

# Reconstructing Bidimensional Scalar Field Theory Models

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## Abstract

In this paper we review how to reconstruct scalar field theories in two dimensional spacetime starting from solvable Schrodinger equations. Three different Schrodinger potentials are analyzed. We obtained two new models starting from the Morse and Scarf II hyperbolic potentials, *i.e.*, the  $U(\phi) = \phi^2 \ln^2(\phi^2)$  model and  $U(\phi) = \phi^2 \cos^2(\ln(\phi^2))$  model respectively.

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# 1 Introduction

Solitons are solutions of nonlinear equations that have the following fundamental properties: the profile is stable, the energy associated with them is finite and also they behave as particles in the sense that multi-solitonic solutions behave as independent one-soliton solutions as time goes to infinity [1]. Also, there are a less restricted class of solutions for non-linear equations that have the same properties of solitonic solutions, except the property that they retain their shape after collision. In this case such solutions are called solitary waves. In general solitons can exist in any  $(d+1)$  dimensional spacetime. In the  $(1+1)$  dimensional case, the static solutions are called kinks. These solutions link two degenerate trivial vacua of the theory. An important property are that these solutions are still stable, if we take into account quantum corrections. A deep analysis of the quantum properties of these solutions were carried out in the seventies. See for example ref.[2] and references therein. On the other hand, there are solutions that become unstable when quantum corrections are taking into account. In the  $(1+1)$  dimensional case these static solutions are called lumps or bounces.

In this paper we reconstruct kink and lump profiles and the scalar field theoretical models that support such kink or lump like solutions starting from exactly solvable Schrodinger equations. We use the fact that quantum corrections around these classical solutions are given by one dimensional Schrodinger equations. The zero mode eigenfunction of this equations is proportional to the derivative of the kink (or lump). This fact is based on translational invariance of the field theory model. Also Bogomol'nyi's condition give us a relation between the density potential and the zero mode eigenfunction. Then solving for the kink (or lump) from the zero mode and replacing in the Bogomol'nyi condition we recover the density potential as a function of the kink (or lump). This is a general strategy to recover the field theoretic model from arbitrary Schrodinger potentials knowing the zero eigenfunctions. Since we will be interested in computing quantum corrections of the field theoretic models, we must study only exactly solvable Schrodinger equations since in this case we are able to perform calculations of the quantum corrections. For our knowledge, the first authors that stressed this fact were Christ and Lee [3]. More recently using supersymmetric quantum mechanics Casahorran *et al* [4] and also Boya and Casahorran [5] continue this research program. For other interesting references see for instance [4],[5] and [6]. We would like to stress that previously to these works, Kumar [7] suggested the construction of solitonic profiles using isospectral Hamiltonians.

In  $(1+1)$  dimensions there are an infinite number of renormalizable scalar field theory models. Nevertheless in the literature it was considered only a very restricted number of them. This fact can be understood by the following reason: only in a few of them we can go beyond the perturbative analysis. There is a spread opinion that in two-dimensional models one can test ideas that can be generalized afterward in the more interesting  $(3+1)$  dimensional case. In  $(1+1)$  dimensions it was obtained amazing results as for example the fractionization of charge [8] or the emergence of fermions from bosons [9], both phenomena that have counterpart in the  $(3+1)$  dimensional case.

From the classical solutions of the non-linear field equations, we can go further and obtain quantum corrections. We have to expand the field operator around the classical solution, and retain only quadratic terms (where the higher order terms can be treated

perturbatively). Then, we will obtain a Schrodinger equation that describes the mode oscillations around the classical solution. This Schrodinger equation admit a zero mode solution with eigenfunction equal to the derivative of the classical solution. In the case in which the classical solution is a kink, we have that this zero mode have the lowest eigenvalue and then all the mode oscillations are real, that is, we have stability. In the case in which the classical solution is a bounce, there will be another mode solution with negative eigenvalue, and consequently there will be an imaginary mode, that signalize instability. As the next step, we can solve the Schrodinger equation for all mode solutions and compute the first quantum corrections. In the case of kinks, one of the quantities of physical interest is the quantum corrections to the mass of such kinks, that is, we compute the "zero point" energy of this configuration. In the case of bounces we compute the decay rate of the unstable vacuum. Any way, in both cases we have to solve a Schrodinger equation that in general can not be solved analytically. In the case of sine-Gordon and  $\phi^4$  stable models, the Schrodinger equations are respectively the  $N = 1$ ,  $N = 2$  cases of the general solvable reflectionless potential  $V(x) = -N(N + 1)/\cosh^2(x)$ . For the most simple polynomial unstable  $\phi^3$  model we have the  $N = 3$  case. This last model was used as a laboratory for computing the decay rate (or the life time) of a system trapped in a false vacuum [10]. Also, recently the  $\phi^3$  model was used as an exactly solvable toy model for tachyon condensation in string field theory [11]. For the case of the  $\phi^6$  model, we have to deal with a complicated Schrodinger equation, that can be reduced to a Heun equation [12]. Unfortunately this equation can not be exactly integrated. Then, we see that although in two dimensions we can have an infinite number of renormalizable scalar field theories, only for a very restricted number of them we can perform an analytic treatment in the non-perturbative sector.

