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GRAVITATIONAL COUPLING OF KLEIN-GORDON AND DIRAC PARTICLES TO MATTER VORTICITY AND SPACETIME TORSION

by

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Abstract

We examine the gravitational coupling of Klein-Gordon and Dirac fields to matter vorticity and spacetime torsion, in the context of Einstein-Cartan theory. The background spacetime is endowed with a Gödel-type metric, characterized by two real parameters (Ω, ℓ^2); the source of spacetime curvature is a Weyssenhoff-Raabe fluid with spin vector parallel to the vorticity field. We show that torsion and matter vorticity have identical effects on the physics of particle fields. Complete sets of solutions are obtained, satisfying boundary conditions connected to the test field character of the solutions. The energy spectrum obtained is discrete in general, except for the case of hyperbolic Gödel-type geometries $(\ell^2 > 0)$ where a continuum region in the energy spectrum may appear: if $0 < \Omega^2 < \ell^2$ a continuum region is present in the upper part of the spectrum; if $\Omega^2 \geq \ell^2$, Dirac solutions may present, under certain conditions, a continuum region in the lower part of the spectrum. The correspondence between classical geodesic motion and Klein-Gordon solutions is established, and used as a guide to select the correct boundary conditions for the test fields.

Matter vorticity and/or spacetime torsion split the energy spectrum of Dirac particles. These effects are additive and result from the existence of the *same* constant of motion for both cases. This constant of motion generates a trivial symmetry of the system in Minkowski spacetime, but whose associated degeneracy is raised by matter vorticity and/or torsion fields, producing the above-mentioned split.

Key-words: Dirac fields; Klein-Gordon fields; Torsion; Matter vorticity; Gödel-type universes; Field theory in curved space.

1 Introduction

The object of this paper is to study the concurrence of the effects of matter vorticity and spacetime torsion on the physics of matter fields, in the context of Einstein-Cartan-Hehl[1,2] theory. The problem is not purely academic, and as motivation we recall that the present observed rotation of galaxies and nebulae can be an indication that matter vorticity played an important role in the dynamics of the primordial universe; on the other hand, although torsion has no observable effects on the present experimental tests of gravitational theories, it could in principle also produce strong effects on the physics of extreme astrophyscial configurations. In this sense the results of our investigation could have some interesting applications in the realm of cosmology and theoretical astrophysics.

For operational simplicity we take the background spacetime endowed with a Gödel-type metric[3,4], and the source of spacetime curvature is a Weyssenhoff-Raabe fluid[5]. As a by-product this paper extends naturally the investigation of the motion of particles in Gödel-type spacetimes[6,7], approaching this problem from the standpoint of Klein-Gordon and Dirac test fields. Also this paper extends previous analysis[8,9,10] of the gravitational coupling of particle fields to matter vorticity, in the General Theory of Relativity.

The paper has the following structure. In Section 2 we present a general characterization of Gödel-type geometries and the admissible sources of spacetime curvature in Einstein-Cartan-Hehl theory. In Section 3, Klein-Gordon equation for a complex scalar field on this background is examined, a general set of solutions is obtained, separated in the invariant modes defined by the Killing vectors of the background geometry. In Section 4 the correspondence between classical geodesic motion and Klein-Gordon solutions is established, and arguments are given to justify the choice of the boundary conditions used in Section 3. In Section 5 we make a similar analysis for Dirac fields and introduce Foldy-Wouthuysen and Cini-Toushek representations of the solutions to interpret the constant of motion that appears in the dynamics of the Dirac field. Finally we conclude in Section 6 by discussing further topics to be examined in a future publication.

2 The Gödel-type Spacetimes in Einstein-Cartan Theory

For completeness we present here a general characterization of the background spacetime, endowed with a homogeneous Gödel-type metric, and the physical sources of this curvature. In the coordinate system (t,r,ϕ,z) the line element of the spacetime can be cast in the form

$$ds^2 = (dt + Hd\phi)^2 - dr^2 - D^2d\phi^2 - dz^2,$$

$$H = \frac{\Omega}{\ell^2}\sinh^2(\ell r) \quad , \quad D = \frac{\sinh 2\ell r}{2\ell} . \tag{2.1}$$

where Ω and ℓ^2 are real parameters, with $-\infty < \ell^2 < \infty$. Throughout the paper, we will assume that Ω is non-negative.

According to (1.1) we can divide our class of two-parameter spacetimes into three families: (i) the *hyperbolic* family ($\ell^2 > 0$), which includes Gödel's universe[11] ($\ell^2 = \Omega^2/2$) as a special case; (ii) the *Som-Raychandhuri* spacetime [12] ($\ell^2 \to 0$), and (iii) the *circular family* ($\ell^2 < 0$). We note that for $\ell^2 < 0$, the hyperbolic functions in (2.1) transform into circular functions. A possible natural choice for the range of the coordinates covering all manifolds in question is $-\infty < t,z < \infty$, $0 \le r < \infty$, $0 \le \phi < 2\pi$, for $\ell^2 \ge 0$; and $-\infty < t,z < \infty$, $0 \le |2\ell|r \le \pi$, $0 \le \phi < 2\pi$, for $\ell^2 < 0$. In any case t will be called the temporal coordinate, ϕ the azimuthal one and z the axial one; as concerns the coordinate r, it is naturally interpreted as a linear magnitude for $\ell^2 \ge 0$, and will be called accordingly the radial coordinate, whereas for $\ell^2 < 0$ it is naturally interpreted as an angular coordinate and so will be called the zenithal coordinate. In this last case ($\ell^2 < 0$) the coordinates r,ϕ are defined on a (topological) 2-sphere, with $|2\ell|r = 0,\pi$ corresponding to the north and south poles, respectively.

From the global point of view the manifold of Gödel-type spacetimes can be characterized[14,16] as the simply connected Lie group $M^3 \times R$ where, (i) for $\ell^2 > 0$, M^3 is the 3-hyperboloid H^3, (ii) for $\ell^2 < 0$, M^3 is the 3-sphere

S^3, and (iii) for $\ell^2 = 0$ M^3 is the Cartesian product $C^2 \times R$, where C^2 is the topological cylinder. By going to the corresponding *universal covering group* for each case we can adopt for Gödel-type spacetimes the topology of R^4.

The constants Ω and ℓ^2 are determined from Einstein-Cartan equations in the framework of Hehl's non-propagating torsion theory[1,2], with a Weyssenhoff-Raabe fluid[5,13] as source of spacetime curvature. We shall comment later (Section 5) the main motivation for the choice a W-R fluid.

We define the torsion tensor $\tau^i{}_{jk}$ as the antisymmetric part of the connection,

$$\tau^i{}_{jk} = \Gamma^i_{[jk]} \ , \tag{2.2}$$

and, for reference, the contorsion tensor

$$K^i{}_{jk} = \tau_k{}^i{}_j - \tau_{jk}{}^i - \tau^i{}_{jk} \ . \tag{2.3}$$

We assume that torsion is generated by the spin $S^i{}_{jk}$ of the Weyssenhoff-Raabe fluid,

$$S^i{}_{jk} = u^i S_{jk} \quad , \quad u^i S_{ij} = 0 \tag{2.4}$$

where u^i is the four-velocity field, and $S_{ij} = - S_{ji}$ is the spin density of the fluid. From the metricity postulate ($g_{ij;k} = 0$) and the field equations[1,2] we obtain for the connection

$$\Gamma^i{}_{jk} = \left\{ {i \atop jk} \right\} - K^i{}_{jk} \ , \tag{2.5a}$$

with

$$K^{i}{}_{jk} = -k(u^{i}S_{jk} - u_{j}S^{i}{}_{k} - u_{k}S^{i}{}_{j}) \tag{2.5b}$$

where k is Einstein's constant and $\left\{ {i \atop jk} \right\}$ is the Christoffel connection constructed with the metric g.

For an observer comoving with the fluid we choose the four velocity

$$u^{i} = \delta^{i}_{0} \tag{2.6}$$

and assume the fluid spin vector $S^{k} = \frac{-1}{2\sqrt{-g}} \varepsilon^{kij\ell} S_{ij} u_{\ell}$ to be constant and directed along the 3-axis,

$$S^{k} = -S\delta^{k}{}_{3} \quad , \quad S = \text{const.} \ , \tag{2.7}$$

that is, $S_{12} \neq 0$, $S_{ij} = 0$ (other indices). We note that the above assumptions are general, the only restrictive one being S = const. which can also be considered as a first approximation to a general case. For the geometries (2.1) the spin vector field is parallel to the vorticity field

$$\omega^{k} = \Omega_{0}\delta^{k}_{3} \ , \tag{2.8}$$

and we have in the realm of Hehl's theory that

$$\Omega = \Omega_{0} + kS \tag{2.9}$$

A realization of Gödel-type solutions (2.1) with torsion was presented by Tiomno et al.[3] A resumé of their results is now given. In General Relativity Theory, if we restrict the matter content to a perfect fluid, an electromagnetic field and a massless scalar field, we can only yield line elements with $-\infty < \ell^{2} \leq \Omega^{2}$ as solutions of the appropriate coupled field equations (Einstein-Maxwell-Klein-Gordon equations), whereas in Einstein-Cartan-Hehl theory it is possible to generate all line elements ($-\infty < \ell^2 < \infty$) taking only a Weyssenhoff-Raabe perfect fluid for matter content (cf. Eqs. (2.4)-(2.9)).

The class of geometries (2.1) admit the five Killing vectors[4]

$$K_{(0)} = \frac{\partial}{\partial t} \quad , \quad K_{(1)} = \frac{\partial}{\partial z} \quad , \quad K_{(2)} = \frac{\partial}{\partial \phi} \tag{2.10a}$$

$$K_{(3)} = \cos\phi \, \frac{\partial}{\partial r} - \sin\phi \left[\frac{\Omega}{\ell} \frac{\sinh \ell r}{\cosh \ell r} \frac{\partial}{\partial t} + 2\ell \, \frac{\cosh 2\ell r}{\sinh 2\ell r} \frac{\partial}{\partial \phi} \right] \tag{2.10b}$$

$$K_{(4)} = - \sin\phi \, \frac{\partial}{\partial r} - \cos\phi \left[\frac{\Omega}{\ell} \frac{\sinh \ell r}{\cosh \ell r} \frac{\partial}{\partial t} + 2\ell \, \frac{\cosh 2\ell r}{\sinh 2\ell r} \frac{\partial}{\partial \phi} \right] \tag{2.10c}$$

with the corresponding Lie algebra

$$\left[K_{(0)}, K_{(i)}\right] = 0 = \left[K_{(1)}, K_{(i)}\right] \quad , \quad i = 0, \ldots, 4$$

$$\left[K_{(2)}, K_{(3)}\right] = K_4 \ , \ \left[K_{(2)}, K_{(4)}\right] = - K_{(3)} \tag{2.11}$$

$$\left[K_{(3)}, K_{(4)}\right] = - 4\ell^2 \left(K_{(2)} + \frac{\Omega}{2\ell^2} K_{(0)} \right)$$

For future reference we define from (2.10) the vector fields

$$L_{(1)} = \frac{1}{2\ell} K_{(3)}$$

$$L_{(2)} = \frac{1}{2\ell} K_{(4)} \tag{2.12}$$

$$L_{(3)} = - i\left(K_{(2)} + \frac{\Omega}{2\ell^2} K_{(0)}\right)$$

They satisfy the angular momentum algebra

$$\left[L_{(a)}, L_{(b)}\right] = - i\varepsilon_{abc} L_{(c)} \tag{2.13}$$

for any value of $\ell^2 \neq 0$. We remark that definition (2.12) cannot be extended to the case of geometries with $\ell^2 = 0$.

