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QUANTUM GROUP GENERALIZATION OF THE HETEROTIC QFT

by

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Abstract

By the use of the generalized Grassmann algebra provided by Quantum Groups we introduce a new class of fields in two dimensions which have deformed commutation relations. We give the action involving this new class of fields and a bosonic scalar field and show that this action has a symmetry that resembles supersymmetry. We also construct the path-integral formalism for this new class of fields and compute the propagator for them in the context of the q-heterotic model.

Key-words: Quantum field theory; Quantum Group; Quasi-triangular Hopf Algebra; Supersymmetry.

1 Introduction

It is well known that, in four or more dimensions, the spin-statistics theorem relates bosons (fermions) to integer (half-integers) values of the spin. However, in two or three dimensions there are possibilities for spin and statistics which do not occur in four or more dimensions. The two dimensional rotation group $SO(2)$ is abelian and so its irreducible representations are characterized by a continuous parameter giving rise to particles with non integral and non half-integral spin. Moreover, the permutation group is not the most general group for continuous interchange of positions of particles. Such group is, in fact, the braid group and particles quantized in such way have braid group statistics. Although the quantum theory associated with these particles is not yet completely understood, it is believed that planar physical systems could show some properties associated with this feature.

In this letter, we will explore this possibility in the two dimensional case by constructing fields which obey different commutation relations from those for fermions and for bosons and with different occupation number. As we will see, they are, under several aspects, generalizations of the Grassmann variables provided by Quantum Groups [1,2,3].

One of the interesting features of Quantum Groups is the fact that these theories can be related to physical problems where the coordinates are non–commutative. On the non–commutative space of a Quantum Group it was shown [4] to be possible to define consistently a non–commutative differential geometry and a simple example was developed [5] on the quantum plane [6].

Recently it was shown [7] that one could understand these non–commutative coordinates as the classical limit of the creation and annihilation operators of the q-deformed Heisenberg Algebra [8]

$$aa^\dagger - qa^\dagger a = q^{-N}, \tag{1}$$

where $[N] = (q^N - q^{-N})/(q - q^{-1}) = a^\dagger a$, and q a complex parameter. If we take $q = \exp(2\pi i/k)$, where k is an integer, we can show that [7]

$$a^k = (a^\dagger)^k = 0, \tag{2}$$

which shows to have a discrete spectrum. If, after a suitable change of variables we consider the classical limit of these operators, we have the classical variables $(\theta, \overline{\theta})$ obeying the relations [7]

$$\theta\overline{\theta} = q\overline{\theta}\theta \tag{3}$$

and

$$(\theta)^k = (\overline{\theta})^k = 0. \tag{4}$$

Notice that for $k = 2$, $q = -1$, these classical variables are nothing more than the anticommutation relations for fermionic variables. For $k > 2$, but finite, they can be seen as generalizing the idea of Grassmann variable. We can construct a differential calculus with such variables, and it is possible to show that there is an underlying Yang-Baxter structure and a covariance under the quantum group $GL_{(q^2,q^{-2})}$ [7].

It is interesting to notice that, for $k = 2$, $q = -1$, eq.(1) becomes the usual anticommutator, which is consistent with eqs.(3) and (4). Taking $k \rightarrow \infty$, eq.(1) becomes the usual commutator. The meaning of this limit in eqs.(3) and (4) is that, if we Taylor expand a function of these variables, it will become a series (obviously, if $\theta^k = 0$, k finite, a Taylor expansion will be a polynomaial of degree $(k - 1)$).

In the following, we address the intermediate cases of these generalized classical variables, specially the case $k = 3$. In the next section we discuss q-commutators for the fields, we construct an action with them and show that such action, plus the action of a scalar field, has symmetries that resemble supersymmetry. In section 3 we discuss the path integral for these fields. In section 4 we write the Functional Generator and compute the propagator. We let some final comments and perpectives for the last section.

2 q-heterotic supersymmetric $2-D$ Field Theory

In this section we discuss the fields that appear in the case $k = 3$ and some of their properties, in the context of two dimensional Field Theory. A similar analysis was performed in the classical mechanical context [9].

