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Suplattices Associated with Convex Sets, Convex Cones and Affine Spaces

by

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ABSTRACT: *A study of natural quasiorder on a convex set whose quotient is a suplattice. When the convex set is a convex cone or an affine space, there are specializations. A typical result is that an affine space is vectorial iff its quasiorder is chaotic.*

KEY WORDS: Supplatice; convexity; vectoriality.

RESUMO: *RETICULADOS ASSOCIADOS A CONJUNTOS CONVEXOS, CONES CONVEXOS E ESPAÇOS AFINS. Um estudo de quasiordem natural num conjunto convexo cujo quociente é um supreticulado. Quando o conjunto convexo é um cone convexo ou um espaço afim, há especializações. Um resultado típico é que um espaço afim é vetorial se e só se sua quasi ordem é caótica.*

PALAVRAS-CHAVE: Supreticulado; convexidade; vetorialidade.

1. INTRODUCTION

A convex set X has a quasiorder on it R that defines an equivalence relation $e(R)$ (see Definition 8). The quotient convex set $X/e(R)$ is a suplattice associated with X (see Proposition 9). A subset T of X is increasing when $t \in T$, $x \in X$, $t \leq x$ imply $x \in T$, it following that T is a convex subset of X (see Definition 12). If V is a strongly convexily independent nonvoid subset of a real vector space E and X is the convex subset of E generated by V, namely a simplex of E, there is a bijection between $X/e(R)$ and the suplattice of all nonvoid finite subsets of V, each such subset of V determining an open face of X (see Example 13). If X is a suplattice considered as a convex set (see Definition 7), then R is an order, $e(R)$ is equality, and the suplattice X is isomorphic with $X/e(R)$. Conversely, if X is a convex set, the convex set structure of $X/e(R)$ derives from its suplattice structure (see Definition 7), hence the suplattice structure of $X/e(R)$ derives from its convex set structure (see Proposition 14). If X is a convex set, $e(R)$ is equality iff the convex set structure of X derives from a necessarily unique suplattice structure on X (see Corollary 15). Every suplattice is isomorphic to the suplattice associated with some convex set (see Remark 16). When the convex set X is a convex cone or an affine space, there are specializations (see Propositions 17 and 19). If $X = A(E)$ is the affine space of all nonvoid affine subspaces of a real vector space E, then $X/e(R)$ is isomorphic to the suplattice $V(E)$ of all vector subspaces of E (see Example 20). An affine space X is vectorial iff its quasiorder R is chaotic, that is $x_1 \leq x_2$ for all $x_1, x_2 \in X$ (see Proposition 24).

2. NOTATION AND TERMINOLOGY

Notation 1. We denote by $\mathbf{N}$ the system of all strictly positive integers, $\mathbf{R}$ the system of all real numbers, $\mathbf{R}^*$ the system of all real numbers different from zero, $\mathbf{R}^*_+$ the system of all strictly positive real numbers, and $\mathbf{J}$ the open interval of $\mathbf{R}$ of extremities $0, 1$.

We refer to the Bibliography at the end for convexity. We review here just a bare minimum.

Definition 2. A *convex set* X is a set in which we are given a *convex combination map* that to every $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \cdots, x_n \in X$ associates

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = \sum_{1 \leq i \leq n} \lambda_i x_i \in X$$

so that the following axioms hold:

Commutativity. If $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \cdots, x_n \in X$ and σ is a permutation of $\{1, \cdots, n\}$, then

$$\sum_{1 \leq i \leq n} \lambda_{\sigma(i)} x_{\sigma(i)} = \sum_{1 \leq i \leq n} \lambda_i x_i.$$

Associativity. If $m, n, m_j \in \mathbf{N}^*$ $(j = 1, \cdots, n)$, λ_{ij}, $\mu_j \in \mathbf{J}$ $(i = 1, \cdots, m_j, j = 1, \cdots, n)$, $\sum_{1\leq i\leq m_j} \lambda_{ij} = 1$ $(j = 1, \cdots, n)$, $\sum_{1\leq j\leq n} \mu_j = 1$, $x_{ij} \in X$ $(i = 1, \cdots, m_j, j = 1, \cdots, n)$, then

$$\sum_{1\leq j\leq n} \mu_j \left(\sum_{1\leq i\leq m_j} \lambda_{ij} x_{ij} \right) = \sum_{\substack{1\leq i\leq m_j \\ 1\leq j\leq n}} (\mu_j \lambda_{ij}) x_{ij}.$$

Distributivity. If $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x \in X$, then $\lambda_1 x + \cdots + \lambda_n x = x$.

Definition 3. A *convex cone* X is a set in which we are given two maps, an *addition* $(x_1, x_2) \in X \times X \mapsto x_1 + x_2 \in X$ and a *multiplication* $(\lambda, x) \in \mathbf{R}_+^* \times X \mapsto \lambda x \in X$, so that the following axioms hold: $x_2 + x_1 = x_1 + x_2$, $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$, $\lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2$, $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$, $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x$, $1x = x$ for all λ, λ_1, $\lambda_2 \in \mathbf{R}_+^*$, x_1, x_2, x_3, $x \in X$. A convex cone is a convex set.

