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BIRKHOFF TYPE THEOREM FOR CONFORMALLY TRANSFORMED METRIC

by

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ABSTRACT:

Ricci flat solutions for conformally transformed Schwarzschild and Kasner metrics are shown to correspond to Schwarzschild and Kasner spacetimes respectively.

Key-words: Gravitation; Conformal symmetry.

The Birkhoff theorem[1] states that a spherically symmetric gravitational field in empty space must be static with the metric given by Schwarzschild solution [2]. We establish here the result (somewhat similar in spirit) that the Ricci flat solution corresponding to conformally transformed Schwarzschild metric describes a Schwarzschild spacetime. This is analogous to the fact that in four spacetime dimensions the vacuum solution corresponding to a conformally flat metric is the flat spacetime. We consider also Kasner[3] metric (with time dependent metric tensor) and demonstrate an analogous result for this case as well. The expressions of the diagonal elements of Ricci tensor are formidable even for the simple cases considered here. To handle the information contained in them we construct suitable linear combinations of these elements using the algebraic computation program SHEEP. This results in very simple differential equations and consequently the whole set of equations may be easily integrated.

The conformally transformed Schwarzschild metric may be described by $g_{tt}= B/W^2$, $g_{rr}= 1/(BW^2)$, $g_{\theta\theta}= (r/W)^2$, $g_{\phi\phi}= (r\mathrm{Sin}\,\theta/W)^2$ where $W= W(t,r,\theta,\phi)$ is the conformal factor and $B=1-L/r$, L=const. We find for the off diagonal components

$$R_{tr} = 2W^{-1}W_{,tr} + r^{-1}W^{-1}W_{,t} - B^{-1}r^{-1}W^{-1}W_{,t}$$

$$R_{t\theta} = 2W^{-1}W_{,t\theta}$$

$$R_{t\phi} = 2W^{-1}W_{,t\phi}$$

$$R_{r\theta} = 2W^{-1}W_{,r\theta} - 2r^{-1}W^{-1}W_{,\theta}$$

$$R_{r\phi} = 2W^{-1}W_{,r\phi} - 2r^{-1}W^{-1}W_{,\phi}$$

$$R_{\theta\phi} = -2W^{-1}\cos(H)\sin^{-1}(H)W_{,\phi} + 2W^{-1}W_{,\theta\phi} \tag{1}$$

The useful combinations of the diagonal components are found to be

$$X1 = W(R_{tt} + B^2 R_{rr})/2$$
$$= B^2 W_{,rr} + W_{,tt}$$

$$X2 = -W(R_{\theta\theta} - R_{\phi\phi}/SIN^2\,\theta)/2$$
$$= \cos(\theta)\sin^{-1}(\theta) W_{,\theta} - W_{,\theta\theta} + \sin^{-2}(\theta) W_{,\phi\phi}$$

$$X3 = W(R_{tt} r^2/B + R_{\theta\theta})/(2r)$$
$$= 3/2 B W_{,r} - 1/2 W_{,r} + r^{-1} W_{,\theta\theta} + B^{-1} r W_{,tt}$$

(2)

Writing $W_{,t}=F$, $W_{,r}=A$, $W_{,\theta}=D$, $W_{,\phi}=C$ and requiring that the Ricci tensor vanish we obtain $F_{,\theta} = D_{,t}=0$, $F_{,\phi}= C_{,t}=0$, $F_{,r}= A_{,t}= F\, B_{,r}/(2B)$, $A_{,\theta} = D_{,r}=D/r$, $A_{,\phi} = C_{,r}= C/r$, $D_{,\phi}= C_{,\theta} = C \operatorname{Cos}\theta / \operatorname{Sin}\theta$ and $C_{,\phi}= (D_{,\theta} \operatorname{Sin}\theta - D \operatorname{Cos}\theta)\operatorname{Sin}\theta$ corresponding to X2 =0. They may be readily integrated to give

$$F= g(t)\, B^{1/2}$$
$$D= K \operatorname{Cos}\theta$$
$$C= K_{,\phi} \operatorname{Sin}\theta$$

(3)

where $K= r(a \operatorname{Cos}\phi + b \operatorname{Sin}\phi)$ and a,b are constants. It follows also that

$$A= K \operatorname{Sin}\theta/r + f(r,t) \qquad (4)$$

On now making use of $F_{,t}+ B^2 A_{,r}=0$ following from X1=0 we find $g_{,tt} = g(t)L(4r-3L)/(4r^4)$ which implies $g = 0$. Consequently, $W_{,t}= F=0$ and $A_{,t}=A_{,r}=0$ (e.g. f=const.). Imposing then X3 =0 leads to

$$3(B-1)\ K\ \mathrm{Sin}\ \theta/r\ +(3B-1)\ f=0 \qquad (5)$$

Hence f= K=0 and W must be spacetime independent for Ricci flat solution. The simplicity of the demonstration here and in the next case owes to the determination of simple combinations of the otherwise very complicated diagonal elements of the Ricci tensor which we do not present here.

