla\lambda^a e^+ \\ & + 2i\mu^a\nabla\mu^a e^- + \lambda^a V^a\zeta - \mu^a V^a\chi + \Pi^a_+\Pi^a_- e^+ e^- - \lambda^a\mu^a\zeta\chi \\ & + \frac{4}{3}iT^{abc}\lambda^a\lambda^b\lambda^c\zeta e^+ - \frac{4}{3}iT^{abc}\mu^a\mu^b\mu^c\chi e^- \\ & - 4iT^{abc}\mu^a\mu^b V^c e^- + 4R^{ab}{}_{cd}\lambda^a\lambda^b\mu^c\mu^d] \\ & + \int_M H \end{aligned} \tag{8.38}$$

Next, we consider the variation of (8.38) in the coordinates X^μ. It is convenient to use the anholonomized variations δX^a (tangent vectors to $\mathcal{M}_{target}$), whose generator is the Lie derivative:

$$\delta_{\delta X} V^a \equiv \ell_{\delta X} V^a = i_{\delta X} dV^a + d(i_{\delta X} V^a) \tag{8.39}$$

where

$$\delta X = \delta X^a P_a, \quad V^a(P_b) = \delta^a_b \tag{8.40}$$

and we have:

$$\delta_{\delta X} V^a = \nabla\delta X^a + 2T^{abc}\delta X^b V^c - (i_{\delta X}\omega^{ab})V^b \tag{8.41}$$

where $(i_{\delta x}\omega^{ab})V^b$ is a field dependent Lorentz transformation on the vielbein. By the same token we have:

$$\delta_{\delta X}\omega^{ab} = \nabla(i_{\delta X}\omega^{ab}) + 2R^{ab}{}_{cd}\delta X^c V^d \tag{8.42}$$

$$\delta_{\delta X}\int_M H = 3\int_{\partial M} H_{abc}\delta X^a V^b V^c \tag{8.43}$$

Varying now (8.38) in δX^a yields, in the $\zeta\zeta$ and $\chi\chi$ sectors, the constraint

$$T^{abc} = -3H^{abc} \tag{8.44}$$

which we anticipated in subsection 8.2. The δX^a variation in those sectors is really a supersymmetry variation, so that the constraint (8.44) is necessary for the supersymmetry invariance of the action. Note that $T^{abc} = const \cdot H^{abc}$ can be obtained from the more general action (8.27), where all the coefficients are still undetermined. The other sectors of $\delta_X S$ simply reproduce the field equations (8.23-8.26).

Finally, we compute the stress-energy tensor E of our final lagrangian (3.12), defined by

$$\delta S = -\int (E_+ \delta e^+ + E_- \delta e^- + E_\bullet \delta\zeta + E_\circ \delta\chi) \tag{8.45}$$

Recalling the transformation rules (8.10) in the special gauge (8.11) and requiring

$$\frac{\delta S}{\delta \Lambda} = \frac{\delta S}{\delta \phi} = 0 \tag{8.46}$$

yields

$$\begin{aligned} &E_{+-} = E_{-+} = E_{\bullet -} = E_{-\bullet} = E_{\circ +} = E_{+\circ} = 0 \\ &E_{+\bullet} = \frac{1}{2} E_{\bullet +} \\ &E_{-\circ} = -\frac{1}{2} E_{\circ -} \end{aligned} \tag{8.47}$$

Therefore, we are left with four independent components:

$$E_{++}, E_{--}, E_{+\bullet}, E_{-\circ}$$

Varying the action S in δe^+, δe^-, $\delta\zeta$, $\delta\chi$, we find

$$E_{++} = V_+^a V_+^a + 2i\lambda^a \nabla_+ \lambda^a \tag{8.48a}$$

$$E_{--} = -V_-^a V_-^a + 2i\mu^a \nabla_- \mu^a + 4iT^{abc} V_-^a \mu^b \mu^c \tag{8.48b}$$

$$E_{+\bullet} = \lambda^a V_+^a - \frac{2}{3} i T^{abc} \lambda^a \lambda^b \lambda^c \tag{8.48c}$$

$$E_{-\circ} = -\mu^a V_-^a + \frac{2}{3} i T^{abc} \mu^a \mu^b \mu^c \tag{8.48d}$$

and it is straightforward (but somewhat laborious) to show that:

$$\nabla_- E_{++} = 0 \tag{8.49a}$$

$$\nabla_+ E_{--} = 0 \tag{8.49b}$$

$$\nabla_- E_{+\bullet} = 0 \tag{8.49c}$$

$$\nabla_+ E_{-\circ} = 0 \tag{8.49d}$$

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