ISSN 0029-3865

CBPF-NF-020/89

BOOTSTRAP VARIABLES IN EINSTEIN'S GENERAL RELATIVITY
(The role of the cosmological constant)

by

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Abstract

We present the proof that the role of the cosmological constant is to allow General Relativity to be described in terms of bootstrap variables. This means that the gravitational field has a potential and the field, on the other hand, is a potential for its potential.

Key-words: Cosmological constant; General relativity; Bootstrap variables.

One of the versions of Mach's principle gives a bootstrap character to the mass : the mass of any single body depends on the masses of all the other bodies in the Universe. This bootstrap idea of the role of the mass can in fact be sustained in standard field theory. Indeed, it is a consequence of any massive field theory that the mass is the entity that allows the description of the field in terms of bootstrap variables.

A theory is said to be described by bootstrap variables if it can be written in terms of a field F which admits a potential A and the field is, on the other hand, a potential for its potential. It follows, on the basis of this definition, that the equation of evolution of the theory can be written as

$$A = \frac{1}{\sigma^2}\, d_{(2)}\, (d_{(1)}\, (A))$$

in which $d_{(1)}$ and $d_{(2)}$ are differential operators (say, linear). The field F is defined in terms of A by

$$F = d_1\, (A)$$

and, on the other hand, the potential A is given in terms of the field

$$A = \frac{1}{\sigma^2}\, d_{(2)}\, F$$

For dimensional considerations $dim\ \sigma = M$ (in natural units $\hbar = c = 1$).

In the absence of the mass σ the knowledge of the potential in terms of the field is possible only through a kind of Kirchoff's integration throughout the whole domain Ω of the space-time in which the field F does not vanish.

The mass, so to speak, eliminates the need for a global knowledge of the field F throughout Ω in order to know the local value of the potential. In this way, the mass contains a sort of intrinsic (Kirchoff) globalization information.

Can this well-known result be applied to non-linear theories like, for instance, Einstein's General Relativity ? The answer is yes and, as one could guess, the role of the mass is played by the cosmological constant Λ. How can this be shown ?

In order to demonstrate this we will follow the work of Grishuk, Petrov and Popova[1] which uses the so-called Palatini first order formalism of General Relativity. These authors gave a very concise and simple demonstration of a result, previously developed by many others[2], that the full non-linear General Relativity Theory can be described in a completely equivalent way in terms of a field theory in an auxiliary Minkowski space-time.

Following the same procedure, as these authors we claim that General Relativity admits as bootstrap variables the tensors $\varphi^{\mu\nu}$ and $K^{\alpha}_{\mu\nu}$. Either $\varphi^{\mu\nu}$ or $K^{\alpha}_{\mu\nu}$ can be considered as the fundamental variable. The other one is then obtained in terms of first order derivatives of the fundamental variable. The cosmological constant Λ is the agent which allows one to demonstrate such an equivalence.

The proof of this is as follows.

Consider the Lagrangian

$$L_{(g)} = \frac{1}{2}\sqrt{-\gamma}\,(\gamma^{\mu\nu}+\varphi^{\mu\nu})\,[K^{\alpha}{}_{\mu\nu;\alpha} - K_{\mu;\nu} + (KK)_{\mu\nu}] +$$

$$+\,\Lambda\,|\gamma|\,\sqrt{|det\,(\gamma^{\mu\nu}+\varphi^{\mu\nu})|} \qquad (1)$$

in which

$$(KK)_{\mu\nu} \equiv K^{\alpha}{}_{\mu\nu}\,K_{\alpha} - K^{\alpha}{}_{\mu\beta}\,K^{\beta}{}_{\nu\alpha}$$