The organization of the paper is the following. In section II we show how to reconstruct scalar field theory models from zero mode solutions. In section III we analyze the models that arise from the Rosen-Morse II hyperbolic potential. In section IV and V we perform the same analysis for the the Morse and the Scarf II hyperbolic potentials respectively. Conclusions are given in section VI. We use throughout this paper  $\hbar = c = 1$ .

## 2 Reconstructing the field theory models

In this section we briefly review how to construct scalar field theory models starting from solvable Schrodinger equations. We start from a lagrangean

$$L = \int dx \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \right) , \quad (1)$$

where  $\mu = 0, 1$  and  $U(\phi)$  with at least two degenerate absolute minima as showed in fig.(1-a) or with a local minima (a false vacuum) as showed in fig.(1-b). The classical equation of motion for static configurations are given from eq.(1)

$$\frac{d^2 \phi}{dx^2} = U'(\phi) . \quad (2)$$

The eq.(2) can be analyzed making use of a particle mechanical analogy. Suppose that  $\phi$  describe the position of a particle and  $x$  is the time. Consequently, eq.(2) is the equation

of motion of a particle in a conservative potential  $-U(\phi)$ . In order to analyze eq.(2) we have to take into account only the possible trajectories of the 'particle' in the inverted potential. Clearly, we are interested only in solutions with a finite energy, in other words, solutions that have a finite interval of motion in  $\phi$  but that are not oscillatory. From the inverted potential  $-U(\phi)$  given by fig.(1-a) it is easy to see that such requirement is satisfied only for those motion that take place between the absolute minima given by points 1 and 2. Using the same argument for the case given by fig.(1-b) we see that we allow the motion that start in point 3 bounce in 4 and return to the point 3. In the first case the static solution is know as kink, while in the second case such solution is called a lump or a bounce. From figures (1-a) and (1-b) we see that these solutions are integrals of motion with zero energy, using the particle mechanical analogy. From eq.(2) we have

$$\frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 = U(\phi) , \quad (3)$$

an equation that is known as Bogomol'nyi condition [13]. From eq.(3) it is straightforward to obtain the kinks or lumps (we denote them as  $\phi_c$ ) solving the integral

$$x - x_0 = \int^{\phi_c(x)} \frac{1}{\sqrt{2U(\phi)}} d\phi \quad (4)$$

and inverting it. With the classical static configuration we can go to the first quantum corrections. For such purpose, we expand the time dependent field  $\phi(x,t)$  around the static configuration, *i.e*  $\phi(x,t) = \phi_c(x) + \eta(x,t)$ . Substituting this expansion in eq.(1) and retaining only quadratic terms in  $\eta$  we obtain the following lagrangean:

$$L = L[\phi_c] + \int dx \left[ \frac{1}{2} \frac{d^2 \eta}{dt^2} - \frac{1}{2} \eta \left( -\frac{d^2}{dx^2} + U''(\phi_c(x)) \right) \eta \right] . \quad (5)$$

As a next step, we use the expansion  $\eta(x,t) = \sum_n q_n(t) \psi_n(x)$ , and choosing a complete basis  $\{\psi_n\}$  as solutions of the Schrodinger equation

$$\left( -\frac{d^2}{dx^2} + U''(\phi_c(x)) \right) \psi_n(x) = \omega_n^2 \psi_n(x) , \quad (6)$$

we reduce the lagrangean given by eq.(5) to

$$L = L[\phi_c] + \sum_n \left( \frac{1}{2} \dot{q}_n^2 - \frac{\omega_n^2}{2} q_n^2 \right) . \quad (7)$$

