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ON THE ATTRACTIVE OR REPULSIVE NATURE OF CASIMIR FORCE IN D-DIMENSIONAL MINKOWSKI SPACETIME

by

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ABSTRACT

The dependence of Casimir energy (associated to a massless scalar field) on spacetime dimensionality (D) is shown to be strongly entangled with the type of the geometric bounds and the kind of macroscopic boundary conditions imposed to the field. In the case of massless scalar field satisfying Dirichlet b.c. in the presence of a hyperparallelepipedal cavity with p sides of finite length L and D-p-1 sides with length much greater than L, a new compact integral formula, more suitable to analyze the nature of the Casimir force, is obtained. The force results to be attractive if p is odd or for very large even values of p, irrespective of D. For each small even p there exists a critical spacetime dimension $D_c(p)$ such that: the force is repulsive if $D < D_c$ and attractive otherwise. As a consequence of these results, the instability of the semiclassical Abraham-Lorentz-Casimir model of the electron is proved to depend on the spacetime dimensionality.

Key-words: Quantum field theory; Vacuum energy; Renormalization.

1. Introduction

The question whether the Casimir effect for different geometries and spacetime dimensions gives rise to an attractive or repulsive force between the configuration boundaries that confine the field (and its physical consequences) is up till now unsolved and will be discussed in this paper.

Actually, in the general case of a D-dimensional spacetime the sign of the Casimir energy may depend on: i) the spacetime dimensionality, ii) the type of boundary conditions, iii) the number (p) of independent directions with finite extension of the space region that constrains the fields, iv) the ratio of these finite characteristic lengths, v) other topological features of spacetime (e.g. compactness), vi) the spacetime metric, vii) the temperature.

In this paper the consequences of (iv-vii) will not be discussed and the interested reader is referred to [1-10]. However, it is useful to note that in Ref. [1] it is argued that the Casimir energy associated with an electromagnetic field quantized inside a perfectly conducting box of sides L_1, L_2 and L_3, may change sign depending on the relative lengths, which indeed suggests a strong dependence on p. In Ref. [2] a similar behavior is shown to occur in the case of a massless scalar field in a three-dimensional parallelepipedal cavity with Dirichlet boundary conditions.

That there is a dependence on the geometric shape of the cavity is evident from early exact computations of Casimir energy at zero temperature, associated with massless scalar and electromagnetic fields, for a few different kinds of geometric configurations, like parallel plates [11-14], a cube [1,15], a cylinder [16-18] and a spherical shell [19-22] in four-dimensional Minkowski spacetime.

The search for a unification scheme of forces in higher dimensional spacetime renders of physical interest the question of what one can learn about particular features of the Casimir effect in going to spaces with an arbitrary number of dimensions [23-26]. For a massless scalar field quantized inside a box with p sides of finite characteristic length of order L and D-1-p sides with characteristic length $\lambda \gg L$, the sign of the Casimir energy density $\bar{\varepsilon}_D^{(p)}$ depends crucially on the boundary conditions [23]. In the case of Neumann and periodic boundary conditions it is straightforward to see that $\bar{\varepsilon}_D^{(p)}$ is always negative. For Dirichlet boundary conditions, numerical computations for different values of D and p up to D=6 seem to indicate that the sign of $\bar{\varepsilon}_D^{(p)}$ depends on whether p is even or odd [23].