The five Killing vectors (2.8) are right-invariant vector fields[15,16] over $M^3 \times R$ and therefore globally defined on the group manifold. We then select (2.10a) to construct the global invariant modes $\phi_{(1)}$ defined by[17]

$$\mathcal{L}_{\partial/\partial z}\phi_{(3)} = - ik_3\phi_{(3)} \quad , \quad \mathcal{L}_{\partial/\partial\phi}\phi_{(2)} = - im\phi_{(2)} \tag{2.14}$$

$$\mathcal{L}_{\partial/\partial t}\phi_{(0)} = - i\varepsilon\phi_{(0)} \tag{2.15}$$

with respective solutions $\phi_{(3)} \sim e^{-ik_3 z}$, $\phi_{(2)} \sim e^{-im\phi}$ and $\phi_{(0)} \sim e^{-i\varepsilon t}$. We interpret (2.15) as the definition of the invariant energy modes, and use $\phi_{(1)}$ to separate field amplitudes in the modes (ε, m, k_3).

3 Klein-Gordon Test Fields

The Klein-Gordon equation for a complex scalar field ϕ with mass M and minimally coupled to gravitation is given by

$$\frac{1}{\sqrt{-g}}\,\partial_i\left[\sqrt{-g}\;g^{ij}\partial_j\phi\right] + M^2\phi = 0 \tag{3.1}$$

We note that the scalar field ϕ has no direct coupling with torsion. For Gödel-type geometries (2.1), equation (3.1) can be expressed

$$\vec{L}^2\phi = \frac{1}{4\ell^2}\left[\frac{\ell^2-\Omega^2}{\ell^2}\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + M^2\right]\phi \tag{3.2}$$

where $\vec{L}^2 = \left(L_{(1)}\right)^2 + \left(L_{(2)}\right)^2 + \left(L_{(3)}\right)^2$ is the square of the angular momentum operator defined in (2.12). We note that, for the case $\ell^2 > 0$, the *positive-definite* scalar product in the vector space of angular momentum algebra has its correspondent as an *indefinite* product in the vector space of the algebra of H^3 isometries.

We consider scalar field solutions in the modes (2.14)-(2.15),

$$\phi = \phi(r)e^{-im\phi}e^{-ik_3z}e^{-i\varepsilon t} \tag{3.3}$$

The field equation (3.2) reduces then to the eigenvalue equation

$$\vec{L}^2\phi = \frac{1}{4}\left[\frac{\Omega^2-\ell^2}{\ell^4}\,\varepsilon^2 + \frac{M^2+k_3^2}{\ell^2}\right]\phi = \frac{n^2-1}{4}\,\phi. \tag{3.4a}$$

Introducing the variable $x = \cosh 2\ell r$, eq. (3.4a) yields

$$\left[4\ell^2(x^2-1)\frac{d^2}{dx^2} + 8\ell^2\, x\,\frac{d}{dx} + 4\ell^2\left(\frac{\frac{\Omega^2\varepsilon^2}{\ell^2} + 2\Omega\varepsilon m}{2\ell^2(x+1)} - \frac{m^2}{x^2-1} - \frac{n^2-1}{4}\right)\right]\phi(r) = 0 \qquad (3.4b)$$

and we distinguish the set of solutions[18] which are *regular* at r = 0,

$$\phi^{I}_{\varepsilon m} = (x^2-1)^{\frac{m}{2}}(x+1)^{\frac{\Omega\varepsilon}{2\ell^2}}\, F\left(a,b,c;\frac{1-x}{2}\right)e^{-i(m\phi+k_3 z+\varepsilon t)} \qquad (3.5a)$$

for $m \geq 0$, and

$$\phi^{II}_{\varepsilon m} = (x^2-1)^{-\frac{m}{2}}(x+1)^{-\frac{\Omega\varepsilon}{2\ell^2}}\, F\left(1-b,1-a,2-c;\frac{1-x}{2}\right)e^{-i(m\phi+k_3 z+\varepsilon t)} \qquad (3.5b)$$

for $m \leq 0$. Here $F\left(a,b,c;\frac{1-x}{2}\right)$ is the hypergeometric function[19] with parameters

$$a = m + \frac{1}{2} + \frac{\Omega\varepsilon}{2\ell^2} + \frac{n}{2} \quad ,$$

$$b = m + \frac{1}{2} + \frac{\Omega\varepsilon}{2\ell^2} - \frac{n}{2} \quad , \qquad (3.6)$$

$$c = m + 1 \quad ;$$

m is an integer and n is defined by the second equality in (3.4a). For $m = 0$, ϕ^I and ϕ^{II} are linearly dependend. The sets (3.5a) and (3.5b) are related by

$$\sigma\phi^{I}(\varepsilon,m,k_3) = \phi^{II}(-\varepsilon,-m,-k_3)^* \qquad (3.7)$$

where $\sigma = 1$ for $\ell^2 > 0$, $\sigma = (-1)^m$ for $\ell^2 < 0$, and $*$ denotes complex-conjugation.

For the case $\ell^2 = 0$ (Som-Raychandhuri if $\Omega \neq 0$, or Minkowski if $\Omega = 0$) we return to the variable r, making the approximation $x \sim 1 + 2\ell^2 r^2$ and taking the limit $\ell^2 \to 0$ in equation (3.4b). The corresponding scalar solutions will be obtained as the limit $\ell \to 0$ of the hyperbolic solutions ($\Omega^2 > \ell^2 > 0$).

On the space of solutions (3.5) we introduce the operators

$$L_\pm = L_{(1)} \pm iL_{(2)} = e^{\mp i\phi}\left[\sqrt{x^2-1}\,\frac{\partial}{\partial x} \mp \frac{ix}{x^2-1}\,\frac{\partial}{\partial\phi} \mp \frac{i\Omega}{2\ell^2}\sqrt{\frac{x-1}{x+1}}\,\frac{\partial}{\partial t}\right]$$

$$L_{(3)} = -\,i\left[\left(\frac{\partial}{\partial\phi}\right) + \frac{\Omega}{2\ell^2}\,\frac{\partial}{\partial t}\right]$$

They satisfy

$$\left[L_+,L_{(3)}\right] = -\,L_+ \quad , \quad \left[L_-,L_{(3)}\right] = L_-$$

$$\left[L_+,L_-\right] = 2L_{(3)} \qquad (3.8)$$

and

$$L_\pm^* = L_\mp \qquad \text{if} \qquad \ell^2 > 0$$

$$L_\pm^* = -\,L_\mp \qquad \text{if} \qquad \ell^2 < 0 \qquad (3.9)$$

Their action on the set (3.5a) is

$$L_+\phi^I_{\varepsilon m} = -\frac{ab}{2c}\,\phi^I_{\varepsilon,m+1}$$

$$L_-\phi^I_{\varepsilon m} = 2m\phi_{\varepsilon,m-1} \tag{3.10}$$

$$L_{(3)}\phi^I_{\varepsilon m} = \left(m + \frac{\Omega\varepsilon}{2\ell^2}\right)\phi^I_{\varepsilon m}$$

and the effect on the set (3.5b) is obtained by complex-conjugation. By using (3.8) we can show that for a given solution $\phi_{\varepsilon m}$ (eigenstate of $L_{(3)}$ with eingenvalue $m + \frac{\Omega\varepsilon}{2\ell^2}$), $L_\pm\phi_{\varepsilon m}$ is also a solution which is eigenstate of L_3 with eigenvalue $m + \frac{\ell\varepsilon}{2\ell^2} \pm 1$. So starting from a solution with a given m it is possible to generate the whole set of solutions (3.5) by successive applications of $L_\pm$. For instance, starting from ϕ^I $(m = m_0)$ we can obtain all solutions $\phi^I(m > m_0)$ by successive applications of L_+. The solutions $\phi^I(m < m_0)$ are obtained by applications of L_-, reaching ϕ^I $(m = 0)$. The application of L_- on $\phi^I(m = 0)$ will result in $\phi^{II}(m = -1)$, since, from (3.10),

$$\lim_{m\to 0} L_-\phi^I(m) = 2\lim_{c-1\to 0}\Big[(c-1)\phi^I(m-1)\Big] \sim \phi^{II}(m = -1)$$

Successive applications of L_- on $\phi^{II}(m = -1)$ will produce all other solutions $\phi^{II}(m < -1)$.

To proceed we impose boundary condition on the set (3.5), connected to the character of test fields of the solutions, namely, that the scalar field solutions are finite perturbations at any spacetime point. We assume

$$\lim_{x\to\infty} \phi^*\phi = 0 \quad , \quad \text{if } \ell^2 > 0 \tag{3.11a}$$

$$\lim_{x\to -1} \phi^*\phi = 0 \quad , \quad \text{if } \ell^2 < 0 \tag{3.11b}$$

These conditions may result in discrete or continuous energy spectra as well as in restrictions on the allowed physical interval of m, and are sufficient to guarantee the normalization of the discrete energy solutions.

On the set of solutions (3.5) we define the scalar product

$$\langle\phi_{m'}|\phi_m\rangle = \int \sqrt{-g}\, d^4x\, \phi^*_{m'}(x)\phi_m(x) \tag{3.12}$$

where the domain of integration extends to the domain of definition of the coordinates (t,r,ϕ,z). Under (3.12) the subset of discrete energy solutions are normalizable, as we shall see, and we have the property

$$\begin{aligned} \langle\phi_{m'}|L_\pm\phi_m\rangle &= -\,\langle L_\mp\phi_{m'}|\phi_m\rangle \quad \text{if} \quad \ell^2 > 0 \\ \langle\phi_{m'}|L_\pm\phi_m\rangle &= \langle L_\mp\phi_{m'}|\phi_m\rangle \quad \text{if} \quad \ell^2 < 0 \end{aligned} \tag{3.13}$$

We remark that (3.13) holds only for the subset of discrete energy solutions. Let us consider the cases

<u>Hyperbolic metrics ($\ell^2 > 0$):</u>

(i) If $n^2 \geq 1$, the boundary condition (3.11a) imposes

$$\phi_{\varepsilon m} = (x^2-1)^{m/2}(x+1)^{\frac{\Omega\varepsilon}{2\ell^2}} F\left(j+m+1,\ m-j+\frac{\Omega\varepsilon}{\ell^2},\ m+1;\ \frac{1-x}{2}\right) e^{-i(\varepsilon t+m\phi+k_3 z)} \tag{3.14a}$$

for $0 < m < \infty$, and

$$\phi_{\varepsilon m} = (x^2-1)^{-\frac{m}{2}}(x+1)^{-\frac{\Omega\varepsilon}{2\ell^2}} F\left(-m-j,\ j-m+1-\frac{\Omega\varepsilon}{\ell^2},\ 1-m;\ \frac{1-x}{2}\right) e^{-i(\varepsilon t+m\phi+k_3 z)} \tag{3.14b}$$

for $-j \le m \le 0$. Here j = non-negative integer, and

$$\varepsilon = (2j+1)\Omega + \left[(\Omega^2-\ell^2)(2j+1)^2 + M^2 + k_3^2 + \ell^2\right]^{1/2} \tag{3.15}$$

The complete set of positive-energy solutions for this case is given by the union of (3.14a) and (3.14b). The range of m is limited to

$$- j \le m < \infty \; . \tag{3.16}$$

We obviously have $L_{-}\phi_{\varepsilon,-j} = 0$. The corresponding negative-energy solutions are obtained from (3.14) by using the transformation (3.7). The solutions are said to have a *discrete energy spectrum*, and are normalizable with respect to (3.12) because they also satisfy $\lim_{x\to\infty} \sqrt{-g}\ \phi^*\phi = 0$.

