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QUANTUM GROUP GENERALIZATION OF THE CLASSICAL SUPERSYMMETRIC POINT PARTICLE

by

J.L. MATHEUS-VALLE and Marco A.R. MONTEIRO

Abstract

Following the generalization of the Grassmann Algebra provided by Quantum Groups, we introduce an extension of fermionic coordinates and an action for the classical point particle mechanics which has a symmetry that resembles a supersymmetric transformation.

Key-words: Point particle; Supersymmetry; Quantum Groups; Quasitriangular Hopf Algebra.

1 Introduction

For the last few years Quasitriangular Hopf Algebras or Quantum Groups [1,2,3] have been attracting a lot of interest from physicists and mathematicians. One of its interesting features is the fact that these theories can be related to physical problems where the coordinates are non-commutative. On the non-commutative space of a Quantum Group it was shown [4] to be possible to define consistently a non-commutative differential geometry and a simple example was developed [5] on the quantum plane [6].

Recently it was shown [7] that one could understand these non-commutative coordinates as the classical limit of the creation and annihilation operators of the deformed Heisenberg Algebra [8] through the introduction of a complex parameter q. In this case it is possible to interpret these coordinates as a generalization of the Grassmann variables [7].

The deformed creation and annihilation operators satisfy the following commutation relations [7,8]:

$$aa^{\dagger} - qa^{\dagger}a = q^{-N} \tag{1}$$

in the case of $q = \exp(2\pi i/k)$, with k an integer, it can be shown that these operators satisfy:

$$a^k = (a^{\dagger})^k = 0 \tag{2}$$

which shows to have a discrete spectrum. If, after a suitable change of variables, we consider the classical limit of the above system, we get [7]

$$z\overline{z} = q\overline{z}z, \tag{3}$$

with

$$z^k = (\overline{z})^k = 0. \tag{4}$$

These classical coordinates, z and $\overline{z}$, can be interpreted as a generalization of the Grassmann variables [7]. The differential calculus with these coordinates was constructed and it was shown to have an underlying Yang-Baxter structure and a covariance of the equation under the action of the two parameter quantum group $GL_{q^2,q^{-1}}$ [7].

Notice that if we take $k \to \infty$, $q \to 1$, eq. (1) becomes the usual oscillator algebra with infinite occupation number, and the classical limit gives the usual commuting coordinates. Taking $k = 2$, eq. (1) gives anticommutation relations for a, $a^{\dagger}$, the occupation number is two and the classical variables are the Grassmannian ones. As for intermediate values of k the variables interpolate bosonic and fermionic cases, it would be interesting to explore these cases.

In this letter, we address the problem of constructing an action for the classical point particle using these generalized Grassmann coordinates for the case $k = 3$, and we show that this action has a symmetry involving these generalized coordinate fields in a way very similar to supersymmetry. In section 2 we introduce these new q-fermionic generalized coordinate fields and construct the action, in section 3 we analyse the symmetry of the action and finally in section 4 we discuss the perspectives, possible consequences of this discovery and some comments.

2 q-fermionic coordinates and the classical generalized supersymmetric point particle action

To understand how these generalized fields appear, we remind ourselves that in supersymmetry we have a superspace with coordinates (z, t), $z^2 = 0$ and t a c-number to be identified with time (all our discussions here will be at the classical mechanical level).

Functions in the superspace are polynomials in z, and this gives us two different types of fields: fermions, that under commutation relations behave like the Grassmann variable z, that is, anticommute among themselves; and bosons, that behave like z^0, that is, commutes with everything. Translations in the superspace give rise to supersymmetry, and fix the dimension of z with respect to t.

If now we consider the case $z^3 = 0$, we will have three different types of fields: those that under commutations relations behave like z^0 (bosons, or a sector-zero field); like z (sector-one field); or like z^2 (sector-two field).

Having introduced these generalized fields, we define now the q-commutator (from now on we call it quommutator) between two fields $A^{(r)}$ and $B^{(s)}$ as:

$$[A^{(r)}, B^{(s)}]_q \equiv q^{(-rs/2)} A^{(r)} B^{(s)} - q^{(rs/2)} B^{(s)} A^{(r)}, \tag{5}$$

where the supperscript indicates the sector of the field. This is the simplest relation which garantees that any boson obeys the usual commutation relation with any other field, because in this case the q-factor disapears. It gives the correct limit for Grassmann variables in the case $k = 2$ and, in the limit $k \to \infty$ we recover the usual commutator.