Definition 4. An *affine space* X is a set in which we are given an *affine combination map* that to every $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{R}^*$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \cdots, x_n \in X$ associates

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = \sum_{1\leq i\leq n} \lambda_i x_i \in X$$

so that the following axioms hold: *Commutativity, Associativity, Distributivity* for affine spaces have the same formulation as in Definition 2 of convex sets provided we replace $\mathbf{J}$ by $\mathbf{R}^*$. An affine space is a convex set.

Definition 5. A *quasiorder* on a set X is a binary relation on it that is reflexive and transitive. An *order* on X is a quasiorder on it that is antisymmetric. A quasiorder on X defines an equivalence relation on it. A *quasisuplattice* is a quasiordered set X in which any two elements x_1, $x_2 \in X$ have a quasisupremum. A *suplattice* is an ordered set X in which any two elements x_1, $x_2 \in X$ have a supremum $x_1 \vee x_2 \in X$.

Definition 6. Let $R \subset X \times Y$ be a binary relation between two sets X, Y. If X, Y are convex sets, we say that R is *compatible* with the convex set structures of X, Y when R is a convex subset of $X \times Y$. Likewise by replacing convex set by convex cone and affine space.

Definition 7. Let X be a suplattice. X becomes a convex set if we define $\lambda_1 x_1 + \cdots + \lambda_n x_n = x_1 \vee \cdots \vee x_n$ for all $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \cdots, x_n \in X$. Also X becomes a convex cone if we define $x_1 + x_2 = x_1 \vee x_2$, $\lambda x = x$ for all $\lambda \in \mathbf{R}_+^*$, x_1, x_2, $x \in X$. Moreover X becomes an affine space if we define $\lambda_1 x_1 + \cdots + \lambda_n x_n = x_1 \vee \cdots \vee x_n$ for all $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{R}^*$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \cdots, x_n \in X$. A convex set X derives from a suplattice in this way iff convex combinations are constant, that is $\lambda_1 x_1 + \cdots + \lambda_n x_n$ is independent of $\lambda_1, \cdots, \lambda_n$ for all $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \cdots, x_n \in X$. A convex cone X derives from a suplattice in this way iff multiplications are constant, that is $\lambda x = x$ for all $\lambda \in \mathbf{R}_+^*$, $x \in X$. An affine space X derives from a suplattice in this way iff affine combinations are constant, that is $\lambda_1 x_1 + \cdots + \lambda_n x_n$ is independent of $\lambda_1, \cdots, \lambda_n$ for all $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{R}^*$, $\lambda_1 + \cdots + \lambda_n = 1$, $x_1, \cdots, x_n \in X$.

3. CONVEX SETS

Definition 8. Let X be a convex set. If x_1, $x_2 \in X$, define $x_1 \leq x_2$ when we may write x_2 as a convex combination in X in which x_1 does appear, that is $x_2 = \sum_{1 \leq i \leq n} \lambda_i t_i$ where $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $t_1, \cdots, t_n \in X$ and $t_i = x_1$ for some i. Equivalently, when we may write $x_2 = \lambda x_1 + (1-\lambda)t$ where $\lambda \in \mathbf{J}$, $t \in X$. This binary relation R on X is a quasiorder. Reflexivity $x_1 \leq x_1$ for $x_1 \in X$ follows from $x_1 = \lambda x_1 + (1-\lambda)x_1$ where $\lambda \in \mathbf{J}$. Transitivity is seen as follows. Let x_1, x_2, $x_3 \in X$, $x_1 \leq x_2$, $x_2 \leq x_3$. Then $x_2 = \lambda_1 x_1 + (1-\lambda_1)t_1$, $x_3 = \lambda_2 x_2 + (1-\lambda_2)t_2$ where λ_1, $\lambda_2 \in \mathbf{J}$, t_1, $t_2 \in X$. Hence $x_3 = \lambda_2\lambda_1 x_1 + \lambda_2(1-\lambda_1)t_1 + (1-\lambda_2)t_2$ and $x_1 \leq x_3$. This quasiorder R defines an equivalence relation $e(R)$ on X by $x_1 \sim x_2$ when x_1, $x_2 \in X$, $x_1 \leq x_2$, $x_2 \leq x_1$. The quotient set $X/e(R)$ is ordered. The quasiorder R, hence the equivalence relation $e(R)$, are compatible with the convex set structure of X (see Definition 6). Indeed, it is enough to check that $(1-\alpha)x_1 + \alpha u \leq (1-\alpha)x_2 + \alpha u$ for all $\alpha \in \mathbf{J}$, x_1, x_2, $u \in X$, $x_1 \leq x_2$. As a matter of fact, we have $x_2 = \lambda x_1 + (1-\lambda)t$ with $\lambda \in \mathbf{J}$, $t \in X$. Then $(1-\alpha)x_2 + \alpha u = \lambda[(1-\alpha)x_1 + \alpha u] + (1-\alpha)(1-\lambda)t + (1-\lambda)\alpha u$, hence $(1-\alpha)x_1 + \alpha u \leq (1-\alpha)x_2 + \alpha u$ as wanted. There is one and only one convex set structure on $X/e(R)$ such that the quotient map $\pi : X \to X/e(R)$ is a convex set map. Let next x_1, x_2, $x \in X$. If x has an expression as a convex combination in X in which both x_1, x_2 do appear, that is $x = \sum_{1 \leq i \leq n} \lambda_i t_i$ where $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{J}$, $\lambda_1 + \cdots + \lambda_n = 1$, $t_1, \cdots, t_n \in X$ and $t_i = x_1$, $t_j = x_2$ for some i, j, then $x_1 \leq x$, $x_2 \leq x$. Conversely, if $x_1 \leq x$, $x_2 \leq x$, then $x = \lambda_1 x_1 + (1-\lambda_1)t_1$, $x = \lambda_2 x + (1-\lambda_2)t_2$ where λ_1, $\lambda_2 \in \mathbf{J}$, t_1, $t_2 \in X$. For $\alpha \in \mathbf{J}$, we have $x = (1-\alpha)x + \alpha x = (1-\alpha)\lambda_1 x_1 + \alpha\lambda_2 x_2 + (1-\alpha)(1-\lambda_1)t_1 + \alpha(1-\lambda_2)t_2$, hence x has an expression as a convex combination in X in which both x_1, x_2 do appear. This extends easily to $x_1, \cdots, x_m$, $x \in X$ and $x_1 \leq x, \cdots, x_m \leq x$ for $m \in \mathbf{N}^*$.