(ii) if $n^2 < 1$, solutions (3.5) satisfy (3.11a) automatically, and are non-normalizable because $\lim_{x\to\infty} \sqrt{-g}\ \phi^*\phi \neq 0$. The range of the integer m is not restricted ($-\infty < m < \infty$), and the solutions have a *continuous energy spectrum*.

We remark that, in fact, the only normalizable solutions for this case $\ell^2 > 0$ are the discrete energy solutions described in (i), eqs. (3.14) - (3.15).

In short, we have

$$\begin{aligned} &\text{for } n^2 \ge 1\text{: discrete energy, and } \pm m \ge - j \quad (\text{for } \pm \varepsilon) \\ &\text{for } n^2 < 1\text{: continuous energy, and } - \infty < m < \infty \end{aligned} \tag{3.17}$$

These results can be restated in terms of the metric parameters (Ω,ℓ). From the expression $n^2 = \frac{\Omega^2-\ell^2}{\ell^4}\varepsilon^2 + \frac{M^2+k_3^2+\ell^2}{\ell^2}$ (cf. (3.4a)) we conclude

1) *For spacetimes with* $\Omega^2 \geq \ell^2$, the solutions have a *discrete energy spectrum* only.

2) *For spacetimes with* $\Omega^2 < \ell^2$, the solutions have a *discrete spectrum* for values of the energy such that $\varepsilon^2 \leq \frac{\ell^2\left(M^2+k_3^2\right)}{\ell^2-\Omega^2}$. For values above this limit the spectrum is continuous . The quantum number j - associated to the discrete part of the spectrum - is restricted by

$$2j + 1 \leq \frac{\Omega}{\ell}\left[\frac{M^2 + k_3^2}{\ell^2 - \Omega^2}\right]^{1/2} . \tag{3.18}$$

In both cases the discrete energy levels are given by (3.15). The quantum number j is a non-negative integer, unrestricted in case (1) and restricted by (3.18) in case (2).

The discrete energy solutions can be easily normalized, starting from the normalization of the particular solution $\phi^{II}(m=-j)$ and using the relations (3.8), 3.10) and (3.13). We obtain

$$\langle\phi_{\varepsilon' m' k_3'}|\phi_{\varepsilon m k_3}\rangle = \frac{(2\pi)^3}{4\ell^2}\, 2^{2m+\frac{\Omega\varepsilon}{\ell^2}+1}\, \frac{j\,!(m!)^2\Gamma\left(\frac{\Omega\varepsilon}{\ell^2}-j\right)}{(m+j)!\left(\frac{\Omega\varepsilon}{\ell^2}-2j-1\right)\Gamma\left(\frac{\Omega\varepsilon}{\ell^2}+m-j\right)}\, \delta_{m'm}\delta(k_3'-k_3)\delta(\varepsilon'-\varepsilon)$$

for $0 < m < \infty$, and

$$<\phi_{\varepsilon' m' k_3'}|\phi_{\varepsilon m k_3}> = \frac{(2\pi)^2}{4\ell^2}\, 2^{-2m\frac{\Omega\varepsilon}{\ell^2}+1}\; \frac{(j+m)!\,[(-m)!]^2\,\Gamma\left(\frac{\Omega\varepsilon}{\ell^2}-j+m\right)}{j!\left(\frac{\Omega\varepsilon}{\ell^2}-2j-1\right)\Gamma\left(\frac{\Omega\varepsilon}{\ell^2}-j\right)}\;\delta_{m'm}\,\delta(k_3'-k_3)\,\delta(\varepsilon'-\varepsilon)$$

for $-j \le m \le 0$.

Circular metrics ($\ell^2 < 0$)

This case demands a more careful analysis in order to encompass also the solutions of the $\Omega = 0$ limit.

On the set of solutions (3.5) we impose the boundary condition (3.11b). In what follows we denote $\ell^2 = -\lambda^2$, and we note that $n^2 - 1 > 0$. We obtain[20]

$$\phi_{\varepsilon m}(x) = (x^2-1)^{-\frac{m}{2}}(1+x)^{\frac{\Omega\varepsilon}{2\lambda^2}}\, F\left(j-m+1+\frac{\Omega\varepsilon}{\lambda^2},\ -m-j,\ 1-m;\ \frac{1-x}{2}\right)e^{-i(m\phi+k_3z+\varepsilon t)} \tag{3.19a}$$

for $-j \le m \le 0$, and

$$\phi_{\varepsilon m}(x) = (x^2-1)^{\frac{m}{2}}(1+x)^{-\frac{\Omega\varepsilon}{2\lambda^2}}\, F\left(m-j-\frac{\Omega\varepsilon}{\lambda^2},\ m+j+1,\ m+1;\ \frac{1-x}{2}\right)e^{-i(m\phi+k_3z+\varepsilon t)} \tag{3.19b}$$

for $0 \le m < \frac{\Omega\varepsilon}{\lambda^2}$. Here j = non-negative integer, and

$$\varepsilon = (2j+1)\Omega + \sqrt{(2j+1)^2(\Omega^2+\lambda^2) + M^2 + k_3^2 - \lambda^2} \tag{3.20}$$

The range of m for the positive-energy set of solutions (3.19) is them

$$-j \le m < \Omega\varepsilon/\lambda^2 \tag{3.21}$$

The corresponding negative-energy solutions are obtained from (3.19) by complex-conjugation (cf. (3.7)). The solutions have a discrete energy spectrum, and are normalizable in the sense of (3.12), *for any* (Ω,λ).

A further set of normalizable solutions is also obtained, that are distinct from, and are to be added to the family (3.19) since they are fundamental in view of the limit $\Omega = 0$. They have the expression

$$\psi_{\varepsilon_{1,m}} = (x^2+1)^{\frac{m}{2}}(1+x)^{-\frac{\Omega\varepsilon_1}{2\lambda^2}} F\left(m-j,\ m+j+1-\frac{\Omega\varepsilon_1}{\lambda^2},\ m+1;\ \frac{1-x}{2}\right)e^{-i(m\phi+k_3z+\varepsilon_1 t)} \tag{3.22}$$

where the energy eigenvalues are

$$\varepsilon_1 = -\ (2j+1)\Omega + \sqrt{(2j+1)^2(\Omega^2+\lambda^2) + M^2 + k_3^2 - \lambda^2}\ , \tag{3.23}$$

and the range of m is given by

$$\frac{\Omega\varepsilon_1}{\lambda^2} < m \le j \tag{3.24}$$

In the limit $\Omega = 0$, $\lambda^2 \neq 0$ (diagonal metric) the restrictions (3.21) and (3.24) disappear. The family of positive energy solutions is then given by (3.19a) and (3.22), which can be expressed in the form

$$\phi_{\varepsilon m} = P_j^m(x)e^{-i(m\phi+k_3z+\varepsilon t)} \tag{3.25}$$

where

$$\varepsilon = \sqrt{(2j+1)^2\lambda^2 + M^2 + k_3^2 - \lambda^2}\ , \tag{3.26}$$

and $P_j^m(x)$ are the associated Legendre Polynomials[19], with $-j \le m \le j$. This result should be expected since Eq. (3.46) reduces to the associated Legendre equation for $\Omega = 0$.

The normalization of the solutions can be performed analogously to the case of hyperbolic metrics, by using the operators $L_\pm$. The results are

$$<\phi_{\varepsilon' m' k_3'}|\phi_{\varepsilon m k_3}> = \frac{(2\pi)^3 2^{-2m+\frac{\Omega\varepsilon}{\lambda^2}+1}}{4\lambda^2\left(\frac{\Omega\varepsilon}{\lambda^2} + 2j + 1\right)} \frac{(j+m)![(-m)!]^2}{j!} \times$$

$$\frac{\Gamma\left(\frac{\Omega\varepsilon}{\lambda^2} + j + 1\right)}{\Gamma\left(\frac{\Omega\varepsilon}{\lambda^2} + j - m + 1\right)} \delta_{m'm}\delta(k_3'-k_3)\delta(\varepsilon'-\varepsilon) \text{ , for } -j \le m \le 0 \quad (3.27a)$$

$$<\phi_{\varepsilon' m' k_3'}|\phi_{\varepsilon m k_3}> = \frac{(2\pi)^3 2^{2m-\frac{\Omega\varepsilon}{\lambda^2}+1}}{4\lambda^2\left(\frac{\Omega\varepsilon}{\lambda^2} + 2j + 1\right)} \frac{j!(m!)^2}{(j+m)!} \times$$

$$\frac{\Gamma\left(\frac{\Omega\varepsilon}{\lambda^2} + j + 1 - m\right)}{\Gamma\left(\frac{\Omega\varepsilon}{\lambda_2} + j + 1\right)} \delta_{m'm}\delta(k_3'-k_3)\delta(\varepsilon'-\varepsilon) \text{ , for } 0 \le m < \frac{\Omega\varepsilon}{\lambda^2} \quad (3.27b)$$

and

$$<\psi_{\varepsilon_1' m' k_3'}|\psi_{\varepsilon_1 m k_3}> = \frac{(2\pi)^3 2^{2m-\frac{\Omega\varepsilon_1}{\lambda^2}+1}}{4\lambda^2\left(-\frac{\Omega\varepsilon_1}{\lambda^2} + 2j + 1\right)} \frac{(j-m)!(m!)^2}{j!} \times$$

$$\frac{\Gamma\left(\frac{\Omega\varepsilon_1}{\lambda^2}+j+1\right)}{\Gamma\left(\frac{\Omega\varepsilon_1}{\lambda^2}+j+m+1\right)}\delta_{m'm}\delta(k_3'-k_3)\delta(\varepsilon_1'-\varepsilon_1),\text{ for }\frac{\Omega\varepsilon_1}{\lambda^2}<m\le j.\quad(3.27c)$$

For the case of diagonal metrics ($\Omega = 0$) the normalization is trivially obtained by making $\Omega = 0$ in (3.27a) and (3.27c):

$$\langle\phi_{\varepsilon'm'k_3'}|\phi_{\varepsilon mk_3}\rangle = \frac{(2\pi)^3 2^{2m+1}}{4\lambda^2(2j+1)}\frac{(|m|!)^2(j-|m|)!}{(j+|m|)!}\delta_{m'm}\delta(k_3'-k_3)\delta(\varepsilon'-\varepsilon)$$

$$\text{for } -j\le m\le j;$$

This define the normalization adopted for the associated Legendre Polynomials used in (3.26).