Let us start by discussing what type of fields can appear in this case. Let us recall that with Grassmann variables, we can have two different types of objects: those that behave like θ (fermions) and those that behave like θ^0 (bosons).

If now we consider the case $k = 3$, we have three different types of fields: bosons (that behave like θ^0), and two different fields, that we call quermions, that behave under commutation relations like θ (a sector-one quermion) or like θ^2 (a sector-two quermion). The commutation relations depend on the factor $q = \exp(2\pi i/k)$ (from now on we call it quommutator).

We define the quommutator between two fields $A^{(r)}$ and $B^{(s)}$ as

$$[A^{(r)}, B^{(s)}]_q \equiv q^{(-rs/2)} A^{(r)} B^{(s)} - q^{(rs/2)} B^{(s)} A^{(r)}, \tag{5}$$

where the supperscript indicates the sector of the fields (we take a boson as a sector zero field). A similar quommutator appeared related to the q-deformed Virasoro Algebra [10], but in that case the powers of q was related to the modes of the Energy-Momentum tensor, while here it is related to the sectors.

This is the simplest relation which garantees that any boson obeys the usual commutation relation with any other field, because in this case the q-factor disapears. It gives the correct limit for Grassmann variables in the case $k = 2$ and, in the limit $k \to \infty$ we recover the usual commutator.

As we are extending the case of bosons and fermions, the product $A^{(r)}B^{(s)}$ belongs to the sector $(r + s) \bmod 3$. With this choice, we can see that

$$[A^{(r)}, B^{(s)}C^{(t)}]_q = q^{(-rt/2)}[A^{(r)}, B^{(s)}]_q C^{(t)} + q^{(rs/2)} B^{(s)}[A^{(r)}, C^{(t)}]_q, \tag{6}$$

which is a q-extension of the commutator formula.

We can take the two dimensional action for the quermions as

$$S_q = \int d^2x q C^{(s)^2} \partial_+ \psi^{(2)} \psi^{(1)}, \tag{7}$$

where x^+ and x^- are the light cone coordinates in Minkowski space and the first order derivative is chosen such that the classical equations of motions for the $\psi^{(i)}$ are similar to the fermionic case. The cocycle-type factor $C^{(s)^2}$ is required because we want that S_q commutes (that is, behave like a sector-zero object) with respect to any field, including $\psi^{(1)}$ and $\psi^{(2)}$. The point here is that, although $\partial_+\psi^{(2)}\psi^{(1)}$ belongs to the sector-zero, it does not commute with $\psi^{(2)}$ or $\psi^{(1)}$, since we choose equal fields at equal points to commute. We stress here that $\psi^{(i)^3} = 0$, but $\psi^{(i)^2} \neq 0$. This cocycle–type factor $C^{(s)}$ actually behaves like a sector–counter, that is,

$$C^{(s)} A^{(i)} = q^i A^{(i)} C^{(s)}. \tag{8}$$

Notice that S_q does not fix the dimensions of $\psi^{(i)}$. They will be determined in the following, by imposing a symmetry similar to supersymmetry.

To do this, we write an action including the quermionic fields and a scalar field

$$S = \int d^2x(\partial_+\phi\,\partial_-\phi - qC^{(s)^2}\,\partial_+\psi^{(2)}\,\psi^{(1)}). \tag{9}$$

whith $\psi^{(i)}$ real.

This action seems to be an extension of the heterotic two dimensional field theory, and we ask if it is possible to write a transformation among the fields that keeps the action invariant. As in supersymmetry, the infinitesimal parameter will not be a boson, and in this case we can take it to belong to the sector-one or to the sector-two. We also suppose that one could define a q-superspace with coordinates (x^+, x^-, θ) and that there exist translations in this q-superspace. For a sector-one parameter, $\epsilon^{(1)}$, the translations are

$$\begin{aligned} \theta &\rightarrow \theta' = \theta + \epsilon^{(1)} \\ x^- &\rightarrow x^{-'} = x^- + q\theta^2\epsilon^{(1)}, \end{aligned} \tag{10}$$

with the choice $[\epsilon^{(1)}, \theta]_q = 0$. This fixes the mass dimensions for θ and $\epsilon^{(1)}$

$$[\theta] = [\epsilon^{(1)}] = -\frac{1}{3}. \tag{11}$$

As $\epsilon^{(1)}$ is a sector-one object, and we are interested in heterotic q-supersymmetry, it is natural to choose $\delta\psi^{(1)} \sim \epsilon^{(1)}\partial_-{}^l\phi$, $\delta\phi \sim \epsilon^{(1)}\partial_-{}^m\psi^{(2)}$ and $\delta\psi^{(2)} \sim \epsilon^{(1)}\partial_-{}^n\psi^{(1)}$, that is, a cyclical combination of the coordinates.