Proposition 9. A convex set X is a quasisuplattice with respect to the quasiorder R. Hence $X/e(R)$ is a suplattice (called the suplattice associated with X).

Proof. The convex set X is filtered to the right as a quasiordered set. In fact, let x_1, $x_2 \in X$. If $\lambda \in \mathbf{J}$, set $x = (1-\lambda)x_1 + \lambda x_2 \in X$. Then $x_1 \leq x$, $x_2 \leq x$ proving the assertion. (Hence the quasiorder of X cannot be equality, unless X is empty or reduced to one point.) Let next x_1, $x_2 \in X$, λ, $\mu \in \mathbf{J}$, $u = (1-\lambda)x_1 + \lambda x_2 \in X$, $v = (1-\mu)x_1 + \mu x_2 \in X$. We claim that $u \sim v$, that is $u \leq v$, $v \leq u$. Let us prove $u \leq v$. We may find $\nu \in \mathbf{J}$ so that $\alpha = (1-\mu) - \nu(1-\lambda) > 0$, $\beta = \mu - \nu\lambda > 0$. Indeed, we are requiring $0 < \nu < 1$, $\nu < \frac{1-\mu}{1-\lambda}$, $\nu < \frac{\mu}{\lambda}$ which is obviously possible. Then $\alpha + \beta = 1 - \nu$ and α, $\beta \in \mathbf{J}$. Therefore $\nu + \alpha + \beta = 1$ and $\nu u + \alpha x_1 + \beta x_2 = v$, hence $u \leq v$. Likewise $v \leq u$ by symmetry. Let finally x_1, $x_2 \in X$, $\lambda \in \mathbf{J}$, $u = (1-\lambda)x_1 + \lambda x_2 \in X$. We have $x_1 \leq u$, $x_2 \leq u$. Assume

$v \in X$, $x_1 \leq v$, $x_2 \leq v$. As we saw (Definition 8), we may write $v = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$ where α_1, α_2, $\alpha_3 \in \mathbf{J}$, $x_3 \in X$. Hence

$$v = (\alpha_1 + \alpha_2)\,\frac{\alpha_1 x_1 + \alpha_2 x_2}{\alpha_1 + \alpha_2} + \alpha_3 x_3, \qquad \frac{\alpha_1 x_1 + \alpha_2 x_2}{\alpha_1 + \alpha_2} \leq v.$$

Since $u = (1-\lambda)x_1 + \lambda x_2 \sim \frac{\alpha_1 x_1 + \alpha_2 x_2}{\alpha_1 + \alpha_2} \leq v$, we get $u \leq v$. This proves that each $u = (1-\lambda)x_1 + \lambda x_2$ for $\lambda \in \mathbf{J}$ is a quasisupremum of x_1, $x_2 \in X$ in X. It extends easily to prove that each $u = \lambda_1 x_1 + \cdots + \lambda_m x_m \in X$ is a quasisupremum of $x_1, \cdots, x_m \in X$ in X for $m \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_m \in \mathbf{J}$, $\lambda_1 + \cdots + \lambda_m = 1$. □

Remark 10. Let X be a convex set. (1) If x_1, x_2, t_1, $t_2 \in X$, t_2 is a quasisupremum of x_1, x_2 in X, $\lambda \in \mathbf{J}$, $t = (1-\lambda)t_1 + \lambda t_2 \in X$, then t is a quasisupremum of x_1, x_2 in X iff $t_1 \leq t_2$. Indeed, assume t is a quasisupremum of x_1, x_2 in X. Then $t \sim t_2$, $t_1 \leq t$ imply $t_1 \leq t_2$. Conversely, let $t_1 \leq t_2$. We have $t_2 \leq t$. Also $t_1 \leq t_2$ implies $t = (1-\lambda)t_1 + \lambda t_2 \leq (1-\lambda)t_2 + \lambda t_2 = t_2$. Thus $t \sim t_2$ and t is a quasisupremum of x_1, x_2 in X. (2) Let x_1, $x_2 \in X$, $\lambda \in \mathbf{J}$, $u = (1-\lambda)x_1 + \lambda x_2 \in X$. Then u is a quasisupremum of x_1, x_2 in X. If $v \in X$, $v \leq u$, $\mu \in \mathbf{J}$, then $(1-\mu)u + \mu v \in X$ is a quasisupremum of x_1, x_2 in X by (1), not necessarily of the form $(1-\lambda)x_1 + \lambda x_2$, $\lambda \in \mathbf{J}$.

