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STUDY OF WAVE EQUATIONS INVOLVING ITERATED LAPLACIAN AND POTENTIAL $r^{-\beta}$ BY THE 1/N METHOD

by

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ABSTRACT

The relation between the potential $r^{-\beta}$ and the asymptotic behavior of the eigenvalues of the ordinary wave equation for space dimension $N \to \infty$ is found. An analogous relation is discussed when the wave equation involves an iterated Laplacian. A criterion is given to determine when a potential should be considered singular, depending on the space dimension and the degree of iteration of the Laplacian. Finally, a modification is suggested for the approximation proposed by Witten which sensibly improves the results for the ground state energy of the hydrogen and helium atoms.

Key-words: Solutions of wave equations: bound states; Field theories in higher dimensions.

We apply the 1/N expansion method[1,2] to the wave equation

$$(-1)^n \Delta^n \psi - \frac{\alpha}{r^\beta} \psi = E\psi \ . \qquad (1)$$

Here Δ is the Laplacian operator, n is an integer, α and β are parameters and E an eigenvalue. For simplicity we restrict ourselves to spherically symmetric solutions, thus we put

$$\Delta = \frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr} \ . \qquad (2)$$

Here N is the space dimension. The transformation

$$\psi = r^a \varphi \quad \text{with} \quad a = \frac{1-N}{2} \qquad (3)$$

eliminates the derivative of order 2n-1. In particular

$$\Delta\psi = r^a \frac{d^2\varphi}{dr^2} - a(a+1)\, r^{a-2}\, \varphi \ . \qquad (4)$$

By making the additional transformation (dilatation)

$$r = N^\tau R \qquad \text{with} \qquad \tau = \frac{2}{2-\beta} \qquad (5)$$

we obtain in the case of second order (n = 1) the following equation for φ

$$-\frac{1}{N^2}\frac{d^2\varphi}{dR^2} + \frac{a(a+1)}{N^2R^2}\varphi - \frac{\alpha}{R^\beta}\varphi = E\frac{N^{2\tau}}{N^2}\varphi. \qquad (6)$$

For large N, the dominant form of Eq. (6) is $V_{eff}\ \varphi = E\ \varphi$ where

$$V_{eff} = \left(\frac{1}{4R^2} - \frac{\alpha}{R^\beta}\right)\frac{1}{N^{\beta\tau}} \ . \qquad (7)$$

The minimum of the effective potential V_{eff} is at

$$R_0 = \left(\frac{1}{2\alpha\beta} \right)^{\frac{1}{2-\beta}} \tag{8}$$

The case of the Schrödinger equation for the hydrogen atom in N dimensions leads to, (introducing now the electron mass m and charge e),

$$V_{eff} = E_0 \frac{4}{N^2} \left(2 - \frac{(N-1)(N-3)}{N^2} \right) \tag{9}$$

where

$$E_0 = - me^4/2 \tag{10}$$

is the exact ground state energy of the hydrogen atom in 3 dimensions. According to the traditional 1/N expansion method[1] the approximate ground state energy in three dimensions is obtained by taking the limit $N \to \infty$ in the parenthesis of Eq. (9) and thereafter putting N = 3, in which case the result $V_{eff} = -4E_0/9$ is obtained. We want to call the attention to the fact that a better result is obtained by a straightforward replacement of N = 3 in Eq.(9) to give $V_{eff} = -8E_0/9$. Note that in Eqs. (4) and (6) the term a(a+1) vanishes for N=3. An analogous procedure leads to a considerably better value for the ground state energy of the helium atom too. In fact, the traditional 1/N expansion method yields the value $-1.217\ me^4$ for the ground state energy of the helium atom while by simply using N = 3 in the corresponding effective potential yields the value $-2.434\ me^4$ which is 84% of the experimental value $-2.90\ me^4$. Comparable accuracy (88%) requires at least three terms in the series expansion in 1/N. Values as close as 99.96% were obtained by extensions of the method which single out singularities responsible for slow convergence[2].

We see from Eqs.(7) and (8) that for large N the ground state energy depends on N like

$$E_g \propto N^{-\frac{2\beta}{2-\beta}}, \tag{11}$$

which shows explicitly the relation between the behavior of the ground state energy for $N \to \infty$ and the power β of the potential.

Eq. (1) for n=1 (second order equation) in three dimensions can be solved approximately by the WKB method which should be reliable for large quantum numbers ν. The eigenvalues behave like

$$E_\nu \propto \nu^{-\frac{2\beta}{2-\beta}} \tag{12}$$

It should be noted that the energy eigenstates for large quantum numbers ν have the same functional dependence on β as the ground state in N dimensions for large N.

Analogous considerations can be made for arbitrary n. For the calculation of the effective potential we are only interested in that part of $\Delta^n\psi = \Delta^n(\varphi/r^{(\nu-1)/2})$ which does not contain derivatives of φ. We then use the following formula[3]

$$(-1)^n\Delta^n r^a = 2^{2n}\ (\lambda+1)\ldots(\lambda+n)(\lambda+N/2)\ldots(\lambda+N/2+n-1)r^{a-2n} \tag{13}$$

with

$$\lambda = (1 - N)/4 - n = a/2 - n \tag{14}$$

to write the effective potential

$$V_{eff} = \left(\frac{\Lambda}{N^{2n}R^{2n}} - \frac{\alpha}{R^\beta} \right) \frac{1}{N^{\beta\tau}} \tag{15}$$

where here $\tau = 2n/(2n-\beta)$ and Λ is a number which depends on n and N; in the special case n=1, $\Lambda(1,N) = a(a+1)$, and for large N the asymptotic behavior is $\Lambda(n,N) \cong (N/2)^{2n}$.

The first derivative of V_{eff} with respect to R is

$$\frac{dV_{eff}}{dR} = \frac{\alpha\beta}{R^{\beta+1}} - \frac{2n\Lambda}{N^{2n}R^{2n+1}}\ , \tag{16}$$

and equating (16) to zero yields, for large N, the extremum

$$R_0 = \left(\frac{2n}{2^{2n}\alpha\beta} \right)^{\frac{1}{2n-\beta}} . \qquad (17)$$

The value of the second derivative of V_{eff} at R_0, for large N, is

$$\left(\frac{d^2V_{eff}}{dR^2}\right)_{R_0} = \frac{\alpha\beta\ (2n-\beta)}{R_0^{\beta+2}} \qquad (18)$$

The ground state energy is obtained by replacing R_0 in (15):

$$E_g = -\frac{\alpha}{N^{\frac{2n\beta}{2n-\beta}} \left(\frac{2n}{2^{2n}\alpha\beta} \right)^{\frac{\beta}{2n-\beta}}} \cdot \frac{2n-\beta}{2n} \qquad (19)$$

In particular, for $\beta = 1$, $n = 2$ we obtain $E_g = -0.36\ \alpha^{4/3}$ which should be compared with the numerical result $-0.3626\ \alpha^{4/3}$ (See Ref.4).

We see from Eq. (18) that when $\beta > 2n$ the extremum in R_0 is a maximum; the effective potential has no minimum and we say that the potential is singular. In the usual case $n = 1$, any $\beta \geq 2$ gives a singular potential.

For $n = 2$, and $\beta < 4$ the potential has a ground state and it is non-singular.

The fact of the potential being singular or not depends on the power of the Laplacian and the space dimension.

The Green's function of Δ^n (its "Coulomb potential), is[5]

$$G(r) \propto \Gamma(-n+N/2)\ r^{2n-N}. \qquad (20)$$

Thus $\beta = N - 2n$ and $2n - \beta = 4n - N$. If $4n > N$ the potential is not singular.

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