The main aim of this paper is to throw some light on this kind of dependence and it is organized as follows: in Sect. 2 the expression for the Casimir energy density $\bar{\varepsilon}_D^{(p)}$ in a D-dimensional Minkowski spacetime is obtained in terms of a sum of Epstein functions. In Sect. 3 the summation is performed yielding a new compact integral representation of $\bar{\varepsilon}_D^{(p)}$ which is more suitable to discuss its sign. In particular, the above conjecture that the sign of $\bar{\varepsilon}_D^{(p)}$ depends on whether p is even or odd is discussed and demonstrated to be only partially true. Indeed, when the number of finite and equal edges of a rectangular box (p) is *odd* it is analytically shown that $\bar{\varepsilon}_D^{(p)}<0$ for *any* D. However when p is *even* we get a new and surprising result, namely: either the Casimir energy is always negative irrespective of the value of D (this happens only when p =30) or there exists a particular critical value of spacetime dimension (D_c), which depends on p, such that for $D<D_c$, $\bar{\varepsilon}_D^{(p)} \geq 0$ and for $D \geq D_c$, $\bar{\varepsilon}_D^{(p)}$ is shown to be always negative. Numerical results, giving an account of the dependence of D_c on p, as well as the Casimir energy densities for several combinations of D and p are given in Table I. Some concluding remarks are presented in Sect. 4. In particular it is argued that, as a consequence of the new results mentioned above, it comes out a new insight on why, in a four-dimensional Minkowski spacetime, the semiclassical Abraham-Lorentz-Casimir model of the electron fails.All these results, for the sake of simplicity, were obtained in a D-dimensional noncompact (Minkowskian) manifold where rectangular cavities are constructed to trap the scalar field.

Useful mathematical results are given in the Appendix.

2. Casimir Energy of a Hermitian massless scalar field in a D-dimensional Minkowski spacetime

In this Section some problems concerning the zero-point energy in Quantum Field Theory are briefly discussed and the expression of the Casimir energy density is obtained for the simple case of the scalar field.

To quantize a Classical Field Theory in a canonical quantization scheme one essentialy needs to know: the operator algebra, the dynamical equations and how to construct physical states (including the vacuum state). In such a scheme, the ordering of non-commuting operators in the field Hamiltonian is not fixed *a priori*, giving rise to zero-point energies which are divergent. Thus, one may wonder how zero-point fluctuations and the associated energy should be interpreted.

This problem may be circumvented by arguing that actually one does not measure absolute energy values, but only energy differences. This is exactly what is done when Wick's normal ordering is imposed to the field operators. Following this prescription, an infinite amount of energy is subtracted from the vacuum state in such a way that the net energy results to be zero. An important argument supporting this procedure is based on the demonstration that the expected value of the energy-momentum tensor in the vacuum state should be zero to ensure that the correct commutation relations of the Lie algebra are indeed satisfied by the generators of the Poincaré group [27]. This result clearly depends on the fact that vacuum state is defined on a space with no bounds. But what happens when fields are to be quantized in a confined spatial region? How does one interpret zero-point fluctuations and the associated energy in such cases where the Poincaré symmetry of spacetime is globally broken?

The interest in these questions was sharpened after Casimir's work [28]. In 1948, he showed that neutral perfectly conducting parallel plates in vacuum attract each other. Experimental verifications of this effect (known as Casimir effect) [29] show how vacuum fluctuations may give rise to measurable quantities and, therefore, are an evidence that, in general, the vacuum state energy of a quantized field may be not well defined by normal ordering. Then the above vacuum state definition is to be revised.

As stressed in [30], Casimir adopted a new concept of vacuum energy, by assuming that "a meaningful definition of the physical vacuum must take into account that in a real situation quantum fields always exist in the presence of external constraints, i.e., in interaction with matter or other external fields. An idealized description of such circumstances is obtained by forcing the field to satisfy certain boundary conditions". In other words, according to Casimir, the energy of the physical vacuum state is defined as the field energy in the presence of minus its value in the absence of such conditions (this is what is often called Casimir energy). So, it is clear that the Casimir energy may, in principle, depend on the particular choice of the geometry defined by the confining configuration, on the topology of space where the field is given and on the type of boundary conditions.

In Minkowski spacetime such a scheme can be implemented for simple geometric configurations as mentioned in Sect. 1.

Different renormalization techniques [30] are used in those calculations and can be classified as particular cases of two general methods. One of them is the Green's function method [13,31,32] and the other one is based on the direct evaluation of an infinite sum over all normal modes, allowed for those particular geometries, which can be implemented by using a cut-off method [11,12,28], the zeta-function technique [14,15] or a dimensional regularization procedure [23]. Although the Green's function method is known to be more fundamental, it presents technical difficulties so far parallelepipedal geometric configurations ($p > 2$) are considered, which are swept away by the zeta-function techniques. Hence the later method will be adopted throughout this paper.