Definition 11. Let X be a convex set. Denote by $x_1 \vee x_2$ the nonvoid set of all suprema of x_1, $x_2 \in X$ in X for R. Then $x_1 \vee x_2$ is an equivalence class in X for $e(R)$ containing all $(1-\lambda)x_1 + \lambda x_2 \in X$ where $\lambda \in \mathbf{J}$. If x_1, x_2, $u \in X$, then $u \in x_1 \vee x_2$ iff we have $u \sim (1-\lambda)x_1 + \lambda x_2$ for some, equivalently for all, $\lambda \in \mathbf{J}$. Hence $X/e(R)$ is a suplattice whose order is compatible with its convex set structure (see Definition 6). If x_1, $x_2 \in X$, we have $x_1 \vee x_2 = \pi(x_1) \vee \pi(x_2)$, that may be written $\pi(x_1 \vee x_2) = \pi(x_1) \vee \pi(x_2)$ since $x_1 \vee x_2 \subset X$, $\pi(x_1 \vee x_2) = x_1 \vee x_2$. This extends easily to the set $x_1 \vee \cdots \vee x_m$ of all suprema of $x_1, \cdots, x_m \in X$ in X for R and $m \in \mathbf{N}^*$.

Definition 12. An *increasing subset* T in a convex set X is a subset such that, if $t \in T$, $x \in X$, $t \leq x$, then $x \in T$; equivalently, if $t \in T$, $x \in X$, $\lambda \in \mathbf{J}$, then $\lambda t + (1-\lambda)x \in T$. It follows that T is a convex subset of X. Clearly $\emptyset$, X are increasing subsets of X. The intersection and the union of a family of increasing subsets of X are also increasing. Every subset T of X generates an increasing subset $i(T)$ of X, the smallest increasing subset of X containing T, namely the intersection of all increasing subsets of X containing T. We have $i(t) = \{x \in X; t \leq x\}$ for $t \in X$, and $i(T) = \bigcup_{t \in T} i(t)$ for $T \subset X$.

Example 13. Let V be a nonvoid subset of a real vector space E. Call X the convex subset of E generated by V. Assume that V is strongly convexily independent in E, that is the expression of every element of X as a convex combination of elements of V is unique. X is called a simplex in E of vertices in V. Every $x \in X$ determines the nonvoid finite subset $v(x)$ of V of all vertices that occur in the unique expression of x as a convex combination of elements of V. We have $x_1 \leq x_2$ for x_1, $x_2 \in X$ iff $v(x_1) \subset v(x_2)$. Each equivalence class for $e(R)$ is the set of all $x \in X$ that determine the same finite subset of V by the map $x \in X \mapsto v(x) \subset V$. Hence we get a suplattice isomorphism between $X/e(R)$ and the suplattice of all nonvoid finite subsets of V. Each nonvoid finite subset U of V determines an open face $F(U)$ of X, namely the subset of X of all $x = \sum_{u \in U} \lambda(u)u$, where $\lambda : U \to \mathbf{J}$, $\sum_{u \in U} \lambda(u) = 1$. The map $U \mapsto F(U)$ is bijective. If V is finite of $n+1$ elements ($n = 0, 1, \cdots$), X is called an n-simplex; then R has $2^{n+1} - 1$ equivalence classes (the number of nonvoid subsets of the set V of $n+1$ elements). This example may be extended to a polyhedron X in a real vector space E, namely the convex subset of E generated by a nonvoid subset V of E assumed to be convexily independent in E, that is every element of V is not a convex combination of elements of V different from the given element.

Proposition 14. (1) Let X be a suplattice considered as a convex set (see Definition 7). Then R is an order, $e(R)$ is equality, and the suplattice structure of X is isomorphic to that of $X/e(R)$ associated with its convex set structure by Proposition 9. (2) Let X be a convex set. The convex set structure of $X/e(R)$ derives from its suplattice structure (see Definition 7), hence the suplattice structure of $X/e(R)$ derives from its convex set structure (by Proposition 9).