In short, the KG solutions for circular metrics ($\ell^2 < 0$) present: (i) discrete energy spectra; (ii) bounded range of m.

Som-Raychandhuri metrics ($\ell^2 = 0$)

As mentioned before, the scalar field solutions for this case are obtained as the limit $\ell \to 0$ of the hyperbolic solutions ($\Omega^2 > \ell^2 > 0$). As $\ell \to 0$ we make the following approximations in (3.14):

$$b = m - j + \frac{\Omega\varepsilon}{\ell^2} \to \frac{\Omega\varepsilon}{\ell^2}$$

$$x = \cosh 2\ell r \to 1 + \frac{2\xi}{b}$$

where $\xi = \Omega\varepsilon r^2$ and

$$\varepsilon = (2j+1)\Omega + \sqrt{(2j+1)^2\Omega^2 + M^2 + k_3^2} \tag{3.28}$$

Taking $b \to \infty$ and using the relation

$$\lim_{b\to\infty} F\left(a,\ b,\ c;\ \frac{-\xi}{b}\right) = e^{-\xi} F\left(c-a,\ c;\ \xi\right) ,$$

where $F(a,b;\xi)$ is the confluent hypergeometric function[19], it finally results (after straightforward manipulations)

$$\phi_{\varepsilon m} = \left[\frac{2\Omega\varepsilon}{(2\pi)^3}\ \frac{j!}{((-m!))^2(j+m)!}\right]^{1/2} \xi^{-\frac{m}{2}}\ e^{-\frac{\xi}{2}}\ F(-m-j,1-m;\xi).$$

$$.e^{-i(m\phi+k_3z+\varepsilon t)} \quad , \quad -j \le m \le 0, \tag{3.29a}$$

$$\phi_{\varepsilon m} = \left[\frac{2\Omega\varepsilon}{(2\pi)^3}\ \frac{(m+j)!}{(m!)^2 j!}\right]^{1/2} \xi^{\frac{m}{2}}\ e^{-\frac{\xi}{2}}\ F(-j,m+1;\xi).$$

$$.e^{-i(m\phi+k_3z+\varepsilon t)} \quad , \quad 0 \le m \le \infty \tag{3.29b}$$

The solutions (3.29) are orthonormalized already.

Finally, concerning the Minkowski case $\Omega = 0 = \ell^2$, the solutions can be derived as the limit of the solutions of the hyperbolic case $\Omega^2 < \ell^2$, when $\ell \to 0$. The limit procedure is similar to the previous one for Som-Raychandhuri metrics, making $\Omega = 0$ first. Only the *continuous energy* solutions of the hyperbolic case $\Omega^2 < \ell^2$ will contribute, and we have

$$\phi_{\varepsilon m} = J_m\left(\sqrt{\varepsilon^2 - M^2 - k_3^2}\; r\right) e^{-i(m\phi + k_3 z + \varepsilon t)} ,$$

$-\infty < m < \infty$, where J_m are the Bessel functions of the first kind. They are regular at $r = 0$, as expected, since they were obtained from solutions with this property.

4 Comparison with Classical (Geodesic) Motion

We now discuss the analogy between geodesic motion of point test particles and the scalar field solutions obtained in Section 3. A detailed study of geodesic motion in Gödel-type spacetimes was performed by Calvão, Soares and Tiomno[6]. The analogy is established by reproducing the results of Ref. [6] through the correspondence of the continuous parameters of the geodesic motion with the quantum numbers of the solutions of Section 3. We adhere to the same notation and definitions of Ref. [6], except that we rescale Ω^2 to $\Omega^2/4$ and take the mass of the particle equal to M.

In the sense of Lagrangean mechanics, the coordinates t, ϕ and z are cyclic, implying the existence of the integrals of motion

$$\left(P_t,\ P_\phi,\ P_z\right) \tag{4.1}$$

The constant η - that characterizes bounded or unbounded geodesics for the hyperbolic family with $\ell^2 > \Omega^2 > 0$ - is defined[1]

[1] Cf. Eq. (40) of Ref. [6]. It coincides with the parameter α also defined in Eq. (31) of Ref. [6] for the cases $\ell^2 > 0$.

$$\frac{P_t^2\eta}{\ell^2} = \left(\frac{\Omega^2-\ell^2}{\ell^4}\right)P_t^2 + P_z^2 + M^2$$

With the correspondence

$$P_\phi \leftrightarrow m$$
$$P_z \leftrightarrow k_3 \qquad (4.2)$$
$$P_t \leftrightarrow \varepsilon \ ,$$

and consequently

$$\frac{P_t^2}{\ell^2}\eta \longleftrightarrow n^2 - 1 \qquad (4.3)$$

the analogy is established completely. Indeed from (4.3), (3.17) and Ref. [6] it follows that geodesic orbits bounded (unbounded) in r correspond to discrete (continuous) positive energy solutions. Also the condition $n^2 - 1 = 0$ (defining the limit energy $\varepsilon = \left[\frac{\ell^2(M^2+k_3^2)}{\ell^2 - \Omega^2}\right]^{1/2}$ between discrete and continuous energy levels) corresponds to the geodesic defined by $\eta = \alpha = 0$, limiting bounded and unbounded geodesics.

The above results clarify and justify the choice of the boundary conditions (3.11). Indeed a detailed examination of (3.11) shows that it implies $n^2 - 1 = 0$ as defining not only the limit energy between discrete and continuous energy solutions but also, through the correspondence (4.3), the limiting geodesic between bounded and unbounded classical motion.

Furthermore the spacetimes with $\ell^2 \le \Omega^2$ $\left(\text{that implies } n^2 > 1 \text{ and } \frac{P_t^2}{\ell^2}\eta > 0\right)$ admit only discrete-energy solutions and geodesics bounded in r, as shown in Section 3 and Ref. [6]. In this case, the range of the quantum number m as well as the range of its classical equivalent P_ϕ have a lower and/or upper bound. Using the correspondence (4.2) we can show[21] that the *continuous classical interval* of γ contains the same integer values m as in the quantum interval. For example, in the hyperbolic case $\Omega^2 > \ell^2 > 0$, the classical interval is given by

$$P_\phi \; : \; \left[-\frac{\Omega P_t}{2\ell^2} + \frac{P_t}{2\ell}\sqrt{\eta} \; , \; \infty \right) \tag{4.4}$$

while the quantum interval,

$$m \; : \; (-j \; , \; \infty).$$

Using (3.15) we may express $j = \dfrac{\Omega\varepsilon}{2\ell^2} - \dfrac{(n+1)}{2}$, and using the correspondence (4.2) - (4.3) we have

$$0 < -j - \left(-\frac{\Omega\varepsilon}{2\ell^2} + \frac{\sqrt{n^2-1}}{2} \right) < \frac{1}{2} \; . \tag{4.5}$$

From (4.5) it follows that (-j) is the smallest integer contained in the classical interval (4.4).

5 Dirac Test Fields

Here we extend our previous analysis to a Dirac field coupled to the gravitation of Gödel-type models, in the context of Hehl's non-propagating torsion theory[1,2]. Gödel-type spacetimes in Hehl's theory are discussed in Section 2, with a Weyssenhoff-Raabe fluid introduced as physical source of spacetime curvature. We also assume here that Dirac fields do not interact with any Maxwell field (possibly present also as source of curvature).

Contrary to the scalar field case, the Dirac field ψ couples directly to the torsion of spacetime, via the spinorial connection, and the problem in question is therefore perfectly fit to investigate the simultaneous effect of torsion and matter vorticity in the physics of particle fields, as partially discussed in Ref. [22].

We introduce Dirac spinors from the point of view of the tetrad formalis. For a general review see Ref. [23]. We choose a tetrad basis $(e^{(1)}(x))$ such that the metric is expressed

$$g = e^{(A)}(x)e^{(B)}(x)\eta_{AB} \tag{5.1}$$

where η_{AB} is the constant Minkowski matrix[24]. The Lagrangean for Dirac four-component wave function with mass M is[25]

$$L = \sqrt{-g}\left[\frac{1}{2}\left(\bar{\psi}\gamma^A D_A\psi - D_A\bar{\psi}\gamma^A\psi\right) - M\bar{\psi}\psi\right] \tag{5.2}$$

In the present formalism γ^A are the constant Dirac matrices[26] and $\bar{\psi} = \psi^+ \gamma^0$, where γ^0 is the constant matrix. The spinor covariant derivatives are given by

$$D_A\psi = e^{\alpha}_{(A)}\partial_\alpha\psi + \Gamma_A\psi \tag{5.3a}$$

$$D_A\bar{\psi} = e^{\alpha}_{(A)}\partial_\alpha\psi - \bar{\psi}\Gamma_A \tag{5.3b}$$

where the spinor connection Γ_A has the form

$$\Gamma_A = \frac{1}{4}\left(\gamma_{BCA} - K_{BCA}\right)\gamma^B\gamma^C . \tag{5.4}$$

Here K_{BCA} is the contorsion tensor (1.3) - (1.5) expressed in the basis

$(e^{(A)})$, and γ_{BCA} are the Ricci rotation coefficients[27] associated to the basis $(e^{(A)})$.

From (5.2) Dirac's equation is given by

$$\left[i\gamma^{A}\left(D_{A} + \frac{1}{2} K_{BA}{}^{B}\right) - M\right]\psi = 0 \qquad (5.5)$$

We choose for (2.1) the tetrad basis

$$e_{0}^{(0)} = e_{1}^{(1)} = e_{3}^{(3)} = 1 \quad , \quad e_{2}^{(0)} = H \quad , \quad e_{2}^{(2)} = D \qquad (5.6)$$

As discussed in Section 2, the source of spacetime curvature is a Weyssenhoff-Raabe (cf. (1.4) - (1.9)) with spin vector (constant and parallel to the vorticity field) and contorsion tensor given respectively by (1.7) and (1.5b). The main motivation to use a Weyssenhoff-Raabe fluid as source of curvature is that we want a class of models which encompass not only torsion fields but also matter vorticity, and for which class we have the limit of Riemann-flatness (flat spacetime metric plus torsion field). Besides, a Weyssenhoff-Raabe fluid is a physically suitable description of a fluid with spin distribution and matter vorticity. Dirac matter would not generate such a class of background geometries, and indeed a Weyssenhoff-Raabe fluid does not encompass Dirac matter. This presents however no difficulty because the Dirac fields considered in this paper are test fields on a given background spacetime. In case of a pure torsion field it is possible to use a totally antisymmetric torsion generated by a constant background Dirac field. The results in this limit are similar to the case $\Omega = 0$.