A Hermitian massless scalar field $\Phi(x^0,x^i)$ defined in a D-dimensional Minkowski spacetime should satisfy a generalized Klein-Gordon equation (in Cartesian coordinates with $\hbar=c=1$);

$$\left[\left(\frac{\partial}{\partial x^0}\right)^2 - \sum_{j=1}^{D-1} \left(\frac{\partial}{\partial x^j}\right)^2 \right] \Phi(x^0,x^i) = 0 \qquad (2.1)$$

If the field is confined in the interior of a (D-1)-dimensional rectangular cavity with edges L_1, L_2, ..., L_{D-1}, it can be expanded over the complete orthonormal set of mode solutions $\{\phi_n, \phi_n^*\}$ as follows:

$$\Phi(x) = \sum_{n_1 n_2 \ldots n_{D-1} = 1}^{\infty} \left[a_{n_1 n_2 \ldots n_{D-1}} \phi_{n_1 n_2 \ldots n_{D-1}} + a^{\dagger}_{n_1 n_2 \ldots n_{D-1}} \phi^*_{n_1 n_2 \ldots n_{D-1}} \right] \qquad (2.2)$$

Imposing Dirichlet boundary conditions on the box surface $\partial\Omega$, i.e., $\Phi(x)|_{\partial\Omega} = 0$, we have

$$\phi_{n_1 n_2 \ldots n_{D-1}} = f_{n_1 n_2 \ldots n_{D-1}}(x^0) \left(\sin \frac{n_1 \pi}{L_1} x_1 \ \sin \frac{n_2 \pi}{L_2} x_2 \ldots \ldots \sin \frac{n_{D-1}\pi}{L_{D-1}} x_{D-1} \right) \qquad (2.3)$$

where the n_j are positive integers. In the canonical quantization scheme $a^{\dagger}_{n_1 n_2 \ldots n_{D-1}}$ and $a_{n_1 n_2 \ldots n_{D-1}}$ are creation and annihilation operators of field quanta with energy spectrum $\omega_{\{n\}}$ given by ($\{n\}$ stands for a short notation of $n_1 n_2 \ldots n_{D-1}$):

$$\omega_{\{n\}} = \left[\left(\frac{n_1 \pi}{L_1}\right)^2 + \left(\frac{n_2 \pi}{L_2}\right)^2 + \ldots + \left(\frac{n_{D-1}\pi}{L_{D-1}}\right)^2 \right]^{1/2} \qquad (2.4)$$

Different boundary conditions can be imposed on the surface ($\partial\Omega$), like Neumann ($\hat{n}_i \ \partial^i \Phi(x)|_{\partial\Omega} = 0$ where $\hat{n}$ is a unitary vector orthogonal to the surface $\partial\Omega$) or periodic conditions (as in a torus). Since, for both conditions, the Casimir energy density is shown to be always negative for any value of D and p [23], the link between the attractive or repulsive nature of the Casimir effect and the geometry will be discussed hereafter only for Dirichlet boundary conditions.

Let us briefly review how the Casimir energy inside a (D-1)-dimensional rectangular box is obtained from the divergent expression

$$E_D(L_1, L_2, \ldots, L_{D-1}) = \frac{1}{2} \sum_{n_1 n_2 \ldots n_{D-1} = 1}^{\infty} \left[\left(\frac{n_1 \pi}{L_1}\right)^2 + \left(\frac{n_2 \pi}{L_2}\right)^2 + \ldots + \ldots + \left(\frac{n_{D-1}\pi}{L_{D-1}}\right)^2 \right]^{1/2} \qquad (2.5)$$

