CNPq

CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Notas de Física

CBPF-NF-020/92

THE ROLE OF THE MÖLLER ENERGY IN THE HAMILTONIAN FORMULATION OF GENERAL RELATIVITY

by

N. Pinto NETO and I. Damião SOARES

ABSTRACT

We introduce a new prescription to deal with boundary terms in the Hamiltonian formulation of General Relativity. Under this prescription the equations of motion are correctly obtained, and the Möller energy integral for asymptotically stationary gravitational fields arises as the natural definition of energy. This definition agrees with the gravitational mass obtained from the geodesic motion of a test particle, in the case of asymptotically flat and asymptotically Anti-DeSitter spacetimes.

Key-words: Gravitational energy; Definition of gravitational mass; General relativity; Hamiltonian formulation; Surface terms; Asymptotic anti-DeSitter spacetimes.

The definition of the gravitational energy is a fundamental and longstanding problem in the Theory of General Relativity of the gravitational field. The absence of an energy-momentum tensor and the arbitrariness in the definition of a conserved energy-momentum pseuso-tensor[1,2,3,4] from one side, and the appearence of various possible surface terms in the Hamiltonian formulation of General Relativity for open spaces[5,6] from the other, shows that a unique definition of the gravitational energy is not obvious.

In a classical paper[7], Regge and Teitelboim tried to solve this problem from the Hamiltonian point of view, arguing that the surface terms that appear in open spaces problems are necessary and can be determined uniquely for each case by consistency of the Hamiltonian formulation. In fact, let us consider the Hamiltonian of General Relativity[8]

$$H_0 = \frac{1}{8\pi}\int d^3x\left\{N(x)\mathcal{H}(x) + N_i(x)\mathcal{H}^i(x)\right\} \tag{1}$$

where

$$\mathcal{H}(x) := G_{ijk\ell}\pi^{ij}\pi^{k\ell} - \gamma^{1/2}\ {}^{(3)}R$$

$$\mathcal{H}^i(x) := -2\pi^{ij}{}_{;j}$$

$$G_{ijk\ell} = \frac{1}{2}\gamma^{-1/2}(\gamma_{ik}\gamma_{j\ell} + \gamma_{i\ell}\gamma_{jk} - \gamma_{ij}\gamma_{k\ell}).$$

${}^{(3)}R$ is the scalar curvature of spacelike hypersurfaces x^0 = const. and γ_{ij} is the 3-metric of these surfaces, π^{ij} its canonical momentum, γ its determinant, N the lapse function and N_i the shift function. The semi-colon denotes a covariariant derivative based on γ_{ij}.

Regge and Teitelboim argue in their paper that variations in the Hamiltonian H_0, defined in (1), lead to surface terms (due to the appearance of first and second order derivatives of the canonical variables) that may be different from zero for arbitrary open spaces. In fact they showed that

$$\delta H_0 = \frac{1}{8\pi} \int d^3x\left(A_{ij}\delta\pi^{ij} + B^{ij}\delta\gamma_{ij}\right) +$$

$$+ \frac{1}{8\pi} \oint d^2S_\ell\left\{- NG^{abc\ell}\delta\gamma_{ab;c} + \left(N_{;c}G^{abc\ell} - 2N^b\pi^{a\ell} +\right.\right.$$

$$\left.\left. + N^\ell\pi^{ab}\right)\delta\gamma_{ab} - 2N_a\delta\pi^{a\ell}\right\} \tag{2}$$

where $G^{abc\ell}$ is the inverse of $G_{ijk\ell}$ and is given by:

$$G^{abc\ell} := \frac{1}{2}\,\gamma^{1/2}\left(\gamma^{ac}\gamma^{b\ell} + \gamma^{a\ell}\gamma^{bc} - 2\gamma^{cb}\gamma^{c\ell}\right)$$

The tensors A_{ij} and B^{ij} correspond to the right-hand side of the first order Einstein's equations written in the form[9]

$$\dot{\gamma}_{ij} = A_{ij} \quad , \quad \dot{\pi}^{ij} = - B^{ij} \tag{3.a,b}$$