$K_{\alpha} \equiv K^{\epsilon}{}_{\alpha\epsilon}$ and $K^{\alpha}{}_{\mu\nu} = K^{\alpha}{}_{\nu\mu}$. The Minkowski metric is defined by $\gamma^{\mu\nu}$ written in an arbitrary system of coordinates and $\gamma = det\ \gamma_{\mu\nu}$. The symbol ; means covariant differentiation in terms of $\gamma_{\mu\nu}$ so that, for an arbitrary tensor F^{μ},

$$F^{\mu}{}_{;\nu} = F^{\mu}{}_{,\nu} + \gamma^{\mu}{}_{\epsilon\nu}\,F^{\epsilon}$$

and

$$\gamma^{\mu}{}_{\epsilon\nu} = \frac{1}{2}\,\gamma^{\mu\lambda}\,(\gamma_{\epsilon\lambda,\nu} + \gamma_{\epsilon\nu,\lambda} - \gamma_{\epsilon\nu,\lambda})$$

is the standard Christoffel symbol for the Minkowski metric. The theory thus written exhibits general covariance.

Following Grishuk et al. it is convenient to define a new quantity $g^{\mu\nu}$ in terms of the gravitational field $\varphi^{\mu\nu}$ by

$$\sqrt{-g}\; g^{\mu\nu} = \sqrt{-\gamma}\; (\gamma^{\mu\nu} + \varphi^{\mu\nu}) \tag{2}$$

and $g \equiv det\; g_{\mu\nu}$ [note : $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$ defined by $g_{\mu\nu}\; g^{\nu\lambda} = \delta^{\lambda}{}_{\mu}$. It is not giving by $g^{\alpha\beta}\; \gamma_{\alpha\mu}\; \gamma_{\beta\nu}$].

Independent variations of $\varphi^{\mu\nu}$ and $K^{\alpha}{}_{\mu\nu}$ yield from $L_{(g)}$ to the bootstrap equations (which, for simplicity, we write in terms of $g_{\mu\nu}$)

$$g_{\mu\nu} = -\frac{1}{\Lambda}\; [K^{\alpha}{}_{\mu\nu;\alpha} - \frac{1}{2}\; K_{(\mu;\nu)} + (KK)_{\mu\nu}] \tag{3a}$$

$$K^{\alpha}{}_{\mu\nu} = \frac{1}{2}\; g^{\alpha\lambda}\; [g_{\mu\lambda;\nu} + g_{\nu\lambda;\mu} - g_{\mu\nu;\lambda}] \tag{3b}$$

Using definition (2) one can show that the contracted curvature tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ associated to the "metric" $g^{\mu\nu}$ is given by

$$R_{\mu\nu} = -K^{\alpha}{}_{\mu\nu;\alpha} + K_{\mu;\nu} - (KK)_{\mu\nu} \tag{4}$$

If one wants then to pass from the first order formalism to the conventional second order formalism, one substitutes the expression for $K^{\alpha}{}_{\mu\nu}$ given by (3b) into the Lagrangian L_g (eq. (1)). This Lagrangian then reduces to

$$L_{(g)} = \frac{1}{2}\; \sqrt{-g}\; R_{\mu\nu}\; g^{\mu\nu} + \Lambda\; \sqrt{-g} \tag{5}$$

which exhibits its standard form, and equations (3) become

$$R_{\mu\nu} + \Lambda\; G_{\mu\nu} = 0$$

Note that the quantity $K^{\alpha}_{\mu\nu}$ is indeed a true tensor under arbitrary change of coordinates. In terms of the Christoffel symbol associated to the metric $g^{\mu\nu}$ defined by its standard expression

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\, g^{\alpha\lambda}\, \left(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right)$$

one finds

$$\Gamma^{\alpha}_{\mu\nu} = \gamma^{\alpha}_{\mu\nu} + K^{\alpha}_{\mu\nu}$$

We have thus given the proof that the role of the cosmological constant is just to provide the bootstrap character of the two tensors $\varphi_{\mu\nu}$ and $K^{\alpha}_{\mu\nu}$ which describes the gravitational field. Matter minimally coupled to gravity does not disturb this property.

Acknowledgements

One of us (M.N.) would like to thank Professor Yvonne Choquet-Bruhat for her hospitality at the Université de Paris and Dr Vincent Moncief for some interesting remarks.

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