In the limit

$$L_1,\ L_2,\ \ldots,\ L_p \ll L_{p+1},\ L_{p+2},\ \ldots,\ L_{D-1}$$

and, for simplicity (without loss of generality for our purpose) assuming $L_1 \simeq L_2 \simeq \ldots \simeq L_p = L$, we define the energy density (energy per unit hyperarea), $\varepsilon_D^{(p)}(L)$, as a function of the number p of finite length edges:

$$\varepsilon_D^{(p)}(L) \equiv \frac{E_D^{(p)}}{\prod_{i=p+1}^{D-1}(L_i)} = \frac{1}{(2\sqrt{\pi})^{D-p-1}} \frac{1}{\Gamma\left(\frac{D-p-1}{2}\right)} \times$$

$$\times \sum_{n_1 n_2 \ldots n_p = 1}^{\infty} \int_0^{\infty} dr\, r^{D-p-2}\left[r^2 + \left(\frac{n_1 \pi}{L}\right)^2 + \ldots + \ldots + \left(\frac{n_p \pi}{L}\right)^2 \right]^{1/2} \qquad (2.6)$$

This divergent density may be regularized using the techniques of Ref. [14,15] giving the finite value for the Casimir energy:

$$\bar{\varepsilon}_D^{(p)}(L) = \frac{L^{p-D}}{2^{D+1}} \sum_{q=0}^{p-1} (-1)^{q+1} C_p^q (\sqrt{\pi})^{q-D}\, \Gamma\left(\frac{D-q}{2}\right) A(1,\ldots,1;D-q) \qquad (2.7)$$

where there are p-q terms (1, ..., 1) as arguments of the Epstein function defined as [33]

$$A(a_1,a_2,\ldots,a_k;2s) = \sum_{n_1 n_2 \ldots n_k = -\infty}^{\infty}{}' \left[a_1 n_1^2 + a_2 n_2^2 + \ldots + a_k n_k^2 \right]^{-s} \qquad (2.8)$$

for $a_k > 0$ and the prime means that the term $n_1 = n_2 = \ldots = n_k = 0$ has to be excluded.

For p=1 the same result as published in Refs. [23,24,25], is obtained:

$$\bar{\varepsilon}_D^{(1)}(L) = -\,(2\sqrt{\pi})^{-D}\, L^{1-D}\, \Gamma(D/2)\, \zeta(D) \qquad (2.9)$$

where $\zeta(D)$ is the Riemann zeta-function.

3. The sign of the Casimir energy density

One can see from Eq.(2.7) that it is not straightforward to ascertain the sign of the Casimir energy density. To carry on this discussion it is convenient to use the following integral representation for the Gamma functions,

$$u^{-s}\,\Gamma(s) = \int_0^\infty dt\; e^{-ut}\; t^{s-1}$$

to write

$$\Gamma(s)\;A(1,\ldots,1;2s) = \int_0^\infty dt\; t^{s-1}\Big[\sum_{n_1\ldots n_k=-\infty}^{\infty}{}' \exp[-(n_1^2+\ldots+n_k^2)t]\Big]$$

$$= \int_0^\infty dt\; t^{s-1}\Big[\Big[\sum_{n=-\infty}^{\infty}{}' e^{-n^2 t}+1\Big]^k - 1\Big] \qquad (3.1)$$

yielding

$$\bar{\varepsilon}_D^{(p)}(L) = \frac{L^{p-D}}{2^{D+1}}\sum_{q=0}^{p-1}(-1)^{q+1}\,C_p^q\,(\sqrt{\pi})^{q-D}\int_0^\infty dt\,(\sqrt{t})^{D-q-2}\;\times$$

$$\times\Big[\Big(2\sum_{n=1}^{\infty} e^{-n^2 t}+1\Big)^{p-q}-1\Big] \qquad (3.2)$$

Now, performing the sum over q and expressing the integrand in terms of the Elliptic theta function [34] $\vartheta_3(0,e^{-t})$, it results from Eq. (3.2) that

$$\bar{\varepsilon}_D^{(p)}(L) = \frac{L^{p-D}}{2^{D+1}}\;\pi^{-D/2}\int_0^\infty dt\;(\sqrt{t})^{D-2}\Big[\Big(1-\sqrt{\frac{\pi}{t}}\Big)^p +$$

$$-\Big(\vartheta_3(0,e^{-t})-\sqrt{\frac{\pi}{t}}\Big)^p\Big] \qquad (3.3)$$

and this new formula allows one to disclose the relationship between D and p, and the sign of the Casimir energy density.