Thus we see that if the surface terms appearing in eq. (2) are zero, which is the case of closed spaces, the Hamiltonian H_0 given by eq. (1) yields the correct Einsten's equations in the Hamiltonian form

$$\dot{\gamma}_{ij} = \frac{\delta H_0}{\delta \pi^{ij}} \tag{4.a}$$

$$\dot{\pi}^{ij} = - \frac{\delta H_0}{\delta \gamma_{ij}} \tag{4.b}$$

However, if we are dealing with open spaces and the surface terms that are present in eq. (2) are not zero, the Hamiltonian H_0 does not give Einstein's equations in the form of eqs. (4) and boundary terms have to be ammended to the Hamiltonian H_0 in order to define a new Hamiltonian H that yields the correct Einstein's equations. For instance, if we are dealing with asymptotically flat spacetimes, the correct Hamiltonian is

$$H = H_0 + \frac{1}{8\pi} \oint d^2S_\ell \sum_{i=1}^{3} \left(\gamma_{i\ell,i} - \gamma_{ii,\ell} \right) \tag{5}$$

where variations of the surface terme appearing in eq. (5) cancels exactly the non-vanishing contributions due to H_0 and exhibited in eq. (2), yielding the equation

$$\delta H = \frac{1}{8\pi} \int d^3x \left\{ A^{ij}(x) \delta\gamma_{ij}(x) + B_{ij}(x) \gamma\pi^{ij}(x) \right\}$$

Thus the correct Hamiltonian obtained by the Regge-Teitelboin procedure for asymptotically flat spacetimes is given by eq. (5) and the energy of such gravitational fields is given by

$$E = \frac{1}{8\pi} \oint d^2S_k \sum_{i=1}^{3} \left(\gamma_{ik,i} - \gamma_{ii,k}\right) \tag{6}$$

The value of H_0 is always zero due to the existence of the gravitational constraints.

With this line of reasoning, it is possible to obtain a unique and necessary surface term for each open space with a well defined asymptotic behaviour and a correspondent gravitational energy. However, this method is not aesthetically appealing because it does not provide a unique formula for the gravitational energy: for each case of asymptotic behaviour we have to evaluate the surface integral coming from eq. (2) that will correspond to the gravitational energy of the space in consideration. This fact makes impossible a general proof of the positivity of energy in General Relativity for every asymptotic behaviour.

Because of that, we propose a new method to obtain the energy of the gravitational field that yields a unique surface integral for every asymptotic behaviour.

First of all define a new variation $\bar{\delta}$

$$\bar{\delta}F(\phi^A(x)) = \left[\frac{\partial F}{\partial \phi^A(x)} - \left(\frac{\partial F}{\partial \phi^A_{,i}}\right)_{,i} + \left(\frac{\partial F}{\partial \phi^A_{,i,j}}\right)_{,i,j} - \ldots\right]\delta\phi^A(x)$$

where $F\left(\phi^A(x)\right)$ is an arbitrary functional of general field $\phi^A(x)$. The action of this $\bar{\delta}$ variation on a surface term is obviously zero and we can write Einstein's equations for every spacetime as:

$$\dot{\gamma}_{ij} = \frac{\bar{\delta} H_0}{\delta \pi^{ij}} = A_{ij} \tag{7.a}$$

$$\dot{\pi}^{ij} = - \frac{\bar{\delta} H_0}{\delta \gamma_{ij}} = - B^{ij} \tag{7.b}$$

We remark that the main feature of the variation $\bar{\delta}$ is to give automatically the equations of motion without having to identify (and eliminate) volume integrals which yield non-zero surface terms.

Now, the Hamiltonian that comes naturally from the Lagrangian of General Relativity

$$L = \frac{1}{8\pi} \int \sqrt{-g}\, R d^4x + \frac{1}{4\pi} \int dt d^3x \left(\gamma^{1/4} K\right)^{\bullet} \tag{8}$$

after making the 3 + 1 splitting of spacetime is given by

$$H = H_0 + \frac{2}{8\pi} \oint d^2 S_{\ell} \left\{ \gamma^{1/2} \gamma^{k\ell} N_{,k} + \pi^{k\ell} N_k + - \frac{1}{2} \pi N^{\ell} \right\} \tag{9}$$

For closed spaces the surface term appearing in (9) is zero and the Hamiltonian reduces to H_0. For open spaces however, this surface term - that does not contribute to eq. (7) because of the definition of the $\bar{\delta}$ variation - may have an absolute value diferent from zero. We thus propose, in the context of this $\bar{\delta}$ variation, that the gravitational energy of every open space is given by the surface integral

$$E = \frac{2}{8\pi} \oint d^2S_\ell \left\{ \gamma^{1/2}\gamma^{k\ell} N_{,k} + \pi^{k\ell} N_k - \frac{1}{2}\, \pi N^\ell \right\} \tag{10}$$

The advantage of this formulation is obvious: the gravitational energy of open spaces is given by a single surface integral which, as can be shown by straightforward calculations, corresponds to the Möller energy[2] for stationary spacetimes.