Factors $C^{(s)}$ must be introduced to ensure that each variation behaves like the field itself under quommutation relations with the other fields, and powers of q are chosen such that the variations are real.

In order to determine (l, m, n), we compute δS and impose it to be at least a total derivative. This implies that n must be even, and the condition that $[\psi^{(i)}]$ is greater than zero implies that $m = n = 0$, $l = 1$, fixing the transformations as

$$\begin{aligned} \delta\phi &= qC^{(s)}\epsilon^{(1)}\psi^{(2)} \\ \delta\psi^{(1)} &= q^2C^{(s)^2}\epsilon^{(1)}\partial_-\phi \\ \delta\psi^{(2)} &= \pm q\epsilon^{(1)}\psi^{(1)}. \end{aligned} \tag{12}$$

This gives us the dimensions of $\psi^{(i)}$,

$$[\psi^{(1)}] = \frac{2}{3}$$

$$[\psi^{(2)}] = \frac{1}{3}, \tag{13}$$

and the variation of S is given by

$$\delta S = \pm \int d^2x \partial_+ (\epsilon^{(1)} \psi^{(1)^2}), \tag{14}$$

a total derivative. Thus the action eq.(9) in invariant under the transformation given by eq.(12).

We can also write a transformation involving $\epsilon^{(2)}$ in the q-superspace, given by

$$\begin{aligned} \theta^2 &\rightarrow \theta^{2'} = \theta^2 + \epsilon^{(2)} \\ x^- &\rightarrow x^{-'} = x^- + q\theta\epsilon^{(2)} \end{aligned} \tag{15}$$

with the choice $[\theta, \epsilon^{(1)}]_q = 0$, which gives

$$[\epsilon^{(2)}] = -\frac{2}{3} \tag{16}$$

The cyclical transformations among $\phi, \psi^{(i)}$ can be constructed in the same way as before. Factors q and $C^{(s)}$ are analogously dertermined and powers in the derivatives are already known, since we know all dimensions. We then get

$$\begin{aligned} \delta\phi &= -q^2 C^{(s)} \epsilon^{(2)} \psi^{(1)} \\ \delta\psi^{(1)} &= \pm q^2 \epsilon^{(2)} \partial_- \psi^{(2)} \\ \delta\psi^{(2)} &= q C^{(s)^2} \epsilon^{(2)} \partial_- \phi. \end{aligned} \tag{17}$$

In this case the variation of S is given by

$$\delta S = \int d^2x (q^2 C^{(s)} \partial_+ (\epsilon^{(2)} \psi^{(1)} \partial_- \phi) \pm q C^{(s)^2} \epsilon^{(2)} \partial_- \psi^{(2)} \partial_+ \psi^{(2)}))). \tag{18}$$

However, the last term is not a total derivative, so the action eq.(9) is not invariant under the transformation given by eq.(18).

3 Path Integral for Quermions

We now want to construct a path integration over the quermionic fields introduced in the last section. This will be used later when we define the Functional Generator and compute the quermionic propagator.

To state the problem, we want to integrate over the quermionic variables $\psi^{(i)}$ that obey the relations

$$\psi^{(1)}\psi^{(2)} = q^2\psi^{(2)}\psi^{(1)} \tag{19}$$

and

$$\psi^{(i)3} = 0. \tag{20}$$

Imposing linearity and translation invariance, it is easy to see that the integration over one variable is of the form [7]

$$\int d\psi^{(i)}\, \psi(i)^n = \delta_{n,2}\alpha, \tag{21}$$

where α is a constant to be determined. If we want to integrate over both variables, $\psi^{(1)}$ and $\psi^{(2)}$, the invariant measure is given by

$$[d\psi] = C^{(s)^2} d\psi^{(1)}\, d\psi^{(2)}. \tag{22}$$

Let us impose the q-gaussian integral

$$I = \int [d\psi] \exp\left[C^{(s)^2} \psi^{(2)}\, \psi^{(1)}\right] \tag{23}$$

as equal to one. Expanding the exponential, only the quadratic term will survive, and this gives

$$\alpha = \sqrt{2q}. \tag{24}$$

From this, it is clear that

$$\int [d\psi] \exp\left[C^{(s)^2} \psi^{(2)}\, a_1\, \psi^{(1)}\right] = {a_1}^2. \tag{25}$$