It is convenient to study separately the case where p = odd and p = even.

For *odd* values of p it is obvious that $\bar{\varepsilon}_D^{(p)}(L) < 0$ for every p (independently of D), since $\vartheta_3(0,e^{-t}) > 1$ and the integrand is always negative between the integration limits.

When p is *even* the above argument clearly does not apply. Numerical calculations show that, in this case, the energy density is positive for D ≤ 6 [23]. However, using Eq. (2.7) for p=2, it can be shown that the energy density becomes negative for integer values D ≥ 7. Its behaviour for an arbitrary p=2*j* will be discussed now and this situation will be shown to be no longer valid for a certain large value of p. Defining

$$g(t) = \left(1 - \sqrt{\frac{\pi}{t}}\right)^{2j} - \left(1 - \sqrt{\frac{\pi}{t}} + 2\sum_{n=1}^{\infty} e^{-n^2 t}\right)^{2j}, \qquad (3.4)$$

an analysis of this function shows that: it has only one real root t_0(the same for every $j > 0$, lying between 0.6π and π), it is positive for $0 < t < t_0$ and negative for $t_0 < t < \infty$ and that $\int_0^\infty dt\, t^\alpha g(t)$ does exist whenever $\alpha \geq \frac{p-1}{2}$.

A qualitative study of the integral of Eq. (3.3) is detailed in the Appendix and the results are summarized here:

a) $\lim_{k\to\infty} \int_0^\infty dt\, t^{\alpha+k} g(t) = -\infty$

b) For $\alpha_2 > \alpha_1 \geq \frac{p-1}{2}$, if $\int_0^\infty dt\, t^{\alpha_1} g(t) \leq 0$

then $\int_0^\infty dt\, t^{\alpha_2} g(t) < 0$.

c) $$\lim_{p=2j\to\infty}\int_0^\infty dt\, t^{(p-1)/2} g(t) = -\infty .$$

So, since the energy density, given by (3.3), for p even is proportional to $\int_0^\infty dt\, t^{\alpha}\, g(t)$ we can conclude that in this case either $\bar{\varepsilon}_D^{(p)}$ is always negative (for $p \geq 30$ as suggested by numerical calculations given in Table I), which comes from *(c)*, or, from *(a)* and *(b)*, there exist a critical value of the spacetime dimension D_c for which

$$p + 1 \leq D < D_C \Rightarrow \bar{\varepsilon}_D^{(p)} \geq 0$$

$$D \geq D_C \Rightarrow \bar{\varepsilon}_D^{(p)} < 0.$$

The values of D_c for p even, $2 \leq p \leq 30$, with the respective energy density values are given in Table I.

4. Discussions

In spite of an impressive literature on the Casimir effect [30], the query whether its attractive or repulsive character changes by going to higher dimensions had never been elucidated for Dirichlet boundary conditions. In this paper the nature of the Casimir force associated to a massless scalar field trapped inside a rectangular box (with p finite and equal edges and D-p-1 infinite edges) in a D dimensional spacetime is discussed for different values of p. For Dirichlet b.c., a very peculiar dependence between the nature of Casimir force and the value of p and the spacetime dimension is shown to exist. When p is odd the force is always attractive whatever the value of D. For p even and not very large, there does exist a critical spacetime dimensionality (D_c) for which: the force is repulsive if $D < D_c$, and it is attractive if $D \geq D_c$. On the other hand, if p is large enough (≥ 30 as suggested by Table I) one is sure that the Casimir force is always attractive. In addition, since any configuration with even $p < 30$ has $D_c < 30$ (see Table I), and from further numerical calculations for $p > 30$, one can infer that the nature of the Casimir force does not depend anymore on p and D for $D \geq 30$; it is always attractive. It is important to stress that these results could only be obtained by using Eq. (3.3).

