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EXACT COSMOLOGICAL SOLUTIONS OF EINSTEIN MAXWELL EQUATIONS AS PERTURBATIONS OF THE BERTOTTI-ROBINSON MODEL

by

R. Portugal* and I. Damião Soares

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brāsil

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Abstract

Two new classes of spatially homogeneous cosmological solutions of Einstein-Maxwell equations are obtained by considering a class of exact perturbations of the static Bertotti-Robinson (BR) model. The BR solution is shown to be unstable under these perturbations, being perturbed into exact cosmological solutions with perfect fluid (equations of state $p = \lambda\rho$, $0 \leq \lambda \leq 1$), isotropic/anisotropic expansion and non-null electric conductivity.

Key-words: Bianchi cosmologies; Kantowski-sachs cosmologies; Magnetohydrodynamic cosmologies.

Exact cosmological solutions of Einstein-Maxwell equations have been extensively examined in the literature (Kramer et al. 1980). Among them the simplest one is the static solution of Bertotti (1959) and Robinson (1959), denoted here by BR, with parallel electric and magnetic fields and without matter. It can actually be shown that the BR solution is the only Einstein-Maxwell field which is homogeneous and has a homogeneous non-null Maxwell field (Kramer et al. 1980). In the present paper we derive two new classes of exact cosmological solutions of Einstein-Maxwell equations. They have the BR solution as a limiting configuration and are obtained by exact perturbations of the BR model. In fact the BR model is shown to be unstable under these perturbations, being perturbed into exact isotropic/anisotropic cosmological solutions with perfect fluid, electromagnetic fields and non-null electric conductivity.

The Bertotti-Robinson solution considered here is the one which has topology $R \times R \times S^2$. Starting from the BR geometry we make a class of time-dependent perturbations, to obtain a new manifold with the same topology $R \times R \times S^2$ and time-dependent geometry of Kantowski-Sachs type (Kantowski and Sachs 1966, Ryan and Shepley 1975) given in local coordinates (t,χ,θ,ϕ) by

$$ds^2 = dt^2 - A^2(t)d\chi^2 - B^2(t)(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1)$$

The dynamics of the perturbed models are described by Einstein-Maxwell equations* (Hawking and Ellis 1973, Misner et al. 1970)

*The Riemann tensor is defined by $V_{\alpha||[\beta||\delta]} = R^\lambda{}_{\alpha\beta\delta}V_\lambda$, and the Ricci tensor by $R_{\alpha\beta} = R^\lambda{}_{\alpha\lambda\beta}$, which implies that the Einstein constant κ is positive.

$$F^{\alpha\beta}{}_{||\beta} = j^{\alpha} \qquad (2)$$

$$F_{[\alpha\beta||\gamma]} = 0 \qquad (3)$$

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta} \qquad (4)$$

We take for the matter content of the model a perfect fluid with matter-energy density ρ and pressure p, as measured by the comoving observers with four-velocity field $\partial/\partial t$. The total-energy-momentum tensor is then given by

$$T_{\alpha\beta} = (\rho + p) v_{\alpha} v_{\beta} - p g_{\alpha\beta} + F_{\alpha\lambda} F^{\lambda}{}_{\beta} + \frac{1}{4} g_{\alpha\beta} F_{\lambda\rho} F^{\lambda\rho} \qquad (5)$$

The electric four current j^{α} is either zero or a pure conducting current satisfying $j^{\alpha} v_{\alpha} = 0$*. We assume that Ohm's law is valid in the local Lorentz frames of the comoving observers. The perfect fluid is a conducting fluid and we have in covariant form

$$j^{\alpha} = - \sigma v_{\beta} F^{\alpha\beta} \qquad (6)$$

where σ is the electric conductivity of the fluid.

From the symmetries of the line element (1) we restrict the Maxwell tensor $F^{\alpha\beta}$ to have the only non-null component

$$F^{01} = - F^{10} = E(t) \qquad (7)$$

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*The space-like character of j^{α} implies that the density of electric charge of the fluid is zero.

In the cases of $\sigma = 0$ a parallel magnetic field can always be introduced by a dual rotation of the Maxwell tensor (Misner and Wheeler 1957).