From eq.(7) we see that the problem was reduced to a system with infinitely uncoupled harmonic oscillators. Now, the quantization program can be implemented in the standard way. In particular, the zero point energy, that in the case of kinks are interpreted as its mass [1], is given by

$$H = H[\phi_c] + \frac{1}{2} \sum_n \omega_n . \quad (8)$$

Taking the derivative of eq.(3) it is easy to see that eq.(6) always admit a solution with eigenvalue  $\omega_0^2 = 0$  and with respective eigenfunction  $\psi_0$  given by

$$\psi_0 \propto \frac{d\phi_c}{dx}. \quad (9)$$

From fig.(1-a) one can see that  $\frac{d\phi_c}{dx}$  is zero only in the limit  $x \rightarrow \pm\infty$ , that is, the eigenfunction  $\psi_0$  has no nodes and then  $\omega_0^2 = 0$  is the lowest eigenvalue. Then all the  $\omega$ 's are real and consequently the kink is stable when quantum corrections are taking into account. On the other hand, from fig. (1-b) one can see that  $\frac{d\phi_c}{dx}$  is zero for some finite  $x = x_0$  in the turning point 4 (we can always choose this point by translational invariance as corresponding to  $x_0 = 0$ ). In this case the eigenfunction  $\psi_0$  has a node, and then  $\omega_0^2$  is not the lowest eigenvalue. There is one negative eigenvalue  $\omega^2 < 0$  and then one imaginary  $\omega$ . In this situation, the lump becomes unstable by quantum corrections. In this case the eq.(8) has no direct physical interpretation, but the imaginary part signalize decay of a false vacuum [21].

In both cases, to go further in the quantization program, we have to solve a one dimensional Schrodinger equation but in general cases this equation can not be analytically solved. Instead to try to solve general Schrodinger equations, we can adopt a different approach. We can start from a exactly solvable Schrodinger equation to obtain the field theory model associated with it. The steps in this program are the following: first, from eq.(9) solve it for  $\phi_c$

$$\phi_c(x) \propto \int^x \psi_0(y) dy. \quad (10)$$

the second one is to invert for  $x$  from eq.(10) obtaining  $x = x(\phi_c)$ . Thirth, we substitute eq.(9) in eq.(3) to obtain  $U(\phi_c)$ , that is

$$U(\phi_c) = \frac{1}{2} \left( \frac{d\phi_c}{dx} \right)^2 \propto \frac{1}{2} (\psi_0(x(\phi_c)))^2. \quad (11)$$

Finally we can remove the subscript 'c', obtaining in this way the scalar field theoretic model. There are some points that we would like to stress. From figures (1-a) and (1-b) we see that in principle we only obtain in this way the part of  $U(\phi)$  for  $\phi$  that lies between the points 1 and 2 in the case of kinks or in the case of lumps for  $\phi$  that lies between the points 3 and 4. Out of this intervals in principle  $U(\phi)$  obtained in this way can be any arbitrary function. From the above discussion there are an infinite number of  $U(\phi)$ . Since we are interested in field theory models that are smooth functions of  $\phi$ , the number of possible  $U(\phi)$  will reduce to one or zero.

Before analyze particular Schrodinger potentials we would like to clarify the reason when and why we can remove the subscript 'c' in eq.(11). There are two different cases. The first one is the case where from eq.(11) which is valid in principle only for those values of  $\phi = \phi_c$ , we use the same expresion for all values of  $\phi$  that are out of  $\phi_c$ , and if the expresion is still well defined for those values of  $\phi$  then we have obtained a unique and well behaved field theoretic model. The second situation is when eq.(11) is not valid for those values of  $\phi$  that lies out of  $\phi_c$  (For example (11) would be imaginary or there will appear some singularities). Consequently in this case there is no a well defined field theoretical model.

As we have noted the eigenfunction  $\psi_0(x)$  as given by eq.(9) is not normalized. That is, we are obtaining the field theoretical models modulo the coupling constants. With the functional form of  $U(\phi)$  in hands we can choose the coupling constants.