Proof. (1) In principle, we have to distinguish between the order on X as a suplattice and the quasiorder on X derived from its convex set structure by Definition 8. If x_1, $x_2 \in X$, write $x_1 \leq x_2$ (SL) or $x_1 \leq x_2$ (CS) depending on whether we mean it in the suplattice or in the convex set senses. We have $x_1 \leq x_2$ (CS) iff $x_2 = \lambda x_1 + (1-\lambda)t$ where $\lambda \in \mathbf{J}$, $t \in X$, that is $x_2 = x_1 \vee t$, or $x_1 \leq x_2$ (SL). Hence we may write $x_1 \leq x_2$ without specifying (SL) or (CS). This proves that R is the order derived from the given suplattice structure and $e(R)$ is equality. Moreover, the suplattice structrure of X is isomorphic to that of $X/e(R)$ associated with its convex set structure by Proposition 9, because $x_1 \leq x_2$ (SL) iff we have $x_1 \leq x_2$ (CS). (2) Let Y_1, $Y_2 \in X/e(R)$. We have $(1-\lambda)Y_1 + \lambda Y_2 = Y_1 \vee Y_2$ for $\lambda \in \mathbf{J}$. Indeed, if $y_1 \in Y_1$, $y_2 \in Y_2$, set $y = (1-\lambda)y_1 + \lambda y_2 \in y_1 \vee y_2$, hence $\pi(y) = Y_1 \vee Y_2$. Then $\pi(y) = (1-\lambda)\pi(y_1) + \lambda\pi(y_2)$ and $Y_1 \vee Y_2 = (1-\lambda)Y_1 + \lambda Y_2$ as claimed. This proves that the convex set structure of $X/e(R)$ derives from its suplattice structure (see Definition 7). Then (1) implies that the suplattice structure of $X/e(R)$ derives from its convex set structure by Proposition 9. □

Corollary 15. Let X be a convex set. Then $e(R)$ is equality iff the convex set structure of X derives from a necessarily unique suplattice structure on X (see Definition 7).

Proof. Uniqueness is clear. Sufficiency is clear by Proposition 14, (1). Let us see necessity. If $e(R)$ is equality, use Proposition 14, (2) and the fact that then $\pi : X \to X/e(R)$ is a convex set isomorphism. □

Remark 16. Every suplattice X is isomorphic to the suplattice associated with some convex set, since it suffices to consider X as a convex set (see Definition 7) and use Proposition 14, (1).

4. CONVEX CONES

Proposition 17. Let X be a convex cone. (1) If x_1, $x_2 \in X$, then $x_1 \leq x_2$ iff $x_2 = \lambda x_1 + t$, where $\lambda \in \mathbf{R}_+^*$, $t \in X$. (2) If λ, $\mu \in \mathbf{R}_+^*$, $x \in X$, then $\lambda x \sim \mu x$. (3) If x_1, $x_2 \in X$, then $x_1 + x_2 \in X$ is a quasisupremum of x_1, x_2 in X. More generally, if also λ_1, $\lambda_2 \in \mathbf{R}_+^*$, then $\lambda_1 x_1 + \lambda_2 x_2 \in X$ is a quasisupremum of x_1, x_2 in X. This extends easily to $x_1, \cdots, x_m \in X$ for $m \in \mathbf{N}^*$. (4) The quasiorder R of X, hence its equivalence relation $e(R)$, are compatible with the convex cone structure of X. (5) An increasing subset of X is a convex subcone of X.

Proof. (1) Assume x_1, $x_2 \in X$, $x_1 \leq x_2$. Thus $x_2 = \lambda x_1 + (1-\lambda)t$ where $\lambda \in \mathbf{J}$, $t \in T$ (see Definition 8). This proves necessity. Let us see sufficiency. Assume $x_2 = \lambda x_1 + t$ where $\lambda \in \mathbf{R}_+^*$, $t \in X$. We may assume $\lambda \in \mathbf{J}$, for it suffices to choose $\mu \in \mathbf{J}$, $\mu < \lambda$, and write $x_2 = \mu x_1 + (\lambda - \mu)x_1 + t$. Thus, if $x_2 = \lambda x_1 + t$ where $\lambda \in \mathbf{J}$, $t \in X$, we have $x_2 = \lambda x_1 + (1-\lambda)u$ with $u \in X$, hence $x_1 \leq x_2$. (2) The assertion is clear if $\lambda = \mu$. Assume $\lambda < \mu$. Choose $\rho \in \mathbf{R}$, $\rho > \mu$, $\nu = \frac{\rho-\mu}{\rho-\lambda} \in \mathbf{J}$. Then $\mu = \nu\lambda + (1-\nu)\rho$, hence $\mu x = \nu(\lambda x) + (1-\nu)(\rho x)$ and $\lambda x \leq \mu x$. Next choose $\rho \in \mathbf{R}$, $\rho < \lambda$, $\nu = \frac{\lambda-\rho}{\mu-\rho} \in \mathbf{J}$. Then $\lambda = \nu\mu + (1-\nu)\rho$, hence $\lambda x = \nu(\mu x) + (1-\nu)(\rho x)$ and $\mu x \leq \lambda x$. (3) Let x_1, $x_2 \in X$. Obviously x_1, $x_2 \leq x_1 + x_2$ by (1). If $u \in X$, $x_1 \leq u$, $x_2 \leq u$, then $u = \lambda_1 x_1 + t_1$, $u = \lambda_2 x_2 + t_2$ where λ_1, $\lambda_2 \in \mathbf{R}_+^*$, t_1, $t_2 \in X$. Choosing $\mu \in \mathbf{J}$, we have $u = (1-\mu)u + \mu u = (1-\mu)\lambda_1 x_1 + \mu\lambda_2 x_2 + (1-\mu)t_1 + ut_2$. Fix $\nu \in \mathbf{R}$, $0 < \nu < (1-\mu)\lambda_1, \mu\lambda_2$. Then $u = \nu(x_1 + x_2) + [(1-\mu)\lambda_1 - \nu]\, x_1 + (\mu\lambda_2 - \nu)x_2 + (1-\mu)t_1 + \mu t_2$. Thus $x_1 + x_2 \leq u$. This proves that $x_1 + x_2$ is a quasisupremum of x_1, x_2 in X. More generally, let also λ_1, $\lambda_2 \in \mathbf{R}_+^*$. Firstly $x_1, x_2 \leq \lambda_1 x_1 + \lambda_2 x_2$ by (1), hence $x_1 + x_2 \leq \lambda_1 x_1 + \lambda_2 x_2$ because $x_1 + x_2$ is a quasisupremum of x_1, x_2 in X. Secondly, choose $\lambda \in \mathbf{R}$, $0 < \lambda < 1/\lambda_1, 1/\lambda_2$. Then $x_1 + x_2 = \lambda(\lambda_1 x_1 + \lambda_2 x_2) + (1-\lambda\lambda_1)x_1 + (1-\lambda\lambda_2)x_2$, therefore $\lambda_1 x_1 + \lambda_2 x_2 \leq x_1 + x_2$ by (1). We conclude that $\lambda_1 x_1 + \lambda_2 x_2$ is a quasisupremum of x_1, x_2 in X. (4) Firstly, let t_1, t_2, x_1, $x_2 \in X$, $t_1 \leq x_1$, $t_2 \leq x_2$. Then $x_1 = \lambda_1 t_1 + u_1$, $x_2 = \lambda_2 t_2 + u_2$ where λ_1, $\lambda_2 \in \mathbf{R}_+^*$, u_1, $u_2 \in X$. Hence $x_1 + x_2 = \lambda_1 t_1 + \lambda_2 t_2 + u_1 + u_2$. Hence $\lambda_1 t_1 + \lambda_2 t_2 \leq x_1 + x_2$ by (1). Since $t_1 + t_2 \sim \lambda_1 t_1 + \lambda_2 t_2$ by (3), we get $t_1 + t_2 \leq x_1 + x_2$. Secondly, let t, $x \in X$, $t \leq x$, $\lambda \in \mathbf{R}_+^*$. Then $x = \mu t + u$ where $\mu \in \mathbf{R}_+^*$, $u \in X$, hence $\lambda x = \mu(\lambda t) + \lambda u$ and $\lambda t \leq \lambda x$ by (1). (5) Let T be an increasing subset of X. Firstly, if $t \in T$, $x \in X$, then $t \leq t + x$ by (1), hence $t + x \in T$. In particular, if $t \in T$, $x \in T$, then $t + x \in T$. Secondly, if $\lambda \in \mathbf{R}_+^*$, $t \in T$, then $t \sim \lambda t$ by (2), hence $t \leq \lambda t$ and $\lambda t \in T$. □