For (1) and (7), Maxwell's equations (2)-(3) yield

$$\sigma = -\frac{d}{dt}\ \ell n(EB^2) \tag{8}$$

and Einstein's equations reduce to the set of independent equations (cf. Appendix)

$$2\kappa\rho = R^0{}_0 - 3R^1{}_1 + \kappa E^2 + 2\Lambda \tag{9a}$$

$$2\kappa p = R^0{}_0 + R^1{}_1 - \kappa E^2 - 2\Lambda \tag{9b}$$

$$R^2{}_2 - R^1{}_1 + \kappa E^2 = 0 \tag{10}$$

Equation (8) is taken as the definition of the electric conductivity, and equations (9a,b) define ρ and p respectively. For all the solutions discussed here we impose the equation of state

$$p = \lambda\rho \quad , \quad 0 \le \lambda \le 1 \tag{11}$$

We distinguish the following cases:

I) <u>The Berttoti-Robinson Universe</u>

$A^2 = B^2 = \lambda^2$, where λ^2 is a constant. We obtain from (8)-(10)

$$\kappa E^2 = \frac{1}{\lambda^2} \quad , \quad 2\Lambda = -\frac{1}{\lambda^2} \tag{12}$$

with $\rho = p = 0$ and $\sigma = 0$.

II) Exact isotropic perturbations of the BR universe

We take $A(t) = B(t)$. Equations (9)-(10) result

$$\kappa E^2 = \frac{1}{B^2}$$

$$\kappa \rho = 3\left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{2B^2} + \Lambda$$

$$\kappa p = -2\,\frac{\ddot{B}}{B} - \left(\frac{\dot{B}}{B}\right)^2 - \frac{1}{2B^2} - \Lambda$$

The equation of state (11) implies the differential equation for $B(t)$,

$$\frac{\ddot{B}}{B} + \frac{3\lambda+1}{2}\left(\frac{\dot{B}}{B}\right)^2 + \frac{(1+\lambda)}{4B^2} + \frac{(1+\lambda)\Lambda}{2} = 0 \qquad (13)$$

Introducing the new variable $\tilde{t}$ defined by $d\tilde{t} = B^{-\frac{3\lambda+1}{2}}\,dt$, equation (13) can be expressed as

$$B'' + \frac{(1+\lambda)}{4}\,B^{3\lambda} + \frac{(1+\lambda)\Lambda}{2}\,B^{3\lambda+2} = 0 \qquad (14)$$

where a prime denotes $\tilde{t}$-derivative. It is easy to check that $\tilde{t}$ is a monotonic function of t. Equation (14) has the first integral

$$(B')^2 + V(B) = C \qquad (15)$$

where C is an integration constant and

$$V(B) = \frac{1+\lambda}{2(1+3\lambda)}\,B^{3\lambda+1} + \frac{\Lambda}{3}\,B^{3\lambda+3} \qquad (16)$$

For this class of solutions we calculate

$$\kappa\rho = \frac{3C}{B^{3\lambda+3}} - \frac{1}{(3\lambda+1)B^2} \tag{17}$$

The dynamics and properties of the model are completely described by equations (15) and (16), and depend critically on the sign of Λ. We consider here the case $\Lambda < 0$ only. The behaviour of V(B) is depicted in Fig. 1. Since $B = 0$ is the physical singularity of the model (cf. expression (17)) we plot the physical relevant part of V(B) only. The maximum of the potential V(B) occurs at

$$B_c = \sqrt{\frac{1}{-2\Lambda}} \tag{18}$$

where V(B) takes the value

$$V_{max} = \frac{1}{3(3\lambda+1)} \left(\frac{1}{-2\Lambda}\right)^{\frac{3\lambda+1}{2}} \tag{19}$$

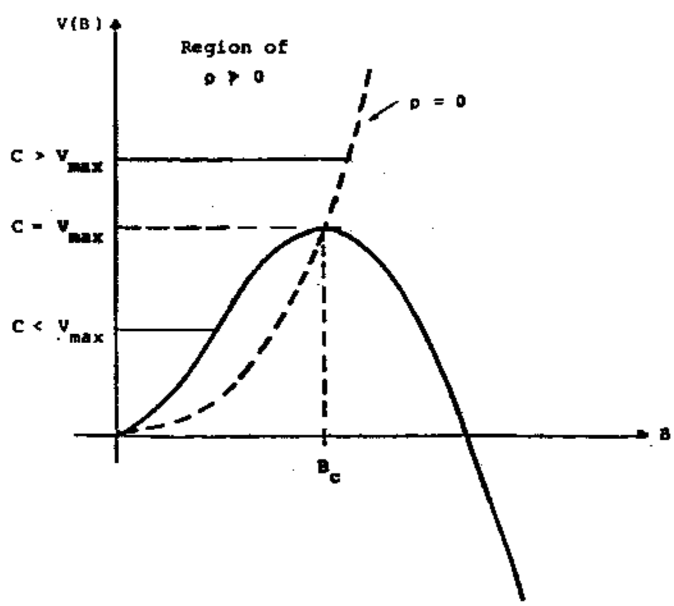

Fig. 1: Graph of the qualitative behaviour of V(B) given in Eqs.(16) and (24). The curve $\rho(C,B) = 0$ is also depicted.