Using (5.6) and the contorsion tensor given by (2.5b) - (2.7) Dirac's equation now reads[26]

$$i\,\frac{\partial\psi}{\partial t} = \left\{\gamma^{5}\left(\Sigma^{1}\pi_{1} + \Sigma^{2}\pi_{2}\right) + \Sigma^{3}\left[\hat{C} + \frac{1}{2}(\Omega - kS)\right]\right\}\psi = \hat{H}\psi \qquad (5.7)$$

with the momentum operators given by

$$\pi_1 = i\,\frac{\partial}{\partial r} - \frac{i}{D}\,\frac{dD}{dr}\ ,\ \pi_2 = i\,\frac{H}{D}\,\frac{\partial}{\partial t} - \frac{i}{D}\,\frac{\partial}{\partial\phi}\ ,\ \pi_3 = -\,i\,\frac{\partial}{\partial z}\ , \tag{5.8}$$

and

$$\hat{C} = \gamma^5\pi_3 + M\gamma^3\gamma^5 \tag{5.9}$$

The operator $\hat{C}$ is a constant of motion with respect to the Hamiltonian defined by (5.7), namely,

$$[\hat{C}\ ,\ \hat{H}] = 0$$

Choosing ψ to be simultaneous eigenstate of π_3 and $\hat{C}$ (π_3 is also a constant of motion, with eigenvalues P_3) we obtain from (5.9)

$$\hat{C}\psi = -\,e\sqrt{M^2+P_3^2}\ \psi\quad ,\quad e = \pm\,1 \tag{5.10}$$

The (two-valued) constant of motion $\hat{C}$ corresponds to a trivial symmetry in Minkowski spacetime, involving the longitudinal motion along a given direction, the z-axis. It ceases to be trivial in the present case because the associated degeneracy is raised by the gravitational coupling of the spin of Dirac particles with matter vorticity and/or spacetime torsion (directed in the present case along the *same* z-axis). The role of this symmetry is important in the coupling of the spin of Dirac particles with matter vorticity and/or spacetime torsion. This symmetry, and the corresponding integral of motion, are trivial in Minkowski spacetime, connected to the longitudinal motion of the system with respect to a given arbitrary direction. Introducing now matter vorticity and/or torsion fields along this direction, the symmetry is preserved but the degeneracy in the energy spectrum is raised. Later on we will show that, by a Foldy-Wouthuysen transformation[28,29] we can show that -

for low values of the momentum π_3 - $\hat{C}$ can be interpreted as proportional to the projection of the spin of the test field along the z-axis. For high values of π_3, or M = 0, $\hat{C}$ is proportional to the helicity operator, and we have

$$e = L \text{ sign } (P_3) \tag{5.11}$$

where L is the eigenvalue of γ^5.

We then select simultaneous eigenstates of $\hat{C}$, of the Hamiltonian defined by (5.7) and of the global momenta defined by the Killing vectors[17] $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \phi}$, described by

$$\psi = A\begin{pmatrix}\phi(r)\\ \eta(r)\end{pmatrix} e^{-i(m\phi+P_3 z+\mathcal{E}t)} \tag{5.12}$$

where the half-integer m is eigenvalue of momentum operator $i\partial/\partial\phi$, and $A = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$. Condition (5.10) requires that the two component spinors ϕ and η have the form

$$\phi = f(r)\begin{pmatrix}1\\ \gamma_+\end{pmatrix} \quad , \quad \eta = g(r)\begin{pmatrix}1\\ \gamma_-\end{pmatrix} \tag{5.13}$$

where

$$\gamma_\pm = \frac{\pm M + e\sqrt{M^2+P_3^2}}{P_3} \tag{5.14}$$

Under (5.12) and (5.13) the second-order equation resulting from Dirac's equation is completely decoupled for each component. The solutions of (5.7) which are regular at r = 0 have the expression

$$\psi = A\begin{pmatrix} \begin{pmatrix}1\\ \gamma_+\end{pmatrix} i\gamma_-(\varepsilon-K)\alpha_+(x) \\ \begin{pmatrix}1\\ \gamma_-\end{pmatrix} \ell(2m+1)\alpha_-(x) \end{pmatrix}\exp\left[-i\left(\varepsilon t+m\phi+P_3z\right)\right] \tag{5.15a}$$

for $m \geq \frac{1}{2}$, and

$$\psi = A\begin{pmatrix} \begin{pmatrix}1\\ \gamma_+\end{pmatrix} \ell\gamma_-(1-2m)\beta_+(x) \\ \begin{pmatrix}1\\ \gamma_-\end{pmatrix} i(\varepsilon+K)\beta_-(x) \end{pmatrix}\exp\left[-i\left(\varepsilon t+m\phi+P_3z\right)\right] \tag{5.15b}$$

for $m \leq -\frac{1}{2}$. We have denoted

$$K = e\sqrt{M^2+P_3^2} - \frac{1}{2}(\Omega-kS) \quad , \quad x = \cosh 2\ell r,$$

and

$$\alpha_\pm = (x^2-1)^{(2m\pm1)/4}(1+x)^{\frac{\Omega\varepsilon}{2\ell^2}\mp\frac{1}{2}} F\left(a,\ b,\ m+1 \pm \frac{1}{2};\ \frac{1-x}{2}\right) \tag{5.16a}$$

$$\beta_\pm = (x^2-1)^{-(2m\pm1)/4}(1+x)^{-\frac{\Omega\varepsilon}{2\ell^2}\pm\frac{1}{2}} F\left(1-a,\ 1-b,\ 1-m \mp \frac{1}{2};\ \frac{1-x}{2}\right) \tag{5.16b}$$

where $F(a,b,c;y)$ is the hypergeometric function[13], with the parameters

$$a = m + \frac{1}{2} + \frac{\Omega\varepsilon}{2\ell^2} + \frac{n}{2} \quad , \quad b = m + \frac{1}{2} + \frac{\Omega\varepsilon}{2\ell^2} - \frac{n}{2} \quad , \quad n = \left(\frac{\Omega^2-\ell^2}{\ell^4}\,\varepsilon^2 + \frac{K^2}{\ell^2}\right)^{1/2} \tag{5.17}$$

Positive and negative energy solutions are related by the symmetry $\psi \to i\gamma^2\psi^*$. In particular, for the sets (5.15a) and (5.15b) we have

$$\psi\left(\varepsilon, m, k_3\right) = \sigma i\gamma^2\psi\left(-\varepsilon,\ -\ m,\ -\ P_3\right)^* \tag{5.18}$$

where $\sigma = 1$ if $\ell^2 > 0$, and $\sigma = (-i)^{m+\frac{1}{2}}$ if $\ell^2 < 0$. We note that in the case of neutrinos ($M = 0$) $L \to -L$ under (5.18) because although γ^5 does not change sign, its eigenvalue changes under (5.18).

On the space of solutions (5.15) we introduce the operators

$$J_\pm = L_\pm + S_\pm \tag{5.19}$$

where $L_\pm$ are the operators (3.8) of the algebra of angular momentum associated to the scalar field, and

$$S_\pm = \pm\ e^{\mp i\phi}\ \frac{1}{2(x^2-1)^{1/2}}\ \Sigma^3 \tag{5.20}$$

We define J_3 by the relation $\left[J_+, J_-\right] = 2J_3$ and obtain

$$J_3 = -\ i\ \left(\frac{\partial}{\partial\phi} + \frac{\Omega}{2\ell^2}\ \frac{\partial}{\partial t}\right) \tag{5.21}$$

Their effect on the set (5.15a), that is, for $m \geq 1/2$, is given by

$$J_-\psi(\varepsilon,m) = (2m+1)\psi(\varepsilon,m-1)$$

$$J_+\psi(\varepsilon,m) = -\ \frac{ab}{2m+3}\ \psi(\varepsilon,m+1) \tag{5.22}$$

$$J_3\psi(\varepsilon,m) = \left(m + \frac{\Omega}{2\ell^2}\ \varepsilon\right)\psi(\varepsilon,m)$$

The effect on the set (5.15b), that is, for $m \leq -1/2$, is derived from (5.22) by complex conjugation and the use of (3.9) and (5.18) - (5.21). We remark that $J_+\psi(m=-1/2) \sim \psi(m=1/2)$, $J_-\psi(m=1/2) \sim \psi(m=-1/2)$.

These relations make possible to span the whole set of solutions (5.15) from a given solution, by successive applications of J_+ and J_-. Analogous to the scalar field case, they will be useful to normalize the solutions.

Defining

$$J_1 = \left(J_+ + J_-\right)/2 \quad , \quad J_2 = \left(J_+ - J_-\right)/2i \tag{5.23}$$

we have that (5.21) and (5.23) satisfy the algebra of angular momentum, and the second-order equation resulting from Dirac's equation is an eigenvalue for the square of the total angular momentum, resulting

$$\left[(J_1)^2 + (J_2)^2 + (J_3)^2\right]\psi(\varepsilon,m) = \frac{n^2-1}{4}\,\psi(\varepsilon,m) \tag{5.24}$$

for the set (5.15). Normalizable solutions must obviously have $n^2 \geq 1$, as we shall see.

We introduce on the space of solutions (5.15) the scalar product

$$<\psi\left(\varepsilon',m',P_3\right)|\psi\left(\varepsilon,m,P_3\right)> = \int\sqrt{-g}\;\psi\left(\varepsilon',m',P_3'\right)^{+}\psi\left(\varepsilon,m,P_3\right)d^4x \tag{5.25}$$

where the domain of integration is the domain of definition of the coordinate system in (2.1), and we will use (5.25) to normalize the solutions. The normalization integral (5.25) is well defined by the following arguments. Let us consider the local classical Dirac current $j^{(A)} = \bar{\psi}\gamma^A\psi = e^{(A)}_{\alpha}(x)\bar{\psi}\gamma^{\alpha}(x)\psi$. Its zeroth component $j^{(0)} = \psi^{+}\psi$ is the local number density of fermions. As expected $j^{(0)}$ transforms as the zeroth component of a Lorentz vector with respect to local Lorentz tranformations, and it is a scalar with respect to coordinate transformations of the spacetime. The local number $j^{(0)}\sqrt{-g}\, d^4x$ is thus a scalar, and integrated over a given volume of the manifold yields a positive definite quantity which is coordinate invariant.

The operators J_1 and J_2, defined in (5.23) satisfy

$$\begin{aligned} \langle\psi'|J_a\psi\rangle &= -\langle J_a\psi'|\psi\rangle \quad \text{if} \quad \ell^2 > 0 \\ \langle\psi'|J_a\psi\rangle &= \langle J_a\psi'|\psi\rangle \quad \text{if} \quad \ell^2 < 0, \end{aligned} \tag{5.26}$$

a = 1,2. We note that relations (5.26) hold only for the subset of (5.15) that are normalizable under (5.25). Since J_1 and J_2 are not Hermitian if $\ell^2 > 0$, it follows that the square of total angular momentum operator J^2 (cf. (5.24)) may have negative or zero eigenvalues ($n^2 \le 1$), related to continuous energy solutions, as we shall see.

The boundary conditions to be adopted for the solutions are that Dirac fields (which are test fields and do not contribute to the curvature) are finite perturbations at any spacetime point. We impose

$$\lim_{x\to\infty} \psi^+\psi = 0 \quad , \quad \text{if} \quad \ell^2 > 0 \tag{5.27a}$$

$$\lim_{x\to -1} \psi^+\psi = 0 \quad , \quad \text{if} \quad \ell^2 < 0 \tag{5.27b}$$

The analysis and implementation of the above conditions are basically analogous to the scalar field case, as to perform linear tranformations[19] on the hypergeometric functions, etc. In the light of these similarities, calculations are omitted. We deal with positive energy solutions only. The corresponding negative solutions may be obtained through the relation (5.18).