That several physical effects may be qualitatively different by varying the spacetime dimensionality is not a new feature [35]. So, in a certain sense, the Casimir effect can be considered as another example, but it is important to make the exception that its dependence on D is strongly entangled with the dependence on the geometric bounds (p) and on the kind of macroscopic boundary conditions imposed to the field. A further unexpected dependence of the Casimir effect on the number of spacetime dimensions comes from Ref. [4], where it is shown that the zero point energy, associated to a massless scalar field defined in an $M^4 x S^{D-4}$ manifold, is well defined if D is odd, but, when D is even this energy is logarithmically divergent. This is perhaps the first example in the literature where "physics seems to prefer D odd".

The results obtained in this paper permit us to understand why the old semiclassical Casimir model for a spinless electron [36] is unstable in a four-dimensional spacetime. In such a scheme, despite the criticism to Casimir model [36], we believe it is useful to deepen our understanding concerning its instability. In a few words, the main point of this model is the suggestion that there should exist a stress of quantum electromagnetic origin (Poincaré stress) to assure the stability of the Abraham-Lorentz electron, modelled as an spherical conducting shell. However, in 1968, the Casimir energy for this configuration was found to be positive [19], giving rise to a repulsive stress (to be added to the Maxwell stress), contrary to what was expected by Casimir (an alternative stable model for the electron was proposed in Ref. [37]).

The bridge connecting our results for scalar fields with the electromagnetic case is the useful formula relating the Casimir energy associated to a massless vector field, in the presence of a rectangular cavity with walls of infinite conductivity and with p equal finite edges, and the Casimir energy of a scalar field satisfying Dirichlet b.c. [23]:

$$^{(e.m)}\varepsilon_D^{(p)}(L) = (D-2)\ \bar{\varepsilon}_D^{(p)}(L) + p\ \bar{\varepsilon}_{D-1}^{(p-1)}\ (L)$$

Use will also be made of the well known fact that if we deform a spherical shell of radius a into a cubic shell of length L with $L \approx 2a$ the magnitude of the Casimir energy almost does not change, as shown in [1,38], allowing us to replace hyperspherical shells S^n by hypercubes with n+1 sides and use the above formula. In the four dimensional spacetime it is easy to see from this equation that the Casimir energy of an S^2 electron is positive [23]. Does this result still hold for higher dimensional flat spacetimes? The answer is no, and it can be shown that the zero-point electromagnetic energy could assure electron's stability in higher dimensional spacetimes if two Casimir's like models for the electron are assumed.

As a first example, the electron could be modelled by a hyperspherical shell (S^{D-2}) with p=D-1, and $p-1 \geq p_c=30$ for D even and $p \geq p_c=30$ for D odd. In such a situation we can infer from our results that $^{(e.m)}\varepsilon_D^{(p)}$ is always negative and, therefore, the electron could be stable. Note that, in this case, the condition of stability will be fulfilled only for a particular electron radius.

Another possible model could be an infinite pipe with a geometry $S^2 \times R^{D-4}$ in a D-dimensional $M^4 \times R^{D-4}$ manifold, where S^2 is contained in the observable three-dimensional space. In this case p=3 and we have:

$$^{(e.m)}\varepsilon_D^{(3)}(L) = (D-2)\ \bar{\varepsilon}_D^{(3)}(L) + 3\ \bar{\varepsilon}_{D-1}^{(2)}\ (L)$$

Looking at this equation it becomes clear that, for $D-1 < D_c$ (p=2) (which is 7), if the positive contribution of the second term of the right hand side is greater than that of the first term (always *negative* as proved in this paper), the Casimir model for the electron is invalidated. This is indeed the case for $D < 8$, generalizing then the proof given in [19] for D=4 (where both models concide).

If, instead, D-1 ≥ D_c (p=2) both $\bar{\varepsilon}_D^{(3)}(L)$ and $\bar{\varepsilon}_{D-1}^{(2)}(L)$ are always negative, which can give rise to a stable semiclassical model for the electron. Thus, the critical dimension for the electromagnetic case (with p=3) is D=8.

In conclusion we have shown that the Poincaré stress could have a quantum electromagnetic origin only if we lived in a higher dimensional flat spacetime.