Our next step is to construct a functional integral that involves two quermions (that is, ${\psi_1}^{(1)}$, ${\psi_1}^{(2)}$, ${\psi_2}^{(1)}$ and ${\psi_2}^{(2)}$). Writing the measure as $[d\psi]$, the integral is of the form

$$I_2 = \int [d\psi] \exp(X) \tag{26}$$

with

$$X = X_1 + X_2 = \psi_1{}^{(2)} A_1 \psi_1{}^{(1)} + \psi_2{}^{(2)} A_2 \psi_2{}^{(1)} \tag{27}$$

where A_α, $\alpha = 1, 2$, are chosen such that each X_α belong to the sector-zero, and in general they will be a product of a c-number a_α times some cocycles.

We must, of course, give the quommutator relations between the different fields $\psi_1{}^{(i)}$ and $\psi_2{}^{(i)}$. We choose them to be

$$\begin{aligned} &[\psi_i{}^{(1)}, \psi_i{}^{(2)}]_q = 0 \qquad (no\, summation\, over\, i) \\ &[\psi_1{}^{(i)}, \psi_2{}^{(j)}]_q = 0 \qquad i, j = 1, 2 \end{aligned} \tag{28}$$

With this choice, it is easy to determine the cocycles needed to make each X_α belongs to the zero-sector. The A_α are

$$\begin{aligned} A_1 &= a_1 C^{(s)^2} C(\psi_2{}^{(1)}) C(\psi_2{}^{(2)})^2 \\ A_2 &= a_2 C^{(s)^2} C(\psi_1{}^{(1)}) C(\psi_1{}^{(2)})^2 \end{aligned} \tag{29}$$

where $C(A)A = qAC(A)$, i.e., the cocycle type factor $C(A)$ q-counts the field A.

We need still to determine the invariant measure. To do it, we simply multiply all the cocycles appearing in A_1 and A_2, obtaining

$$[d\psi] = C^{(s)} C(\psi_2{}^{(1)}) C(\psi_2{}^{(2)})^2 C(\psi_1{}^{(1)}) C(\psi_1{}^{(2)})^2 d\psi_1{}^{(1)} d\psi_1{}^{(2)} d\psi_2{}^{(1)} d\psi_2{}^{(2)}. \tag{30}$$

In fact, with this choice the measure can be broken in two pieces

$$[d\psi] = [d\psi_1][d\psi_2] \tag{31}$$

where

$$[d\psi_i] = C^{(s)^2} C(\psi_j{}^{(1)}) C(\psi_j{}^{(2)})^2 d\psi_i{}^{(1)} d\psi_i{}^{(2)} \tag{32}$$

where $i \neq j$, and each $[d\psi_i]$ is invariant

With this measure, we have

$$I_2 = \int [d\psi] \exp(X_1 + X_2) = a_1{}^2 a_2{}^2 \tag{33}$$

Notice that the eq.(27) can be writen in a matricial form, with A being a diagonal matrix with diagonal coefficients a_α, in this case we can rewrite eq.(33) as

$$I_2 = (\det A)^2. \tag{34}$$

The generalization for $N > 2$ follows in the same way. We take the quommutators of the fields

$$[\psi_i{}^{(1)}, \psi_i{}^{(2)}]_q = 0 \qquad i = 1, 2 \ldots, N (no\, summation\, over\, i)$$

$$[\psi_i{}^{(l)}, \psi_j{}^{(m)}]_q = 0 \qquad i, j = 1, 2 \ldots, N;\ i < j;\ l, m = 1, 2 \tag{35}$$

where X now is given by

$$X = \sum_{\alpha=1}^{N} X_\alpha = \sum_{\alpha=1}^{N} \psi_\alpha{}^{(2)} A_\alpha \psi_\alpha{}^{(1)} \tag{36}$$

with

$$A_\alpha = a_\alpha C^{(s)^2} \prod_{\beta=1, \beta\neq\alpha}^{N} C(\psi_\beta{}^{(1)}) C(\psi_\beta{}^{(2)})^2. \tag{37}$$