Hyperbolic metrics ($\ell^2 > 0$)

(i) For $n^2 \ge 1$, the boundary condition (5.27a) implies

$$\psi = A \begin{pmatrix} \begin{pmatrix} 1 \\ \gamma_+ \end{pmatrix} i\gamma_-(\varepsilon-K)\alpha_{(+)}(x) \\ \begin{pmatrix} 1 \\ \gamma_- \end{pmatrix} \ell(2m+1)\alpha_{(-)}(x) \end{pmatrix} e^{-i(\varepsilon t+m\phi+P_3 z)} \tag{5.28a}$$

for $\frac{3}{2} \le m < \infty$, and

$$\psi = A \begin{pmatrix} \begin{pmatrix} 1 \\ \gamma_+ \end{pmatrix} \ell\gamma_-(1-2m)\beta_{(+)}(x) \\ \begin{pmatrix} 1 \\ \gamma_- \end{pmatrix} i(\varepsilon+K)\beta_{(-)}(x) \end{pmatrix} e^{-i(\varepsilon t+m\phi+P_3 z)} \tag{5.28b}$$

for $-j \le m \le \frac{1}{2}$, where

$$\varepsilon = (2j+1)\Omega + \left[(\Omega^2-\ell^2)(2j+1)^2 + \left[e\sqrt{M^2+P_3^2} - \left(\frac{\Omega-kS}{2}\right)\right]^2\right]^{1/2} \tag{5.29}$$

and

$$\alpha_{(\pm)} = (x^2-1)^{\frac{2m\pm1}{4}}(x+1)^{\frac{\Omega\varepsilon}{2\ell^2}\mp\frac{1}{2}} F\left(m-j+\frac{\Omega\varepsilon}{\ell^2},\ m+j+1,\ m+1\pm\frac{1}{2};\ \frac{1-x}{2}\right)$$

$$\beta_{(\pm)} = (x^2-1)^{-(\frac{2m\pm1}{4})}(x+1)^{-\frac{\Omega\varepsilon}{2\ell^2}\pm\frac{1}{2}} F\left(-m-j,\ j+1-m-\frac{\Omega\varepsilon}{\ell^2},\ 1-m\mp\frac{1}{2};\ \frac{1-x}{2}\right)$$

Here

$$j = \text{half-integer} \ge -\frac{1}{2} \tag{5.30}$$

The solutions are said to have a discrete energy spectrum, in the sense of (5.29), and are normalizable since they satisfy also $\lim_{x\to\infty} \sqrt{-g}\,\psi^{+}\psi = 0$. The complete range of m is given by

$$- j \leq m < \infty \tag{5.31}$$

We obviously have $J_{-}\psi(m = - j) = 0$.

(ii) For $n^2 < 1$, condition (5.27a) is automatically satisfied. The range of m is not restricted ($-\infty < m < \infty$) and the solutions have a continuous energy spectrum.

It is easy to see that the only normalizable solutions (with respect to the scalar product (5.52)) are those for which $n^2 \geq 1$, that is, the *discrete energy* solutions.

From the expression (5.29) for the energy spectrum, we see that the term $(\Omega-kS)/2$ is responsible for the splitting of each energy level into a doublet. These splitting effects produced by matter vorticity and spacetime torsion are additive, due to the existence of the *same* constant of motion (5.9) for Dirac test particles in both cases. This constant of motion generates a trivial symmetry of the system in Minkowski spacetime, but whose associated degeneracy in the energy spectrum is raised by the gravitational coupling to matter vorticity and/or spacetime torsion, producing the above-mentioned split. In the Riemannian limit ($S = 0$) this splitting does remain but it disappears in a Riemann-Cartan spacetime with $\Omega = kS$. The presence (or absence) of the splitting does not therefore imply the presence (or absence) of torsion or matter vorticity. What matters is the simultaneous effect of torsion and vorticity.

The splitting is present even for the energy levels in the continuous region. Although not observable, it has the effect of doubling the number density of states. This will be easily seen when we consider the Foldy-Wouthuysen representation of the solutions.

The above discussion is integrally valid for $\ell^2 \leq 0$, as we will see from the energy spectrum for those cases.

The conditions (i) and (ii) for discrete and continuous energy can be reexpressed in terms of the cosmological parameter Ω and ℓ^2:

(1) $\Omega^2 > \ell^2$ (this case includes Gödel's geometry)

- if $\varepsilon^2 \geq \dfrac{\ell^2(\ell^2-K^2)}{\Omega^2-\ell^2}$, discrete energy;

- continous energy, otherwise.

Unlike the scalar field case, we have here a continuous energy region lying in the lower part of the spectrum.

(2) $\Omega^2 = \ell^2$

- if $K^2 \geq \ell^2$, discrete energy;
- if $K^2 < \ell^2$, continuous energy.

(3) $\Omega^2 < \ell^2$

- if $\varepsilon^2 \leq \dfrac{\ell^2(K^2-\ell^2)}{\ell^2-\Omega^2}$, discrete energy;

- continuous energy, otherwise.

The conditions supra, (1) and (3), impose restrictions on the values of j in the spectrum (5.29). Such restrictions ensure the reality of (5.29).

The normalization of the discrete energy solution (5.28) with respect to (5.25) can be performed in a manner analogous to the scalar field case. Starting from the normalization of $\psi(m = -j)$ and using the operators (5.29) and their properties defined in (5.23), (5.26) it results

$$<\psi_{\varepsilon' m' P_3'}|\psi_{\varepsilon m P_3}> = (2\pi)^3(1+\gamma_-^2)2\varepsilon(\varepsilon-K)2^{1+2m+\frac{\Omega\varepsilon}{\ell^2}} \times$$

$$\frac{\left(j-\frac{1}{2}\right)!\left[\left(m+\frac{1}{2}\right)!\right]^2\Gamma\left(\frac{\Omega\varepsilon}{\ell^2}-j-\frac{1}{2}\right)}{4\ell^2(m+j)!\left(\frac{\Omega\varepsilon}{\ell^2}-2j-1\right)\Gamma\left(\frac{\Omega\varepsilon}{\ell^2}-j+m\right)}\;\delta_{m',m}\delta(\varepsilon'-\varepsilon)\delta\left(P_3'-P_3\right)$$

for $\frac{3}{2} \le m < \infty$, and

$$\langle\psi_{\varepsilon' m' P'_3}|\psi_{\varepsilon m P_3}\rangle = (2\pi)^3(1+\gamma_-^2)2\varepsilon(\varepsilon+K)2^{1-2m-\frac{\Omega\varepsilon}{\ell^2}} \times$$

$$\frac{(j+m)!\left[\left(\frac{1}{2}-m\right)!\right]^2\Gamma\left(\frac{\Omega\varepsilon}{\ell^2}-j-m\right)}{4\ell^2\left(j+\frac{1}{2}\right)!\left(\frac{\Omega\varepsilon}{\ell^2}-2j-1\right)\Gamma\left(\frac{\Omega\varepsilon}{\ell^2}-j+\frac{1}{2}\right)}\,\delta_{m',m}\delta(\varepsilon'-\varepsilon)\delta\left(P'_3-P_3\right)$$

for $-j \le m \le \frac{1}{2}$.

<u>Circular metrics ($\ell^2 < 0$)</u>

The analysis in this case must account for the solutions of the $\Omega = 0$ limit. In what follows we denote $\lambda^2 = -\ell^2$. On (5.15) we impose the boundary condition (5.27b), and obtain the set of independent solutions (positive energy)

$$\psi = A\begin{pmatrix}\begin{pmatrix}1\\ \gamma_+\end{pmatrix} i\gamma_-(\varepsilon-K)\alpha_+(x)\\ \begin{pmatrix}1\\ \gamma_-\end{pmatrix}\lambda(2m+1)\alpha_-(x)\end{pmatrix} e^{-i(\varepsilon t+m\phi+P_3 z)}, \tag{5.32}$$

for $\frac{1}{2} \le m < \frac{\Omega\varepsilon}{\lambda^2} - \frac{1}{2}$;

$$\psi = A\begin{pmatrix}\begin{pmatrix}1\\ \gamma_+\end{pmatrix}\lambda\gamma_-(1-2m)\beta_+(x)\\ \begin{pmatrix}1\\ \gamma_-\end{pmatrix} i(\varepsilon+K)\beta_-(x)\end{pmatrix} e^{-i(\varepsilon t+m\phi+P_3 z)}, \tag{5.33}$$

for $-j \le m \le -\frac{1}{2}$; and

$$\psi = A\begin{pmatrix}\begin{pmatrix}1\\ \gamma_+\end{pmatrix} i\gamma_-(\varepsilon_1-K)\sigma_+(x)\\ \begin{pmatrix}1\\ \gamma_-\end{pmatrix}\lambda(2m+1)\sigma_-(x)\end{pmatrix} e^{-i(\varepsilon t+m\phi+P_3z)} \tag{5.34}$$

for $j \geq m > \frac{\Omega\varepsilon_1}{\lambda^2} + \frac{1}{2}$. Here

j = half-integer $\geq \frac{1}{2}$,

$$\varepsilon = (2j+1)\Omega + \left[(2j+1)^2(\Omega^2+\lambda^2) + \left[e\sqrt{M^2+P_3^2} - \frac{1}{2}\,(\Omega-kS)\right]^2\right]^{1/2}, \tag{5.35a}$$

$$\varepsilon_1 = -\,(2j+1)\Omega + \left[(2j+1)^2)(\Omega^2+\lambda^2) + \left[e\sqrt{M^2+P_3^2} - \frac{1}{2}\,(\Omega-kS)\right]^2\right]^{1/2}, \tag{5.35b}$$

and

$$\alpha_\pm(x) = (1-x^2)^{\frac{2m+1}{4}}(1+x)^{-\frac{\Omega\varepsilon}{2\lambda^2}\mp\frac{1}{2}} F\left(m+j+1,\ m-j-\frac{\Omega\varepsilon}{\lambda^2},\ m+1\pm\frac{1}{2};\ \frac{1-x}{2}\right)$$

$$\beta_\pm(x) = (1-x^2)^{-(\frac{2m\pm1}{4})}(1+x)^{-\frac{\Omega\varepsilon}{2\lambda^2}\pm\frac{1}{2}} F\left(-m-j,\ j+1-m+\frac{\Omega\varepsilon}{\lambda^2},\ 1\mp\frac{1}{2}-m;\ \frac{1-x}{2}\right)$$

$$\sigma_\pm(x) = (1-x^2)^{\frac{2m\pm1}{4}}(1+x)^{-\frac{\Omega\varepsilon_1}{2\lambda^2}\mp\frac{1}{2}} F\left(m-j,\ j+1+m-\frac{\Omega\varepsilon_1}{\lambda^2},\ 1\pm\frac{1}{2}+m;\ \frac{1-x}{2}\right)$$

In the limit $\Omega = 0$, $\lambda^2 \neq 0$, the set (5.32) disappears, while (5.34) is crucial to complete the solutions in this limit. The energy eigenvalues (5.35a) and (5.35b) become equal, and the range of m is given by $-j \le m \le j$. Contrary to the scalar field case, the solutions (5.33) - (5.34) in the limit $\Omega = 0$ cannot be expressed as associated Legendre polynomials.