5. AFFINE SPACES

Lemma 18. If X is an affine space, $\lambda \in \mathbf{R}_+^*$, x_1, $x_2 \in X$, then $x_2 + \lambda x_1 - \lambda x_1 = x_2 + x_1 - x_1$. (we recall the simplified notation $x_2 + \lambda x_1 - \lambda x_1 = 1x_2 + \lambda x_1 + (-\lambda)x_1$, in particular $x_2 + x_1 - x_1 = 1x_2 + 1x_1 + (-1)x_1$.)

Proof. We have $x_2+\lambda x_1-\lambda x_1 = x_2+(\lambda x_1+x_1-x_1)-\lambda x_1 = x_2+x_1-(x_1+\lambda x_1-\lambda x_1) = x_2+x_1-x_1$. □

Proposition 19. Let X be an affine space. (1) If x_1, $x_2 \in X$, then $x_1 \leq x_2$ iff we may write x_2 as an affine combination in X in which x_1 does appear, that is $x_2 = \sum_{1\leq i\leq n} \lambda_i t_i$ where $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{R}^*$, $\lambda_1 + \cdots + \lambda_n = 1$, $t_1, \cdots, t_n \in X$ and $t_i = x_1$ for some i. (2) If x_1, $x_2 \in X$, then $x_1 \leq x_2$ iff we may write $x_2 = \lambda x_1 + (1-\lambda)t$ where $\lambda \in \mathbf{R}$, $\lambda \neq 0, 1$, $t \in X$. (3) If x_1, $x_2 \in X$, then $x_1 \leq x_2$ iff $x_2 = x_2 + x_1 - x_1$. (4) If x_1, $x_2 \in X$, $\lambda \in \mathbf{R}$, $\lambda \neq 0, 1$, then $(1-\lambda)x_1 + \lambda x_2 \in X$ is a quasisupremum of x_1, x_2 in X. This extends easily to $x_1, \cdots, x_m \in X$ for $m \in \mathbf{N}^*$. (5) The quasiorder R of X, hence its equivalence relation $e(R)$, are compatible with the affine space structure of X. (6) An increasing subset of X is an affine subspace of X.