To fix the dimension of z, we introduce the following transformation in the q-superspace:

$$\begin{aligned} z &\rightarrow z' = z + \epsilon^{(1)} \\ t &\rightarrow t' = t + q z^2 \epsilon^{(1)}, \end{aligned} \tag{6}$$

with $\epsilon^{(1)}$ an infinitesimal constant of the sector one, ensured by homogeneity of δz .

The factor z^2 is chosen in the t-transformation because in general, the product of a sector-two with a sector-one field yields something that lives in the sector–zero. We extend the case of bosons and fermions so that, in general, the product $A^{(r)}B^{(s)}$ belongs to the sector $(r+s) \bmod 3$ in the case of $k=3$. We will discuss in a moment some subtleties related to the product of these fields. In this t-transformation the factor q is needed to preserve the reality of t, and this is decided by the choice $[\epsilon^{(1)}, z]_q = 0$.

Equation 6 fixes the dimensions of z and $\epsilon^{(1)}$, which are:

$$[z] = [\epsilon^{(1)}] = 1/3. \tag{7}$$

We can also construct a transformation with an infinitesimal parameter which belongs to the sector–two:

$$\begin{aligned} z^2 &\rightarrow z'^2 = z^2 + \epsilon^{(2)} \\ t &\rightarrow t' = t + qz\epsilon^{(2)}, \end{aligned} \tag{8}$$

where we chose $[z, \epsilon^{(2)}]_q = 0$, and this gives $[\epsilon^{(2)}] = 2/3$.

Our next step is to construct an action which extends the supersymmetric point particle through the use of these generalized fields. This generalized particle is described by the coordinates $(x(t), \psi^{(1)}(t), \psi^{(2)}(t))$, in the same way as a supersymmetric point particle is described be the coordinates $(x(t), \psi(t))$. We call the $\psi^{(i)}(t)$ the q-fermionic generalization of the coordinates or, simply, the quermionic coordinates.

The action involving the quermions is given by

$$S = \int dt(\frac{1}{2}\dot{x}^2 - qC^{(s)^2}\dot{\psi}^{(2)}\psi^{(1)}), \tag{9}$$

where we choose the mass equal to one. The first derivative in the quermionic term is chosen such that the classical equations of motion for $\psi^{(i)}$ resemble that for fermions. The cocycle-type factor $C^{(s)^2}$ is required because when we multiply two objects of different sectors, $A^{(r)}B^{(s)}$, it must behave like an object of the sector $(r+s) \bmod 3$, but if this factor is not inserted this product would not quommute correctly with $A^{(r)}$ or $B^{(s)}$, and underlying this point is the fact that in the fermionic case the fields anticommute even if they are equal but in this generalized case we choose equal fields at equal points as commuting ones.

This cocycle–type factor $C^{(s)}$ actually behaves like a sector–counter, that is,

$$C^{(s)}A^{(i)} = q^i A^{(i)}C^{(s)}. \tag{10}$$

Finally, with the choice $[\psi^{(1)}, \psi^{(2)}]_q = 0$, taking all the fields as real, the second term in the action, eq. (9), becomes real and is a representative of the zeroth sector.

3 Analysis of the symmetry of the action

We now want to show that the action, eq. (9), has a symmetry, by using the infinitesimal parameter $\epsilon^{(1)}$, which resembles a supersymmetric transformation. As $\epsilon^{(1)}$ is a sector–one object, it is natural to choose $\delta\psi^{(1)} \sim \epsilon^{(1)}\partial_t{}^l x$, $\delta x \sim \epsilon^{(1)}\partial_t{}^m\psi^{(2)}$ and $\delta\psi^{(2)} \sim \epsilon^{(1)}\partial_t{}^n\psi^{(1)}$, that is, a cyclical combination of the coordinates.

To fix the multiplicative factors in the variations we should keep in mind that each of these variations must be real and behave like the field itself under quommutation with the other fields. This determines the powers of $C^{(s)}$ and q in each variation.

Now, in order to determine (l, m, n) we compute δS and impose it to be at least a total derivative. This gives that n is even, and the condition that $[\psi^{(i)}]$ is greater than zero implies that $m = n = 0$, $l = 1$, and the dimension of $\psi^{(i)}$ is

$$[\psi^{(i)}] = i/3. \tag{11}$$

The transformation is given by

$$\begin{aligned} \delta x &= qC^{(s)}\epsilon^{(1)}\psi^{(2)} \\ \delta\psi^{(1)} &= q^2 C^{(s)^2}\epsilon^{(1)}\dot{x} \\ \delta\psi^{(2)} &= \pm q\epsilon^{(1)}\psi^{(1)} \end{aligned} \tag{12}$$

and the action submitted to this transformation behaves as

$$\delta S = \pm\int dt \frac{d}{dt}(\epsilon^{(1)}\psi^{(1)^2}), \tag{13}$$

where we used $[\epsilon^{(1)}, \psi^{(1)}]_q = [\psi^{(2)}, \epsilon^{(1)}]_q = 0$.