As a last remark, we would like to note that, although in the classical level the electrostatic energy inside a cavity does not depend on whether Dirichlet or Neumann b.c. are imposed to the field, the quantum zero-point energy strongly depends on the choice of the macroscopic boundary conditions.

The results presented in this paper, together with other open questions stressed in [30], compel us to share the opinion that the Casimir effect is still, in essence, a poorly understood phenomenon.

Appendix:

Proof of a): For $k > 0$ and $\varepsilon > 0$

$$\int_0^\infty dt\ t^{\alpha+k}\ g(t) = \int_0^{t_0} dt\ t^{\alpha+k}\ g(t) + \int_{t_0}^{t_0+\varepsilon} dt\ t^{\alpha+k}\ g(t) + \int_{t_0+\varepsilon}^{\infty} dt\ t^{\alpha+k}\ g(t) <$$

$$< \int_0^{t_0} dt\ t^{\alpha+k}\ g(t) + \int_{t_0+\varepsilon}^{\infty} dt\ t^{\alpha+k}\ g(t) <$$

$$< t_0^k \int_0^{t_0} dt\ t^{\alpha}\ g(t) + (t_0+\varepsilon)^k \int_{t_0}^{\infty} dt\ t^{\alpha}\ g(t) =$$

$$= (t_0+\varepsilon)^k \left[\left(\frac{t_0}{t_0+\varepsilon} \right)^k a - b \right]$$

where $a \equiv \int_0^{t_0} dt\ t^{\alpha}\ g(t) > 0$

and $b \equiv - \int_{t_0+\varepsilon}^{\infty} dt\ t^{\alpha}\ g(t) > 0$

Thus

$$\lim_{k \to \infty} \int_0^\infty dt\ t^{\alpha+k}\ g(t) < \lim_{k \to \infty} (t_0+\varepsilon)^k \left[\left(\frac{t_0}{t_0+\varepsilon} \right)^k a - b \right] =$$

Since c is arbitrary, it can be conveniently chosen to make $(t_0+ c)^k$ greater than the coefficient of the integral of Eq. (3.3) and, therefore, the renormalized energy $\bar{\varepsilon}_D^{(p)} \to -\infty$ in this limit.

Proof of b): For $k = \alpha_2 - \alpha > 0$,

$$\int_0^\infty dt\ t^{\alpha_2}\, g(t) = \int_0^{t_0} dt\ t^{\alpha+k}\, g(t) + \int_{t_0}^\infty dt\ t^{\alpha+k}\, g(t) <$$

$$< t_0^k \int_0^{t_0} dt\ t^{\alpha}\, g(t) + t_0^k \int_{t_0}^\infty dt\ t^{\alpha}\, g(t) <$$

$$< t_0^k \int_0^\infty dt\ t^{\alpha}\, g(t) \le 0$$

Proof of c):

We want to discuss the behaviour of the integral below with p;

$$I_p = \int_0^{t_0} dt\ t^{(p-1)/2}\, g(t) + \int_{t_0}^\infty dt\ t^{(p-1)/2}\, g(t)$$

For $0 < t < t_0$

$$g(t) = \left(1 - \sqrt{\frac{\pi}{t}}\right)^p - \left(2\,h(t) + 1 - \sqrt{\frac{\pi}{t}}\right)^p <$$

$$< \left(1 - \sqrt{\frac{\pi}{t}}\right)^p < \left(\sqrt{\frac{\pi}{t}}\right)^p$$

where $h(t) = \sum_{n=1}^{\infty} e^{-n^2 t}$. So

$$\int_0^{t_0} dt\, t^{(p-1)/2}\, g(t) < 2\, \pi^{p/2}\, t_0^{1/2}$$

$$I_p < 2\, \pi^{p/2}\, t_0^{1/2} - \int_{t_0}^{\infty} dt\, t^{(p-1)/2}\, |g(t)|$$