The measure is

$$[d\psi] = \prod_{\alpha=1}^{N} [d\psi_\alpha], \tag{38}$$

with

$$[d\psi_\alpha] = C^{(s)^2} \prod_{\beta=1, \beta\neq\alpha}^{N} C(\psi_\beta{}^{(1)}) C(\psi_\beta{}^{(2)})^2 d\psi_\alpha{}^{(1)} d\psi_\alpha{}^{(2)}, \tag{39}$$

giving for I_N:

$$I_N = \prod_{\alpha=1}^{N} (a_\alpha)^2 = (\det A)^2. \tag{40}$$

Notice that, as we are considering generalizations of the Grassmann variables, this could be an expected result. In fact, if we consider generalized Grassmann variables, $\psi^k = 0$, k integer, linearity and translation invariance impose that

$$\int d\psi \psi^n = \delta_{n,k-1} \tag{41}$$

and the q-gaussian integral behaves as

$$\int [d\psi] \exp(\psi A \psi) \sim (\det A)^{(k-1)} \tag{42}$$

with the correct limit for $k = 2$.

Analogously to the case of fermions we can generalize eq.(40) to an infinite number of degres of freedom obtaining in this way the path integral for the quermionic field action.

4 The Functional Generator and the quermionic propagator

Let us now write the Functional Generator and compute the propagator for the $\psi^{(i)}$. To do this, we need to introduce sources $J^{(1)}$ and $J^{(2)}$, coupled to the quermions. We choose their quommutators to be

$$\begin{aligned} [J^{(1)}, J^{(2)}]_q &= 0 \\ [J^{(i)}, \psi^{(1)}]_q &= 0 \\ [\psi^{(2)}, J^{(i)}]_q &= 0 \end{aligned} \tag{43}$$

with $i = 1, 2$.

The functional generator can be written as

$$Z[J^{(1)}, J^{(2)}] = \frac{1}{\mathcal{N}} \int [d\psi] \exp[i \int d^2x (qC^{(s)^2} \psi^{(2)} A \psi^{(1)} + + q^2 C^{(s)} C(J^{(2)})^2 \psi^{(2)} J^{(1)} + q^2 C^{(s)} C(J^{(1)})^2 J^{(2)} \psi^{(1)})] \tag{44}$$

where A is some invertible operator ($A = \frac{1}{\pi}\partial_+$, if we consider the action eq.(7)) and the powers of q and the cocycles in the last two terms are fixed by imposing the reality condition and that they belong to the sector-zero. We now follow the usual trick to compute this integral. First, we write the integrand of the argument of the exponential in eq.(44) as

$$Q(\psi^{(1)}, \psi^{(2)}) = qC^{(s)^2} (\psi^{(2)} + \psi_0{}^{(2)}) A (\psi^{(1)} + \psi_0{}^{(1)}) + X \tag{45}$$

with

$$\begin{aligned} \psi_0{}^{(1)} &= q^2 C^{(s)^2} C(J^{(2)}) A^{-1} J^{(1)} \\ \psi_0{}^{(2)} &= qC^{(s)^2} C(J^{(1)})^2 J^{(2)} A^{-1} \end{aligned} \tag{46}$$

and

$$X = -q^2 C(J^{(1)})^2 C(J^{(2)}) J^{(2)} A^{-1} J^{(1)} \tag{47}$$

As X commutes with all the fields and sources (and with the measure), we can take it out of the integral, remaining only the q-gaussian integral already discussed in the last section. Normalizing Z such that $Z[J^{(1)} = 0, J^{(2)} = 0] = 1$ we get

$$\mathcal{N} = (\det A)^2 \tag{48}$$

and Z becomes

$$Z[J^{(1)}, J^{(2)}] = \exp\left(q^2 C(J^{(1)})^2 C(J^{(2)}) \int d^2x J^{(2)} A^{-1} J^{(1)}\right) \tag{49}$$