The normalization of (5.32) - (5.34) is given by

$$\langle\psi_{\varepsilon' m' p_3'}|\psi_{\varepsilon m p_3}\rangle = (2\pi)^3(1+\gamma_-^2)2\varepsilon(\varepsilon-K)2^{2m+1-\frac{\Omega\varepsilon}{\lambda^2}} \times$$

$$\frac{\left(j+\frac{1}{2}\right)!\left[\left(m+\frac{1}{2}\right)!\right]^2\Gamma\left(\frac{\Omega\varepsilon}{\lambda^2}+j+1-m\right)}{\left(\frac{\Omega\varepsilon}{\lambda^2}+2j+1\right)(j+m)!\,\Gamma\left(\frac{\Omega\varepsilon}{\lambda^2}+j+\frac{1}{2}\right)}\,\delta_{m'm}\delta(\varepsilon'-\varepsilon)\delta(p_3'-p_3)$$

for $\frac{1}{2} \le m < \frac{\Omega\varepsilon}{\lambda^2} - \frac{1}{2}$;

$$\langle\psi_{\varepsilon' m' p_3'}|\psi_{\varepsilon m p_3}\rangle = (2\pi)^3(1+\gamma_-^2)2\varepsilon(\varepsilon+K)2^{1-2m+\frac{\Omega\varepsilon}{\lambda^2}} \times$$

$$\frac{(2j+1)(j+m)!\left[\left(\frac{1}{2}-m\right)!\right]^2\Gamma\left(\frac{\Omega\varepsilon}{\lambda^2}+j+\frac{3}{2}\right)}{\left(\frac{\Omega\varepsilon}{\lambda^2}+2j+1\right)(j+\frac{1}{2})!\,\Gamma\left(\frac{\Omega}{\lambda^2}+j+1-m\right)}\,\delta_{m'm}\delta(\varepsilon'-\varepsilon)\delta(p_3'-p_3)$$

for $-j \le m \le -\frac{1}{2}$;

$$\langle\psi_{\varepsilon_1' m' p_3'}|\psi_{\varepsilon_1 m p_3}\rangle = (2\pi)^3(1+\gamma_-^2)2\varepsilon_1(\varepsilon_1-K).2^{1-2m+\frac{\Omega\varepsilon_1}{\lambda^2}} \times$$

$$\frac{(2j+1)(j-m)!\left[\left(\frac{1}{2}+m\right)!\right]^2\Gamma\left(-\frac{\Omega\varepsilon_1}{\lambda^2}+j+\frac{3}{2}\right)}{\left(-\frac{\Omega\varepsilon_1}{\lambda^2}+2j+1\right)\;(j+\frac{1}{2})!\,\Gamma\left(-\frac{\Omega\varepsilon_1}{\lambda^2}+j+1+m\right)}\;\delta_{m'm}\delta(\varepsilon'-\varepsilon)\delta(p'_3-p_3)$$

for $j \geq m \geq \frac{\Omega\varepsilon_1}{\lambda^2} + \frac{1}{2}$.

Som-Raychandhuri metrics ($\ell^2 = 0$).

We use the same limit procedure, as in the scalar field case, on taking $\ell \to 0$ in (5.28) - (5.29). Only discrete energy solutions will be obtained,

$$\psi = A\begin{pmatrix}\begin{pmatrix}1\\ \gamma_+\end{pmatrix} i\gamma_-(\varepsilon_1-K)\alpha_+(\xi)\\ \begin{pmatrix}1\\ \gamma_-\end{pmatrix}\sqrt{\Omega\varepsilon}(2m+1)\alpha_-(\xi)\end{pmatrix}e^{-i(\varepsilon t+m\phi+P_3z)}$$

for $\frac{3}{2} \leq m < \infty$;

$$\psi = A\begin{pmatrix}\begin{pmatrix}1\\ \gamma_+\end{pmatrix}\sqrt{\Omega\varepsilon}\;\gamma_-(1-2m)\beta_+(\xi)\\ \begin{pmatrix}1\\ \gamma_-\end{pmatrix} i\gamma_-(\varepsilon+K)\beta_-(\xi)\end{pmatrix}e^{-i(\varepsilon t+m\phi+P_3z)}$$

for $-j \leq m \leq \frac{1}{2}$. We have denoted $\xi = \Omega\varepsilon r^2$ and

$$\varepsilon = \Omega(2j+1) + \left[(2j+1)^2\Omega^2 + \left[e\sqrt{M^2+P_3^2} - \frac{1}{2}(\Omega-kS)\right]^2\right]^{1/2}, \tag{5.36}$$

$$\alpha_\pm(\xi) = \xi^{\frac{2m\pm1}{4}}\,e^{-\frac{\xi}{2}}\,F\left[-j+\frac{1}{2},\; m+1\pm\frac{1}{2};\; \xi\right)$$

$$\beta_{\pm}(\xi) = \xi^{-\frac{2m\pm1}{4}} e^{-\frac{\xi}{2}} F\left[-m-j,\ 1\mp\frac{1}{2}-m;\ \xi\right]$$

where F(a,b;ξ) is the confluent hypergeometric function.

For the Minkowski case (limit $\ell^2 \to 0$, $\Omega = 0$), we obtain

$$\psi = A\left(\begin{pmatrix}1\\ \gamma_+\end{pmatrix} i\gamma_-(\varepsilon-K) J_{m+\frac{1}{2}}\left(\sqrt{\varepsilon^2-K^2}\ r^2\right) \atop \begin{pmatrix}1\\ \gamma_-\end{pmatrix}\sqrt{\varepsilon^2-K^2}\ J_{m-\frac{1}{2}}\left(\sqrt{\varepsilon^2-K^2}\ r^2\right)\right) e^{-i(\varepsilon t+m\phi+P_3 z)} ,$$

$-\infty < m < \infty$, where J_ν are Bessel functions of first kind and

$$K = e\sqrt{M^2+P_3^2} + \frac{kS}{2}$$

The curvature effect on the Dirac field is contained in K. To see its effect on the energy levels, we make $\Omega = H = 0$ and $D = r$ in Dirac's equation (5.7). Afterwards we perform the unitary transformation $U = \cos\frac{\phi}{2} - i\Sigma^3\sin\frac{\phi}{2}$, followed by a coordinate transformation to Cartesian coordinates. Dirac's equation becomes

$$i\ \frac{\partial\phi}{\partial t} = \left[\gamma^5\left(\Sigma^1\hat{P}_1 + \Sigma^2\hat{P}_2\right) + \Sigma^3\left(\hat{C} + \frac{kS}{2}\right)\right]\psi \tag{5.37}$$

where the momentum operators are given by

$$\hat{P}_i = -\ i\ \frac{\partial}{\partial x^i}$$

As the energy and momenta are constants of motion we can perform the separation of variables

$$\psi = \mathring{\psi}\exp[-i(\varepsilon t + \vec{p}.\vec{x})] \tag{5.38}$$

where $\mathring{\psi}$ is a constant four spinor. Choosing ψ to be eigenstate of $\hat{C}$, the number of independent components of $\mathring{\psi}$ reduces to two,

$$\mathring{\psi} = \begin{pmatrix} \psi^0_1 \\ \psi^0_2 \\ \gamma_+\psi^0_1 \\ \gamma_-\psi^0_2 \end{pmatrix} \quad , \quad \gamma_\pm = \left(\pm M + e\sqrt{M^2+P^2_3}\right)/P_3 \tag{5.39}$$

Substituting (5.38) amd (5.39) into (5.37), we obtain a system of algebraic equations for the two independent spinor components ψ^0_1 and ψ^0_2. The compatibility condition for this system gives

$$\varepsilon^2 = P^2_1 + P^2_2 + \left[e\sqrt{M^2+P^2_3} + \frac{1}{2}\,kS\right]^2 \tag{5.40}$$

We see that if $S \neq 0$ we have for a given value of momenta a splitting of the energy levels into two levels, one for each value of the quantum number e, analogous to the previous cases where matter vorticity was also present. The energy spectrum (5.40) is continuous, and the splitting is therefore unobservable although it has the effect of doubling the number density of states. A result similar to (5.40) was also derived by Kerlick[30] and Rumpf[31] (the first for $\vec{p} = 0$) in the approximation of Riemann-flatness and a totally antisymmetric constant torsion generated by a background Dirac field (matter vorticity is not considered). They did not notice, however, the existence of the constant of motion (5.9), responsible for the split.

Expression (5.40) suggests that the split of the energy levels occurs even at the continuum region of the spectrum appearing in hyperbolic geometrics ($\ell^2 > 0$). This is in fact true, as we shall show by introducing a Foldy-Wouthuysen representation[28,29] of the solutions.

Foldy-Wouthuysen and Cini-Toushek Representation[28,29,32] of the Solutions

The purpose of this topic is to give representations of the solutions (for the low momentum and high momentum limits) that exhibit clearly the physical meaning of the constant of motion $\hat{C}$.

Let us start from the original Hamiltonian (5.7)

$$H = P + \Sigma^3\left[\hat{C} + \frac{1}{2}(\Omega - kS)\right]$$

where we have denoted $P = \gamma^5\left(\Sigma^1\pi_1 + \Sigma^2\pi_2\right)$, and consider the following unitary transformation

$$U = e^{\sigma\gamma^3} \tag{5.41}$$

where σ is a real parameter to be specificed for each case.

1) FW transformation

The parameter σ in (5.41) is specified by $\sigma = \frac{1}{2}\,\mathrm{arctg}(-P_3/M)$, $-\frac{\pi}{2} < 2\sigma\,\mathrm{sign}(P_3) < 0$. This transformation diagonalizes the constant of motion $\hat{C}$, namely

$$\hat{C}' = U\hat{C}U^{-1} = \sqrt{M^2+P_3^2}\,\gamma^0\Sigma^3 \quad , \tag{5.42}$$

and from (5.10) we have

$$\gamma^0\Sigma^3\psi' = -e\psi' \tag{5.43}$$

where $\psi' = U\psi$. The transformed Hamiltonian is given by

$$H' = P + \Sigma^3\left[\sqrt{M^2+P_3^2}\,\gamma^0\Sigma^3 + \frac{1}{2}\,(\Omega-kS)\right]$$

Since γ^0 and Σ^3 commute with P and $\gamma^0\Sigma^3$ is a constant of motion, by a further FW transformation the Hamiltonian (5.43) is diagonalized to the form

$$H'' = \left[P^2 + \left(\sqrt{M^2+P_3^2} + \frac{1}{2}(\Omega-kS)\gamma^0\Sigma^3\right)^2\right]^{1/2}\gamma^0 , \tag{5.44a}$$

where $P^2 = \pi_1^2 + \pi_2^2 - i\Sigma^3\left[\pi_1,\pi_2\right]$. We remark that the constant of motion $\hat{C}'$ commutes with the latter FW transformation, and we have from (5.43) and (5.44)

$$H''\psi'' = \left[P^2 + \left(e\sqrt{M^2+P_3^2} - \frac{1}{2}(\Omega-kS)\right)^2\right]^{1/2}\gamma^0\psi'' = \varepsilon\psi'' \tag{5.45}$$

The sign of the energy is given by the eigenvalue of γ^0, as usual.

The transformed solutions ψ'' obtained from (5.15) are eigenstates of P^2; in particular for the discrete energy solutions its eigenvalues are given by $P^2 = 2(2j+1)(\Omega|\varepsilon|-(j+1/2)\ell^2)$. Thence it follows from (5.45) that the splitting effect is general, occurring also for the continuous energy levels, and doubling the number of states density in the continuum region.