We will now compare the energies given by eq. (2) and eq. (10) for different asymptotic behaviours of space-time.

We will examine two cases: asymptotically flat and asymptotically Anti-DeSitter spacetimes.

i) Asymptotically Flat Spacetime

Any solution of Einstein's equations with finite energy which is asymptotically flat behaves at spatial infinity like

$$ds^2 \underset{r\to\infty}{\sim} -\left(1 - \frac{2m}{r}\right)dt^2 + \left(\delta_{ij} + 2m\,\frac{x^i x^j}{r^3}\right)dx^i dx^j \tag{11}$$

A straighforward calculation of the surface integral given by equation (6) gives

$$E = 2m \tag{12}$$

Using now our definition of energy in equation (10) we obtain

$$E = m \tag{13}$$

differing from the former by a multiplicative factor. The significance of this difference will be discussed later.

ii) Asymptotically Anti-DeSitter (ADS) spacetimes

We will consider the Kerr-Anti-DeSitter solution which has the Sckwarzschild-Anti-DeSitter solution as a special case.

The Anti-DeSitter line element is given by

$$ds_0^2 = -\left[1 + \left(\frac{r}{R}\right)^2\right]dt^2 + \left[1 + \left(\frac{r}{R}\right)^2\right]^{-1}dr^2 + d\Omega^2 \tag{14}$$

where $d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$ and R is the radius of curvature.

The deviations from the Anti-DeSitter background at the spatial infinity are given by[10]:

$$h_{tt} = \frac{2m}{r}\left(1 - \alpha^2\sin^2\theta\right)^{-5/2} + O(r^{-3}) \tag{15.a}$$

$$h_{t\phi} = -\frac{2am\sin^2\theta}{r}\left(1 - \alpha^2\sin^2\theta\right)^{-5/2} + O(r^{-3}) \tag{15.b}$$

$$h_{\phi\phi} = \frac{2ma^2\sin^4\theta}{r}\left(1 - \alpha^2\sin^2\theta\right)^{-5/2} + O(r^{-3}) \tag{15.c}$$

$$h_{rr} = \frac{2mR^4}{r^5}\left(1 - \alpha^2\sin^2\theta\right)^{-3/2} + O(r^{-7}) \tag{15.d}$$

$$h_{\theta r} = -\frac{2mR^2a^2}{r^4}\left(1 - \alpha^2\sin^2\theta\right)^{-5/2}\sin\theta\cos\theta + O(r^{-6}) \tag{15.e}$$

$$h_{\theta\theta} = \frac{2ma^4}{r^3}\left(1 - \alpha^2\sin^2\theta\right)^{-7/2}\sin^2\theta\cos^2\theta + O(r^{-5}) \tag{15.f}$$

where $\alpha = a/R$ and a is related to the angular momentum per unit mass. The non-vanishing components of the gravitational momentum are:

$$\pi^{r\phi} = -\frac{3am\sin\theta}{r^2}\left(1 - \alpha^2\sin^2\theta\right)^{-5/2} + O(r^{-4}) \tag{16.a}$$

$$\pi^{\theta\phi} = O(r^{-5}) \tag{16.b}$$

In ref. [10], the non-vanishing surface terms appearing in eq, (2) are calculated for asymptotically Anti-DeSitter spacetimes and the energy for the Kerr-ADS metric is calculated yielding

$$E = \frac{2m}{(1 - \alpha^2)^2} \tag{17}$$

Note that as R goes to infinity (α goes to zero) this energy goes to its value for asymptotically flat spacetime as should be expected.

In our definition of energy, the nonvanishing terms are given by

$$E = \frac{1}{8\pi} \oint d^2S_\ell \; 2\gamma^{1/2}\gamma^{\ell k}N_{,k} \tag{18}$$

where $\gamma_{ij} = \overset{(0)}{g}_{ij} + h_{ij}$,

$\overset{(0)}{g}_{ij}$ denoting the spatial metric of ADS spacetime. The surface integral has divergent volume dependent terms coming from the infinite gravitational energy of the ADS background that are authomatically excluded from the former case because the variations considered in eq. (2) are supposed to preserve the background structure. Thus, taking solely the finite value of the integral (18) we obtain

$$E = m \frac{(1 + \alpha^2)}{(1 - \alpha^2)^2} \tag{19}$$

which also goes to the asymptotically flat case when R goes to infinity (and consequently α goes to zero). Comparison with eqs. (19) and (17) shows different values for the gravitational energy, arising from the two definitions.