Consider next the conformally transformed Kasner metric with $g_{tt}=1/W^2$, $g_{xx}=(t^P/W)^2$, $g_{yy}=(t^Q/W)^2$, $g_{zz}=(t^R/W)^2$ where $W=W(t,x,y,z)$ and $P+Q+R = P^2+Q^2+R^2 = 1$ which ensures that we have a vacuum solution when W is a constant. It is convenient to parameterize P,Q,R as $P = -U/(1+U+U^2)$, $Q=(1+U)/(1+U+U^2)$, $R=U(1+U)/(1+U+U^2)$ where $U \geq 1$ and we note that $P(1/U)=P(U)$, $Q(1/U)=R(U)$, $R(1/U)=Q(U)$. The nondiagonal elements are

$$R_{tx} = -2Pt^{-1}W^{-1}W_{,x} + 2W^{-1}W_{,tx}$$

$$R_{ty} = -2Qt^{-1}W^{-1}W_{,y} + 2W^{-1}W_{,ty}$$

$$R_{tz} = -2Rt^{-1}W^{-1}W_{,z} + 2W^{-1}W_{,tz}$$

$$R_{xy} = 2W^{-1}W_{,xy}$$

$$R_{xz} = 2W^{-1}W_{,xz}$$

$$R_{yz} = 2W^{-1}W_{,yz}$$

(6)

For these components to vanish we find $W_{,x}=F(x)t^P$, $W_{,y}=G(y)t^Q$, $W_{,z}=H(z)t^R$ where F,G,H are arbitrary functions. The following set of combinations of the otherwise quite messy diagonal components result in simple expressions

$$X1 = R_{tt} + R_{xx}/t^{2P}$$

$$= -2Pt^{-1}W^{-1}W_{,t} + 2W^{-1}W_{,tt} + 2t^{-P}W^{-1}F_{,x}$$

$$X2 = R_{yy} - R_{zz}t^{2(P+2Q-1)}$$

$$= -2Qt^{2Q-1}W^{-1}W_{,t} - 2t^{P+3Q-1}W^{-1}H_{,z} + 2t^{Q}W^{-1}G_{,y}$$

$$X3 = R_{xx}t^{-2P} - R_{yy}t^{-2Q}$$

$$= -2Pt^{-1}W^{-1}W_{,t} + 2Qt^{-1}W^{-1}W_{,t} - 2t^{-Q}W^{-1}G_{,y} + 2t^{-P}W^{-1}F_{,x}$$

$$X4 = R_{tt}/3 + R_{zz}\,t^{2(P+Q-1)}$$

$$= -2F^{2}W^{-2} - 2G^{2}W^{-2} - 2H^{2}W^{-2} + 2Pt^{-1}W^{-1}W_{,t} +$$

$$+2Qt^{-1}W^{-1}W_{,t} + 8/3t^{P+Q-1}W^{-1}H_{,z} + 2W^{-2}(W_{,t})^{2} - 8/3t^{-1}W^{-1}W_{,t} +$$

$$+2/3t^{-Q}W^{-1}G_{,y} + 2/3t^{-P}W^{-1}F_{,x} \qquad (7)$$

where the substitution $W_{,t} = [-t^{(1-Q)}G_{,y} + t^{(1-P)}F_{,x}]/(P-Q)$ corresponding to X3=0 has been made in X1, X2, X4. On requiring X1=0 identically we find

$$G_{,y} + F_{,x}\,t^{(2U+1)/(1+U+U^{2})} = 0 \qquad (8)$$

It follows that $G_{,y} = F_{,x} = 0$ which implies $W_{,t} = 0$ and X2 reduces to a term proportional to $H_{,z}$ which must then also vanish. Inserting these results in X4 leads to $F^2+G^2+H^2=0$ implying F=G=H=0 and for this case as well only a spacetime independent conformal factor is allowed .

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