We also draw in Fig. 1 the curve $\rho = 0$, ρ being considered as a function of B and C. We call attention to the remarkable fact that for $C = V_{max}$, ρ is equal to zero at $B = B_c$.

The following physical situation are possible (cf. Fig. 1):

IIi) $C = V_{max}$

For this case the dynamical system (14) admits the point ($B = B_c$, $\dot{B} = 0$) as a critical unstable point. This point corresponds to the static configuration with density ρ, pressure p and conductivity σ equal to zero, which is the Berttoti-Robinson solution with topology $R \times R \times S^2$.

IIii) From Fig. 1 we easily see that this solution is unstable under perturbations involving the electric field, the density ρ and the conductivity σ. By perturbations of $C = V_{max}$ into $C = V_{max} - \varepsilon^2$ we are led from the Berttoti-Robinson solution to physical solutions of Einstein-Maxwell equations contracting isotropically from the radius B_0 (solution of $V(B_0) = C > 0$) towards the point-like singularity $B = 0$. These solutions have a pure electric field, and positive conductivity σ as can be calculated from (8). The density ρ is positive always.

At this point the meaning of <u>exact</u> perturbations and instability of the BR solution becomes clear. Consider for instance the perturbation $C = V_{max} \to C = V_{max} - \varepsilon^2$, ε^2 infinitesimal. The critical point of the system $P_0 = (B = B_c, \dot{B} = 0)$ - which corresponds to the static BR configuration - is perturbed to a point $P_\varepsilon = (B = B_c + 0(\varepsilon), \dot{B} = 0)$ infinitesimally close to P_0. The point P_ε is now taken as the initial conditions of the perturbed gravitational system. The time development of the system as given by (14) with the prescribed initial conditions will depart largely from the unperturbed configuration P_0. This trajectory corresponds to an exact solution of Einstein-Maxwell equations.

We exclude the cases $C > V_{max}$ because the values of B must be restricted by $B \leq B_M$, where $B_M = (3C(\lambda+1))^{\frac{1}{3\lambda+1}}$ is the value of B where $\rho = 0$. Since beyond B_M the density ρ would become negative these expanding solutions are not physically satisfactory.

For the above cases $0 < C \leq V_{max}$, equation (15) can be integrated in terms of analytic functions. The general solution has the expression

$$\tilde{t} - \tilde{t}_0 = \int \frac{dB}{\sqrt{C - V(B)}}$$

For $\lambda = 0$, 1/3, 2/3, 1 the solutions $(\tilde{t} - \tilde{t}_0)$ are given by Jacobian elliptic functions (Abramowitz et al. 1965). In the general case the solutions are given by hypergeometric functions (Erdélyi et al. 1953). The detailed expressions will be published elsewhere.

We finally remark that although these models have an isotropic expansion the presence of electric fields implies that the spatial sections t = const. have a privileged direction locally defined by $\partial/\partial\chi$.

III) Exact anisotropic perturbations of the BR universe

We consider $A^2 = \lambda^2$, where λ^2 is a constant. Equations (9)-(10) result then

$$\kappa E^2 = \frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{B^2} \qquad (20)$$

$$\kappa \rho = -\frac{\ddot{B}}{B} + \frac{\kappa}{2} E^2 + \Lambda \qquad (21a)$$

$$\kappa p = -\frac{\ddot{B}}{B} - \frac{\kappa}{2}E^2 - \Lambda \tag{21b}$$

Imposing the equation of state (11) and introducing the variable η defined by $d\eta = B^{-\frac{1+\lambda}{3-\lambda}} dt$, we obtain the differential equation for B(t),

$$\frac{3-\lambda}{1+\lambda} B'' + B^{\frac{3\lambda-1}{3-\lambda}} + 2\Lambda B^{\frac{5+\lambda}{3-\lambda}} = 0 \tag{22}$$

where a prime denotes η-derivative. Equation (22) has the first integral

$$\frac{1}{2}(B')^2 + V(B) = C \tag{23}$$

where

$$V(B) = \frac{1}{2} B^{\frac{2(\lambda+1)}{3-\lambda}} + \frac{2\Lambda(1+\lambda)}{8} B^{\frac{8}{3-\lambda}} \tag{24}$$