Proof. (1) Assume x_1, $x_2 \in X$. Necessity in (1) obviously follows from necessity in (2). Let us prove sufficiency in (1). Assume that $x_2 = \sum_{1\leq i\leq n} \lambda_i t_i$ where $n \in \mathbf{N}^*$, $\lambda_1, \cdots, \lambda_n \in \mathbf{R}^*$, $\lambda_1 + \cdots + \lambda_n = 1$, $t_1, \cdots, t_n \in X$ and say $t_1 = x_1$. If $\mu \in \mathbf{J}$, $\mu \neq \lambda_1$ is chosen, write $x_2 = \lambda_1 x_1 + \sum_{2\leq i\leq n} \lambda_i t_i = \mu x_1 + (\lambda_1 - \mu)x_1 + \sum_{2\leq i\leq n} \lambda_i t_i = \mu x_1 + (1-\mu)t$ with a suitable $t \in X$. Thus $x_1 \leq x_2$. (2) Assume x_1, $x_2 \in X$. Sufficiency in (2) obviously follows from sufficiency in (1). Necessity in (2) is clear because $x_1 \leq x_2$ means that we may write $x_2 = \lambda x_1 + (1-\lambda)t$ where $\lambda \in \mathbf{J}$, $t \in X$ by Definition 8. (3) Assume x_1, $x_2 \in X$. If $x_2 = x_2 + x_1 - x_1$, fix $\lambda \in \mathbf{J}$ and write $x_2 = \lambda x_1 + (1-\lambda)x_1 + x_2 - x_1 = (1-\lambda)x_1 + \lambda t$ with a suitable $t \in X$. Thus $x_1 \leq x_2$, by Definition 8. Conversely, let $x_1 \leq x_2$. Then $x_2 = \lambda x_1 + (1-\lambda)t$ where $\lambda \in \mathbf{J}$, $t \in X$. Hence $x_2 + x_1 - x_1 = x_2 + \lambda x_1 - \lambda x_1$ (by Lemma 18) $= \lambda x_1 + (1-\lambda)t + \lambda x_1 - \lambda x_1 = (\lambda x_1 + \lambda x_1 - \lambda x_1) + (1-\lambda)t = \lambda x_1 + (1-\lambda)t = x_2$ as wanted. (4) If x_1, $x_2 \in X$, λ, $\mu \in \mathbf{R}$, λ, $\mu \neq 0$, 1, set $u = (1-\lambda)x_1 + \lambda x_2 \in X$, $v = (1-\mu)x_1 + \mu x_2 \in X$. We claim that $u \sim v$, that is $u \leq v$, $v \leq u$. To check $u \leq v$, let $\nu \in \mathbf{R}^*$, define $\alpha = (1-\mu) - \nu(1-\lambda)$, $\beta = \mu - \nu\lambda \in \mathbf{R}$ and choose ν so that α, $\beta \neq 0$. Then $\nu + \alpha + \beta = 1$ and $\nu u + \alpha x_1 + \beta x_2 = v$ because $\mu \neq 0, 1$. Thus $u \leq v$. Similarly $v \leq u$ because $\lambda \neq 0, 1$. We know (proof of Proposition 9) that v is a quasisupremum of x_1, x_2 in X if $\mu \in \mathbf{J}$. Thus u is a quasisupremum of x_1, x_2 in X for $\lambda \in \mathbf{R}$, $\lambda \neq 0, 1$. (5) It is enough to check that $(1-\lambda)x_1 + \lambda u \leq (1-\lambda)x_2 + \lambda u$ for $\lambda \in \mathbf{R}^*$, x_1, x_2, $u \in X$, $x_1 \leq x_2$. As a matter of fact, $x_2 = x_2 + x_1 - x_1$ by (3), $u = u + u - u$, hence $(1-\lambda)x_2 + \lambda u = (1-\lambda)x_2 + (1-\lambda)x_1 - (1-\lambda)x_1 + \lambda u + \lambda u - \lambda u = [(1-\lambda)x_2 + \lambda u] + [(1-\lambda)x_1 + \lambda u] - [(1-\lambda)x_1 + \lambda u]$. Thus $(1-\lambda)x_1 + \lambda u \leq (1-\lambda)x_2 + \lambda u$ as wanted. (6) If T is an increasing subset of X, we have $\lambda t + (1-\lambda)x \in T$ if $\lambda \in \mathbf{R}$, $\lambda \neq 0, 1$, $t \in T$, $x \in X$ by (2). It follows that T is an affine subspace of X. □

Example 20. Let $X = A(E)$ be the affine space of all nonvoid affine subspaces of a real vector space E. It $T \in X$, let $v(T) = T - T$ be the vector subspace of E associated with T. Then $T_1 \leq T_2$ for T_1, $T_2 \in A(E)$ iff $v(T_1) \subset v(T_2)$. Hence $X/e(R)$ is isomorphic to the suplattice of all vector subspaces of E.

Lemma 21. If X is an affine space, the following conditions are equivalent: (1) X satisfies the cancellation rule for convex sets. (2) $x_2 = x_2 + x_1 - x_1$ for all $x_1, x_2 \in X$. (3) If $\lambda \in \mathbf{R}$, $\lambda > 1$, $x_1, x_2, x_3 \in X$, then $x_3 = (1-\lambda)x_1 + \lambda x_2$ is equivalent to $x_2 = (1-1/\lambda)x_1 + (1/\lambda)x_3$.