As in supersymmetry, the action transforms as a total derivative. Another similar feature is that one of the fields, $\psi^{(1)}$, transforms as a total derivative which can be taken as indicating that $\psi^{(1)}$ is the highest term in a z–expansion of some superfield, in fact, a naive expansion of a scalar superfield could be written as $\Phi = x + z\psi^{(2)} + z^2\psi^{(1)}$. Similar arguments can be used to construct a transformation involving the sector–two infinitesimal parameter $\epsilon^{(2)}$. Once more, powers of $C^{(s)}$ and q are fixed by constraints of homogeneity and reality of the transformation. The powers of the derivatives are immediately determined since we know all dimensions. We then get

$$\begin{aligned} \delta x &= -q^2 C^{(s)} \epsilon^{(2)} \psi^{(1)} \\ \delta \psi^{(1)} &= \pm q^2 \epsilon^{(2)} \dot{\psi}^{(2)} \\ \delta \psi^{(2)} &= q C^{(s)^2} \epsilon^{(2)} \dot{x}, \end{aligned} \tag{14}$$

with the choice $[\epsilon^{(2)}, \psi^{(1)}]_q = [\psi^{(2)}, \epsilon^{(2)}]_q = 0$.

However, in this case δS is not a total derivative. In fact its variation is given by

$$\delta S = \int dt \left(-q^2 C^{(s)} \epsilon^{(2)} \frac{d}{dt} (\dot{x} \psi^{(1)}) \pm q C^{(s)^2} \epsilon^{(2)} \dot{\psi}^{(2)^2} \right). \tag{15}$$

The fact that in this case a field squared does not vanish implies that $\epsilon^{(2)}$ cannot be taken as an infinitesimal parameter of a transformation under which the action is invariant.

Concludingly, the action, eq. (9), is invariant under the generalized supersymmetry eq. (6) which takes the bosonic component into a quermionic one and a quermionic component into another quermionic one.

4 Concluding Remarks

Using the generalization of the Grassmann variables provided by Quantum Groups [7] we introduce in this letter a class of fields which behave differently from bosons and fermions which we call quermions. With quermionic coordinates we construct an action with a symmetry which is an extension of the standard supersymmetric classical point particle.

We believe that the role of these quermions in the relativistic case, specifically in two or three dimensional Quantum Field Theory, it will be that representing particles with arbitrary spin, particularly in three dimensions without the need of the Chern–Simons term [9], the reasons are the following. Firstly, as we mentioned in the introduction, the variables we considered correspond to an extension of the Grassmann variables and for $2 < k < \infty$ interpolate between Grassmannian and commuting variables. Secondly, the way these variables and fields quommute could be understood as a generalization of the commutator through the R-matrix of $GL_{q^2,q^{-2}}$ [5,7] and it is believed that particles quantized in such a way would have Braid Group statistics [3]. Lastly, the dimension of the quermionic coordinates depends on k, and for the standard supersymmetric case it corresponds to the conformal spin in two dimensional conformal field theories.

Finally, we consider that it would be interesting to investigate the field theoretical approach in two [10] and three dimensions, the superspace formalism [11] and the possible consequences of these theories for string theories and in condensed matter physics.

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References

[1] E. K. Sklyanin and L. D. Fadeev, Sov. Phys. Dokl. **23** (1978) 902

[2] V.G. Drinfeld, Sov. Math. Dokl. **27** (1983) 68

[3] S. Majid, Int. Jour. Mod. Phys. A **5** (1990) 1,
S. Sciuto, Lectures given at Chia Conference on "Lower Dimensional QFT and Condensed Matter", Chia, Cagliari, Italy (1990)

[4] S.L. Woronowicz, Comm. Math. Phys. **111** (1987) 613

[5] J.Wess and B. Zumino, "Covariant differential calculus on the quantum hyperplane", preprint CERN-TH-5697/90, LAPP-TH-284/90

[6] Yu.I. Manin, Comm. Math. Phys. **123** (1989) 163

[7] L. Baulieu and E. Floratos, Phys. Lett. B **258** (1991) 271

[8] A.J. Macfarlane, J. Phys. A Math. Gen.**22** (1989) 4581,
L.C. Biedernharn, J. Phys. **A22** (1989) L873

[9] A. Lerda, Lectures given at CBPF-RJ Brazil (1991)

[10] J.L. Matheus-Valle and M.A.R. Monteiro, work in progress

[11] J.L. Matheus-Valle and M.A.R. Monteiro, work in progress