For this class of solutions we calculate from (20) and (21a)

$$\kappa\rho = \frac{4C}{3-\lambda} B^{\frac{-8}{3-\lambda}} + \Lambda \tag{25}$$

and

$$\kappa E^2 = \frac{4(1-\lambda)}{3-\lambda} C B^{\frac{-8}{3-\lambda}} - (1+\lambda)\Lambda \tag{26}$$

We see from equation (26) that for $\lambda = 1$ we must have $\Lambda < 0$. For all other cases we restrict ourselves to $\Lambda < 0$ in order to have κE^2 positive definite.

The graph of V(B) and of the curve $\rho = 0$ (ρ being considered as a function of B and C) are qualitatively the same as in case II.

The maximum of the potential V(B) occurs at $B_c = \sqrt{\frac{1}{-2\Lambda}}$ where V(B) has the value $V_{max} = \frac{3-\lambda}{8}\left(\frac{1}{-2\Lambda}\right)^{\frac{1+\lambda}{3-\lambda}}$. Also for this case we note that for $C = V_{max}$, ρ is equal to zero at $B = B_c$.

Analogously we have the following physical situations:

IIIi) $C = V_{max}$

The point ($B = B_c$, $\dot{B} = 0$) is a solution of (22) and corresponds to the Berttoti-Robinson static solution, with density ρ, pressure p and conductivity σ equal to zero. From the graph of Fig. 1 we can see that this configuration is uns table.

IIIii) By perturbation of $C = V_{max}$ into $C = V_{max} - \varepsilon^2$ we are led from the Bertotti-Robinson solutions into physical solutions of Einstein-Maxwell equations contracting anisotropically towards the singularity B = 0. These solutions also have a pure electric field and positive conductivity σ as can be calculated from (8). The density ρ is positive always and goes to infinity as B goes to zero. The singularity B = 0 has the structure of a infinite line, up to identification of points.

We must finally comment about the sign of the conductivity σ. From equation (8) we obtain by a straightforward calcula tion that the sign of σ is opposite to the sign of expansion parameter θ, of the four velocity field of matter $\partial/\partial t$. For both classes (II) and (III) presented here σ is positive in the contracting phase of the model. There is however a bold distinction between solutions (II) and (III) concerning the interpretation of σ. For a closed system in flat space-time it can be shown (Landau and Lifshitz 1960) that σ must be positive in order that the entropy of the system increases. In the curved space-time of a cosmological model the concept of entropy of a closed system is not in general well defined. If we adhere to the orthodox principle that locally the entropy of any system must increase and assume that the sign of the conductivity σ is related to the local rate of change of entropy, then σ must be greater than zero. For the anisotropic models (III) discussed here this view can actually be sustained by local thermodynamics considerations: from the local conservation of $T^{\mu\nu}$ we can derive (Ellis 1971) that the time derivative of the specific entropy is given by $\phi = \pi_{\mu\nu}\sigma^{\mu\nu}$ where $\pi_{\mu\nu}$ is the traceless anisotropic pressure tensor and $\sigma_{\mu\nu}$ is the shear of the matter velocity field $\partial/\partial t$, and a simple calculation results $\phi = -\frac{2}{3}E^2\left(\frac{\dot{B}}{B}\right)$. Using expressions (8) and (26) we show immediately that the sign of σ is equal to the sign of ϕ.

For the isotropic solutions (II) however $\sigma_{\mu\nu} = 0$. We have obviously $\phi = 0$ and the above interpretation fails. It then remains to be given a physical criterion for defining the sign of σ if the concept of increasing entropy (even from a local point of view) has any meaning at all for a macroscopic system in interaction with the cosmological background.

Appendix

For the metric (1) the Ricci tensor $R^{\mu}{}_{\nu}$ has non-null components

$$R^{0}{}_{0} = -\frac{\ddot{A}}{A} - 2\frac{\ddot{B}}{B}$$

$$R^{1}{}_{1} = -\frac{\ddot{A}}{A} - 2\frac{\dot{A}\dot{B}}{AB}$$

$$R^{2}{}_{2} = R^{3}{}_{3} = -\frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} - \left(\frac{\dot{B}}{B}\right)^{2} - \frac{1}{B^{2}}$$

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