Proof. (1) $\Longrightarrow$ (2). Fix $\theta \in \mathbf{J}$. If $x_1, x_2 \in X$, we have $(1-\theta)x_1 + \theta(x_2 + x_1 - x_1) = [(1-\theta)x_1 + \theta x_1 - \theta x_1] + \theta x_2 = (1-\theta)x_1 + \theta x_2$, hence $x_2 + x_1 - x_1 = x_2$ by the cancellation rule for convex sets. (2) $\Longrightarrow$ (3). If $\lambda \in \mathbf{R}$, $\lambda > 1$, $x_1, x_2, x_3 \in X$, assume $x_3 = (1-\lambda)x_1 + \lambda x_2$. Then $(1-1/\lambda)x_1 + (1/\lambda)x_3 = (1-1/\lambda)x_1 + (1/\lambda)[(1-\lambda)x_1 + \lambda x_2] = (1-1/\lambda)x_1 + (1/\lambda - 1)x_1 + x_2 = x_2$ (by Lemma 18). Conversely, assume $x_2 = (1-1/\lambda)x_1 + (1/\lambda)x_3$. Then $(1-\lambda)x_1 + \lambda x_2 = (1-\lambda)x_1 + \lambda[(1-1/\lambda)x_1 + (1/\lambda)x_3] = (1-\lambda)x_1 + (\lambda - 1)x_1 + x_3 = x_3$ (by Lemma 18). (3) $\Longrightarrow$ (1). In particular, if $\lambda \in \mathbf{R}$, $\lambda > 1$, we see that $x_2 = (1-1/\lambda)x_1 + (1/\lambda)x_3$ implies that $x_3 = (1-\lambda)x_1 + \lambda x_2$, hence X satisfies the cancellation rule for convex sets. □

Remark 22. Keep (1), (2) as in Lemma 21. In the notation of (3), call (3a) the condition that $x_3 = (1-\lambda)x_1 + \lambda x_2$ implies $x_2 = (1-1/\lambda)x_1 + (1/\lambda)x_3$, and (3b) the condition that we have the inverse implication, so that (3) = (3a) $\cap$ (3b). We may give direct proofs of (2) $\Longrightarrow$ (1), (3a) $\Longleftrightarrow$ (2), (3b) $\Longleftrightarrow$ (2) as follows. (2) $\Longrightarrow$ (1). Let $\theta \in \mathbf{J}$, x_1, x_2, $x_3 \in X$, $(1-\theta)x_1 + \theta x_2 = (1-\theta)x_1 + \theta x_3$. If $x \in X$, we have

$$\frac{1}{\theta}\ [(1-\theta)x_1 + \theta x] + \left(1 - \frac{1}{\theta}\right)\ x_1 = \left(\frac{1}{\theta} - 1\right)\ x_1 + \left(1 - \frac{1}{\theta}\right)\ x_1 + x = x$$

by (2) and Lemma 18. If we set $x = x_2, x_3$, we then get $x_2 = x_3$ as wanted. (3a) $\Longleftrightarrow$ (2). In fact, (3a) means that $x_2 = (1-1/\lambda)x_1 + (1/\lambda)[(1-\lambda)x_1 + \lambda x_2] = (1-1/\lambda)x_1 + (1/\lambda - 1)x_1 + x_2$ which is $x_2 = x_1 - x_1 + x_2$ by Lemma 18. (3b) $\Longleftrightarrow$ (2). In fact, (3b) means that $x_3 = (1-\lambda)x_1 + \lambda[(1-1/\lambda)x_1 + (1/\lambda)x_3] = (1-\lambda)x_1 + (\lambda-1)x_1 + x_3$ which is $x_3 = x_1 - x_1 + x_3$ by Lemma 18.

Lemma 23. An affine space X is vectorial as an affine space iff it is vectorial as a convex set.

Proof. Necessity is clear. To prove sufficiency, let X be vectorial as a convex set. We may then assume X to be a convex subset of a real vector space E. We claim that, if $\lambda \in \mathbf{R}$, $\lambda > 1$, x_1, x_2, $x_3 \in X$, then $x_3 = (1-\lambda)x_1 + \lambda x_2$ in the affine space X implies that equality in the real vector space E. Indeed, we then have $x_2 = (1-1/\lambda)x_1 + (1/\lambda)x_3$ in the affine space X because (1) $\Longrightarrow$ (3) in Lemma 21, once X is vectorial as a convex set, hence it satisfies the cancellation rule for convex sets. This is an equality in the convex set X, therefore in the convex set E, hence in the real vector space E. It follows that $x_3 = (\lambda - 1)x_1 + \lambda x_2$ in the real vector space E, as claimed. This proves that the inclusion map $X \to E$ is an affine space map if X has its given affine space structure and E its real vector space structure (since we also know that this map is a convex set map, once X is a convex subset of E). It follows that X is an affine subspace of E and that the given affine space structure of X coincides with that induced on X by E. Hence X is vectorial with its given affine space structure. □

Proposition 24. An affine space X is vectorial iff its quasiorder is chaotic, that is $x_1 \leq x_2$ for all $x_1, x_2 \in X$.

Proof. Necessity is clear. Let us prove sufficiency. Assume the quasiorder is chaotic, hence $x_2 = x_2 + x_1 - x_1$ for all $x_1, x_2 \in X$ (see Proposition 19, (3)). Then X satisfies the cancellation rule for convex sets because (2) $\Longrightarrow$ (1) in Lemma 21, hence X is vectorial as a convex set, and finally X is vectorial as an affine space by Lemma 23. $\square$

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