The two-point function is defined as

$$< \psi^{(2)}(x)\psi^{(1)}(y) >= \frac{1}{(\det A)^2}\left\{\int [d\psi]\psi^{(2)}(x)\psi^{(1)}(y)\exp\left(\int d^2x i Q(\psi^{(1)},\psi^{(2)})\right)\right\}\Bigg|_{J^{(i)}=0}$$

$$= C^{(s)} C(J^{(2)})^2 C(J^{(1)}) \frac{\delta^2}{\delta J^{(1)}(y)\delta J^{(2)}(x)} F(J^{(1)}, J^{(2)})\Bigg|_{J^{(i)}=0} \tag{50}$$

with

$$F(J^{(1)}, J^{(2)}) = \exp\left(q^2 C^{(s)^2} C(J^{(1)})^2 C(J^{(2)}) \int d^2x J^{(2)} A^{-1} J^{(1)}\right) \tag{51}$$

Computing the derivatives, we get

$$< \psi^{(2)}(x)\psi^{(1)}(y) >= q^2 C^{(s)} A^{-1}(x-y). \tag{52}$$

For the free quermions of the action eq.(7), $A = \frac{1}{\pi}\partial_+$ and the propagator is

$$\psi^{(2)}(x^-)\psi^{(1)}(y^-) >= \frac{q^2 C^{(s)}}{(x^- - y^-)}. \tag{53}$$

We can interpret this expression as an extension of the fermionic propagator, since taking the limit $k = 2$ (or $q = -1$), apart from the cocycle, we get the well known result.

5 Concluding Remarks

Using the generalization of the Grassmann variables provided by Quantum Groups [7] we introduce in this letter a class of fields in two dimensions which behave differently from bosons and fermions. These fields have different commutation relations among themselves, indicating that they would have to obey braid group statistics.

We construct an action involving these quermionic fields and show that, after including a scalar term, the total action has an amusing symmetry relating the bosonic and the quermionic fields which resembles supersymmetry. This action, in fact, can be interpreted as the generalization of the heterotic two dimensional Field Theory. We determine also the mass dimension of the quermionic fields, and as expected it is not half-integral as it is in the fermionic case.

As we believe to be interesting to analyse the quantisation of quermionic variables, we start to develop the path-integral formalism for these new fields and we compute the quermionic propagator, within this approach, in the context of the q-heterotic model. The propagator is very simple and, apart from the cocycle and a q-multiplicative term, is similar to that for the $bc\beta\gamma$ system [11] even if the physics is different from that case which can have only bosonic or fermionic statistics.

Finally we hope that the approach we have introduced could bring new ideas in understanding the role of Quantum Groups in string theories and in planar physical systems.

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References

[1] E. K. Sklyanin and L. D. Fadeev, Sov. Phys. Dokl. **23** (1978) 902

[2] V.G. Drinfeld, Sov. Math. Dokl. **27** (1983) 68

[3] S. Majid, Int. Jour. Mod. Phys. A **5** (1990) 1,
S. Sciuto, Lectures given at Chia Conference on "Lower Dimensional QFT and Condensed Matter", Chia, Cagliari, Italy (1990)

[4] S.L. Woronowicz, Comm. Math. Phys. **111** (1987) 613

[5] J.Wess and B. Zumino, "Covariant differential calculus on the quantum hyperplane", preprint CERN–TH–5697/90, LAPP–TH–284/90

[6] Yu.I. Manin, Comm. Math. Phys. **123** (1989) 163

[7] L. Baulieu and E. Floratos, Phys. Lett. B **258** (1991) 271

[8] A.J. Macfarlane, J. Phys. A Math. Gen.**22** (1989) 4581,
L.C. Biedernharn, J. Phys. **A22** (1989) L873

[9] J.L. Matheus-Valle and M.A.R. Monteiro, "Quantum Group Generalization of the Supersymmetric Point Particle" preprint CBPF may 1992

[10] N. Aizawa and H. Sato, Phys. Lett. **B256** (1991) 185

[11] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B **271** (1985) 93