From (5.42) and (5.43) we have that - in the low momentum limit - the quantum number e is the projection of the spin of the Dirac field along the z-axis, namely, along the direction of the torsion and/or matter vorticity fields.

2) CT transformation

It can be accomplished by (5.41) with the parameter σ defined by $\sigma = \frac{1}{2}\,\mathrm{arctg}\,\frac{M}{P_3}$, with $0 < 2\sigma\,\mathrm{sign}(P_3) < \pi/2$. The transformed Hamiltonian results

$$H' = P + \Sigma^3\left[-\text{sign}(k_3)\sqrt{M^2+P_3^2}\ \gamma^5 + \frac{1}{2}(\Omega-kS)\right]$$

In the new representation γ^5 is a constant of motion, with

$$\gamma^5\psi' = L\psi' \quad , \quad L^2 = 1 \tag{5.46}$$

where L is the chirality of ψ'. The operator γ^5 corresponds in the old representation to

$$U^{-1}\gamma^5 U = \frac{-\text{sign}(k_3)}{\sqrt{M^2+P_3^2}}\ \hat{C} \tag{5.47}$$

and from (5.10), (5.46) and (5.47) we derive

$$L = e\ \text{sign}(k_3)\ , \tag{5.11}$$

As mentioned previously, the result (5.11) holds in the limit of high momentum ($P_3^2 >> M^2$) or zero mass ($M = 0$), that is, the limit where the CT representation is well defined.

From (5.46), (5.47) and (5.11) we have that the quantum number e - in the limit of high momentum or zero mass - is the projection of the spin of the Dirac field along the direction of the momentum $\vec{\pi}$.

Finally, we comment that the spinorial solutions - whether continuous or discrete energy solutions - must also be in correspondence with the classical motion of particles with spin, as expected. This correspondence would be the correct guide to decide among the various equations of motion of particles with spin in a curved spacetime with torsion, appearing in the literature[33-36]. We shall come to this issue in a future publication.

6 Conclusions

In the present paper we examine the gravitational coupling of Klein-Gordon and Dirac fields to matter vorticity and spacetime torsion, in the context of Einstein-Cartan theory. We show that - from the theoretical point of view - torsion and matter vorticity have identical effects on the physics of particle fields (Klein-Gordon or Dirac). For technical simplicity, the geometry of the spacetime is taken to be the family of Gödel-type metrics characterized by two real parameter (Ω,ℓ^2). In the framework of Hehl's non-propagating torsion theory, these are the simplest known solutions with matter vorticity; the source of spacetime curvature is a Weyssenhoff-Raabe fluid with spin vector constant and parallel to the vorticity field. The main motivation to use a WR fluid is that we want a class of models which encompass not only torsion but also matter vorticity, and for which class we have the limit of flat spacetime metric plus torsion field.

A complete set of solutions is obtained - for both Klein-Gordon and Dirac equations - that can be generated from a particular solution by successive applications of angular momentum operators derived from the Killing vectors of the spacetime. Boundary conditions are imposed, connected to the test field character of the set of solutions. The energy spectrum is derived there upon, and we distinguish for each class of Gödel-type metrics:

1) if $\ell^2 < 0$ (circular case), the energy spectrum is discrete, given by (3.20) and (3.23) for Klein-Gordon fields, and (5.35) and (5.36) for Dirac fields.

2) if $\ell^2 = 0$ (Som-Raychandhuri case), the energy spectrum is also discrete, given by (3.28) for Klein-Gordon fields, and (5.36) for Dirac fields.

3) if $\ell^2 > 0$ (hyperbolic case) we have to distinguish:
   - (3a) for metrics such that $\Omega^2 < \ell^2$, the discrete energy levels are given by (3.15) for Klein-Gordon fields, and (5.29) for Dirac fields. The energy spectrum presents also a continuum region in the upper part of the spectrum. We note that this class of Gödel-type metrics $(0 < \Omega^2 < \ell^2)$ presents no closed time-like curves.
   - (3b) for metrics such that $\Omega^2 \geq \ell^2$ the Klein-Gordon solutions have a

discrete energy spectrum only, with energy given by (3.15). Dirac solutions, under certain conditions, may present a continuum region in the lower part of the spectrum.

In general the reality of the eigenvalues of the square of the total angular-momentum operator imply discrete energy solutions. The eigenvalues of the operator $i\partial/\partial\phi$ (component of the angular momentum operator along the direction of vorticity/torsion) for continuuous energy solutions have no upper or lower bound, contrary to the discrete energy cases.

The analogy between classical geodesic motion and Klein-Gordon solutions is completely established. It follows that bounded (unbounded) geodesics correspond to discrete (continuous) energy solutions. The solution with limit energy between discrete and continuous energy levels corresponds exactly to the geodesics limiting bounded and unbounded motion. This correspondence is a guide to choose the boundary condition (3.11a) as the correct one.

Matter vorticity and/or torsion fields split the energy spectrum of Dirac particles in the same manner. These effects are additive, and result from the existence of the same constant of motion for both cases. The constant of motion generates a trivial symmmetry of the system in Minkowski spacetime, but whose associated degeneracy in the energy spectrum is raised by the gravitational coupling to matter vorticity and/or spacetime torsion, producing the above mentioned split[37]. The split effect is general: for discrete energy levels we can see immediately from the expression of the energy (cf. (5.29), (5.35) and (5.36)), while for the continuum region it can be made explicit by a Foldy-Wouthuysen transformation. Although not observable, the latter has the effect of doubling the number of states density of the continuum.

In the light of the results of this paper, two points remain to be examined for Dirac particles:

1) the effect of spin precession, occurring in the presence of torsion[31], and also in a background with matter vorticity[8,10], the precession being about the torsion/vorticity field direction. If torsion and vorticity are simultaneously present and aligned (as in the present case) the effects are

expected to be additive and cancelations can possibly occur;

2) the correspondence between Dirac field solutions of Section 4 and the classical motion of particles with spin, that would be crucial to decide among various classical equations of motion of particles with spin in a curved spacetime with torsion.

References

1) F.W. Hehl, Gen. Rel. Grav. *4*, 333 (1973); *5*, 491, (1973).
2) F.W. Hehl, in: Proc. 6th Course of the International School of Cosmology and Gravitation, Erice (1979), eds. P.G. Bergmann and V. de Sabatta (Plenum, NY, 1980).
3) J.D. de Oliveira, A.F.F. Teixeira and J. Tiomno, Phys. Rev. *D34*, 3661 (1986) and references therein.
4) M.J. Rebouças and J. Tiomno, Phys. Rev. *D28*, 1251 (1983).
5) J. Weyssenhoff and A. Raabe, Acta Phys. Pol. *9*, 7 (1947).
6) M.O. Calvão, I. Damião Soares and J. Tiomno, Gen. Rel. Grav. *22*, 683 (1990).
7) M. Novello, I. Damião Soares and J. Tiomno, Phys. Rev. *D27*, 779 (1983); Phys. Rev. *D28*, 1561 (E), 1983.
8) I. Damião Soares, Phys. Rev. *D23*, 272 (1981).
9) I. Damião Soares and L.M.C.S. Rodrigues, Phys. Rev. *D31*, 422 (1985).
10) I. Damião Soares and J. Tiomno, Rev. Bras. Fís., Suppl. vol. *14*, 372 (1984).
11) K. Gödel, Rev. Mod. Phys. *21*, 447 (1949).
12) M.M. Som and A.K. Raychaudhuri, Proc. R. Soc. London *A304*, 81 (1968).
13) Y.N. Obukov and V.A. Korotky, Clas. Q. Grav. 4, 1633 (1987).
14) I. Damião Soares, J. Math. Phys. *21*, 521 (1980).
15) C. Chevalley, Theory of Lie Groups I (Princeton University Press, Princeton, 1946).
16) I. Damião Soares and L.M.C.S. Rodrigues, in Proceedings of the III Brazilian School of Cosmology and Gravitation (ed. J.A.F. Pacheco, Sociedade Astronomica Brasileira, São Paulo, 1982).
17) G. Gibbons, Comm. Math. Phys. *44*, 245 (1975); M. Henneaux, Gen. Rel. Grav. *9*, 1031 (1978).
18) Equation (2.4) cna be solved by the standard substitution

$\phi(x) = (x^2-1)^{m/2}(x+1)^A g(x)$

where the parameter A is such that g satisfies the hypergeometric equation.

19) A. Erdélyi et al., Higher Transcendental Functions (McGraw Hill, New York, 1960).

20) For this analysis in the limit $x \to -1$, we make use of the transformation

$$F(a,b,c;y) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, y^{-a} F\left(a,\ a-c+1,\ a+b-c+1;\ 1-\frac{1}{y}\right) +$$

$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\, (1-y)^{c-a-b} y^{a-c}\, F\left(c-a,\ 1-a,\ c-a-b+1;\ 1-\frac{1}{y}\right)$$

21) B.D.B. Figueiredo, D. Sc. Thesis, Centro Brasileiro de Pesquisas Físicas, 1987.

22) B.D.B. Figueiredo, I. Damião Soares and J. Tiomno, Phys. Lett. *A137*, 99 (1989). We call attention to printing errors in this paper.

23) F.W. Hehl and B.K. Datta, J. Math. Phys. *12*, 1334 (1971).

24) Capital Latin indices are tetrad indices; they are raised and lowered with Minkowski metric $\eta^{AB}, \eta_{AB} = \mathrm{diag}(+1,-1,-1,-1)$.

25) We use units such that $\hbar = c = 1$.

26) We use a representation such that $(\gamma^i)^+ = \gamma^0\gamma^i\gamma^0$ with $(\gamma^0)^2 = -(\gamma^k)^2 = 1$, $k = 1,2,3$. The spin matrices are defined $\Sigma^j = \gamma^5\gamma^0\gamma^j$.

27) They are defined by $\gamma_{BCA} = -\, e^{\alpha}_{(B)||\beta}\, e_{\alpha(C)}\, e^{\beta}_{(A)}$, where $(||)$ denotes covariant derivative with respect to the metric (Christoffel) connection.

28) L.L. Foldy and S.A. Wouthuysen, Phys. Rev. *78*, 29 (1950).

29) J. Tiomno, Physica *53*, 58 (1971).

30) G.D. Kerlick, Phys. Rev. *D12*, 3004 (1975).

31) H. Rumpf, Gen. Rel. Grav. *10*, 647 (1979).

32) M. Cini and B. Toushek, N. Cim. 7, 422 (1958).

33) F.W. Hehl, Phys. Lett. *36A*, 225 (1975).

34) A. Trautman, Bull. Acad. Pol. Sci. *20*, 895 (1972).

35) G. Gognola, R. Soldati, L. Vanzo and S. Zerbini, Phys. Rev. *D25*, 3109 (1982).

36) R. Amorim, Phys. Rev. *D31*, 3099 (1985).

37) These effects differ in order of magnitude. While the split due to torsion is beyond any experimental possibility (cf. W.R. Stoeger, Gen. Rel. Grav. *17*, 981 (1985)) the split due to matter vorticity may be observable for known astrophysical configurations (I. Damião Soares, in Jayme Tiomno Festschrift, ed. R. Salmeron, to be published).