For $t > t_0$, it is used the mean value theorem

$$|g(t)| = \left(2\, h(t) + 1 - \sqrt{\frac{\pi}{t}} \right)^p - \left(1 - \sqrt{\frac{\pi}{t}} \right)^p =$$

$$= 2\, p\, h(t) \left[\left(1 - \sqrt{\frac{\pi}{t}} \right) + \theta_{p,t}\, 2\, h(t) \right]^{p-1}$$

with $0 < \theta_{p,t} < 1$. So

$$t^{(p-1)/2}\, |g(t)| > 2\, p\, t^{(p-1)/2} \left(1 - \sqrt{\frac{\pi}{t}} \right)^{p-1} h(t)$$

For $t > 4\pi > t_0$,

$$t^{(p-1)/2}\, |g(t)| > 2^{2-p}\, p\, h(t)\, t^{(p-1)/2}$$

Thus

$$\int_{t_0}^{\infty} dt\, t^{(p-1)/2}\, |g(t)| > \int_{4\pi}^{\infty} dt\, t^{(p-1)/2}\, |g(t)| > \frac{p}{2^{p-2}} \int_{4\pi}^{\infty} dt\, t^{(p-1)/2}\, h(t)$$

$$> \frac{p}{2^{p-2}} \int_{64\pi}^{\infty} dt\, t^{(p-1)/2}\, h(t) > 2p\, (4\sqrt{\pi})^{p-1} \int_{64\pi}^{\infty} dt\, h(t)$$

Thus

$$I_p < 2\, \pi^{p/2}\, t_0^{1/2} - 2p\, (4\sqrt{\pi})^{p-1} \int_{64\pi}^{\infty} dt\, h(t)$$

$$I_p < 2\,(4\sqrt{\pi})^p \left(4^{-p}\sqrt{\bar{\varepsilon}_0} - 2\pi^{-1/2}p \int_{64\pi}^{\infty} dt\ h(t) \right)$$

Then, for p (even) large enough,

$$\int_0^{\infty} dt\ t^{(p-1)/2} g(t) < 0$$

and $\bar{\varepsilon}_D^{(p)} < 0$ for any $D \geq p-1$ (this result does not depend on the coefficient of Eq. (3.3)). In this case there is no critical D since all $\bar{\varepsilon}_D^{(p)}$ (for D = p+1, p+2, ...) are negative.

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p \ u	0	1	2	3	4	D_c
2	4.1×10^{-2}	4.8×10^{-3}	8.1×10^{-4}	1.2×10^{-4}	-1.9×10^{-5}	7
4	6.2×10^{-3}	5.0×10^{-4}	6.2×10^{-5}	4.7×10^{-6}	-3.9×10^{-6}	9
6	1.1×10^{-3}	6.8×10^{-5}	6.4×10^{-6}	7.3×10^{-8}	-7.0×10^{-7}	11
8	2.2×10^{-4}	1.0×10^{-5}	7.8×10^{-7}	-6.4×10^{-8}		12
10	4.4×10^{-5}	1.8×10^{-6}	9.8×10^{-8}	-2.7×10^{-8}		14
12	9.4×10^{-6}	3.2×10^{-7}	1.0×10^{-8}	-1.0×10^{-8}		16
14	2.0×10^{-6}	6.0×10^{-8}	-1.6×10^{-10}			17
16	4.5×10^{-7}	1.1×10^{-8}	-8.4×10^{-10}			19
18	1.0×10^{-8}	1.9×10^{-9}	-5.4×10^{-10}			21
20	2.2×10^{-8}	2.4×10^{-10}	-3.1×10^{-10}			23
22	5.0×10^{-9}	-4.4×10^{-11}				24
24	1.1×10^{-9}	-6.9×10^{-11}				26
26	2.3×10^{-10}	-5.6×10^{-11}				28
28	3.0×10^{-11}	-4.4×10^{-11}				30
30	-1.1×10^{-11}					31

TABLE I

TABLE I - Casimir energy densities for massless scalar fields satisfying Dirichlet boundary conditions inside a hyperparallelepipedal cavity with p even unit sides and u sides much greater than one in a D-dimensional spacetime with D=p+u+1. D_c is the critical dimension for each value of p.

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