We now argue in favor of the latter, by using a definition of the gravitational mass from a completely different point of view: the geodesic motion of a test particle in Anti-de Sitter and Kerr-ADS backgrounds.

We will proceed as in the Newtonian limit of General Relativity. As we know, the concept of mass in General Relativity is well defined for asymptotically flat spacetimes and mass is defined in comparison to the

Newtonian limit. Here we extend this definition for asymptotically ADS spacetimes. We impose two limiting conditions:

i) at spatial infinity ($r \to \infty$) the background is ADS plus small Kerr-ADS deviations (as given by (15)).

ii) the velocities of all bodies along geodesics of ADS asymptotic background are small, and we will take the approximation which consists in keeping only first-order terms in the deviations (i) and/or (ii).

Under these conditions the asymptotic ADS group reduces to the Galileo group, and the gravitational mass of the Newtonian-like field - which is a deviation of the ADS asymptotic background - produces the deviation in acceleration given by the four-vector.

$$\Delta^{\alpha} = \frac{d^2x^{\alpha}}{ds^2} - \frac{d^2x^{\alpha}}{ds^{(0)2}} \tag{20}$$

which, in the approximation considered, has the non-zero components

$$\Delta^{i} = - \left[\left\{ {i \atop 0\ 0} \right\} - \overset{(0)}{\left\{ {i \atop 0\ 0} \right\}} \right] \frac{1}{\overset{(0)}{g}_{00}} + \overset{(0)}{\left\{ {i \atop 0\ 0} \right\}} \frac{h_{00}}{\overset{(0)2}{g}_{00}} \tag{21}$$

Here s and $s^{(0)}$ are, respectively the proper-time along a geodesic of the Kerr-ADS metric and the proper-time along a geodesic of ADS background metric. In general a superscript (0) denotes a geometrical quantity of the ADS background metric.

Δ^{i} is a measure of the deviation of the aceleration of bodies along ADS geodesics, produced by the Kerr-ADS perturbations. Thus, the asymptotic gravitational mass can be defined by

$$M_g = \frac{1}{4\pi} \oint_{r=\text{cons}\to\infty} \sqrt{-g}\ \Delta^\alpha d^2S_\alpha = \frac{1}{4\pi} \oint_{r=\text{cont}\to\infty} \sqrt{-g}\ \Delta^1 d^2S_1$$

which gives, for (21),

$$M_g = m \frac{(1 + \alpha^2)}{(1 - \alpha^2)^2} \tag{22}$$

which is the value of the Möller energy (19).

Thus, our definition of energy given in eq. (10) agrees with the mass definition from geodesic motion while the Regge-Teitelboim definition that comes from eq. (2) does not. The difference by a constant factor of the values of the Regge-Teitelboim and Möller energies *for asymptotically flat spacetime* (eqs. (12) and (13)) is irrelevant: it is a matter of definition of the non-gravitational energy-momentum tensor, which fixes the multiplicative constant of the gravitational action by the Newtonian approximation. However, the difference in the values given in eqs. (17) and (19) is fundamental and the geodesic motion definition of mass is *crucial* in order to decide which definition of energy is correct. To our knowledge, it was the first time that a mass definition from geodesic motion was given for non-asymptotically flat spacetimes.

We are now undertaking the evaluation of the energy integral (eq. (9)) for other asymptotic behaviours, and for a class of non-stationary spacetimes which present gravitational waves at asymptotic infinity. We also intend to examine the problem of positiveness of this energy integral.

REFERENCES

1) A. Einstein, Ann, Physik Ser. 4, 769 (1916)

2) C. Möller, Ann. Phys. 4, 347 (1958)

3) P.C. Vaidya, J. Univ. Bombay 21, 1 (1952)

4) L.D. Landau and E.M. Lifshtz, "The classical Theory of Fields" (Pergamon, Oxford, 1985) p. 280.

5) B.S. DeWitt, Phys. Rev. 160, 1113 (1967)

6) P.A.M. Dirac, Phys. Rev. 114, 924 (1959)

7) T. Regge and C. Teitelboim, Ann. Phys. 88, 286 (1974)

8) Throughout the paper we use units such that G = c = 1.

9) R. Arnowitt, S. Deser and C.W. Misner, in "Gravitation: An Introduction to Current Research" (L. Witten, Ed.), John Wiley and Sons, New York, 1962.

10) M. Henneaux and C. Teitelboim, Commun. Math. Phys. 98, 391 (1985).