In next sections we will construct field theory models starting from the following integrable Schrodinger equations with three different potentials [14]:

Rosen-Morse II hyperbolic

$$A^2 + B^2/A^2 - A(A+1)/\cosh^2(x) + 2B \tanh(x), \quad B < A^2, \quad (12)$$

the Morse potential:

$$V(x) = A^2 + B^2 \exp(-2x) - 2B(A+1/2) \exp(-x) \quad (13)$$

and finally the Scarf II hyperbolic potential:

$$V(x) = A^2 + (B^2 - A^2 - A)/\cosh^2(x) + B(2A+1) \tanh(x)/\cosh(x). \quad (14)$$

### 3 The Rosen-Morse II hyperbolic potential

As we discussed in the in the last section, the Rosen-Morse II hyperbolic potential is given by

$$V(x) = A^2 + B^2/A^2 - A(A+1)/\cosh^2(x) + 2B \tanh(x), \quad B < A^2, \quad (15)$$

where  $A, B$  are constants. For this potential the eigenfunctions are given by [14]

$$\psi_n(x) = (1-y)^{\alpha/2} (1+y)^{\beta/2} P_n^{(\alpha,\beta)}(y), \quad (16)$$

with eigenvalues

$$\omega_n^2 = A^2 - (A-n)^2 + \frac{B^2}{A^2} - \frac{B^2}{(A-n)^2}, \quad n = 0, 1, 2.. \quad (17)$$

In eq.(16) we have  $\alpha = A - n + B/(A - n)$ ,  $\beta = A - n - B/(A - n)$ ,  $y = \tanh(x)$  and  $P_n^{(\alpha,\beta)}(y)$  are Jacoby polynomials [15]. To reconstruct field theory models that support kink-like solutions (stable models) from this potential have to work with the ground state, the zero node eigenfunction  $\psi_0(x)$ . On the other hand, to reconstruct field theoretic models that support lump-like solutions we have to consider the eigenfunction with one node  $\psi_1(x)$ .

#### 3.1 Stable Models

To obtain the kinks from the potential given by eq.(15) we have to integrate the ground state eigenfunction  $\psi_0(x)$  that can be obtained from eq.(16),

$$\psi_0(x) = (1-y)^{\alpha/2} (1+y)^{\beta/2}. \quad (18)$$

Using eq.(9) we obtain the kink

$$\phi_c(x) = \int^{\tanh(x)} \frac{(1-y)^{\alpha/2} (1+y)^{\beta/2}}{1-y^2} dy, \quad (19)$$

and from eq.(11) we obtain for  $U(\phi_c)$

$$U(\phi_c) = (1 - y)^\alpha (1 + y)^\beta . \quad (20)$$

Note that in general the integral given by eq.(19) can not be performed analytically. Consequently, we will be restrict the following cases:

**3.1.a**  $\alpha = \beta = A$  : In this case  $B = 0$ , and eq.(19) becomes

$$\phi_c(x) = \int^{\tanh(x)} \frac{(1 - y^2)^{A/2}}{1 - y^2} dy . \quad (21)$$

The above integral can ben performed in terms of elementary functions only when  $A$  is an integer. In eq.(20) we have in principle that  $-1 \leq y \leq 1$ , (this follows from the definition  $y = \tanh(x)$ ). In this interval we only obtain  $U(\phi)$  for those values that take  $\phi_c$ . As was mentioned in last section, to obtain  $U(\phi)$  for all  $\phi$  we have to extend the eq.(20) for all values of  $\phi$  and see if it is still well behaved. For integer  $A$  we see that eq.(20) is a well behaved funtion for all values of  $y$ . Then extending eq.(20) for all  $\phi$  is equivalent to extend it for all  $y$ . Then if we invert eq.(21) for  $y = \tanh(x)$  as function of  $\phi_c$ , extend such result for all values of  $\phi$  (and then for all values of  $y$ ) in a well behaved way and replacing it in eq.(20) we will obtain a well behaved field theory model. But for  $A > 2$  from eq.(19) we can solve  $d\phi/dy = (1 - y^2)^{(A-2)/2} = 0$  and we find that the points  $y = \pm 1$  signalizes maxima or minima for  $\phi$  as function of  $y$ , that is  $\phi$  as function of  $y$  is not singled valued, then it will be not possible to invert  $y$  as function of  $\phi$  in unique way. One can try to circumvent this difficulty supressing those values of  $y$  in which  $\phi$  is not single valued. Nevertheless, we will be able only to obtain  $U(\phi)$  by parts, generating in this way discontinuities, for example in the derivatives of  $U(\phi)$ . The same type of singularities has been noted in [3]. For  $A < 2$  *i.e.*,  $A = 1, 2$  this problem does not occur. One can perform the integral given by eq.(21) and replace in eq.(20) to obtain the well know sine-Gordon and  $\phi^4$  models respectively.

We would like to stress the following point. We are considering the case in which  $V(x) = -A(A + 1)/\cosh^2(x)$  for  $A$  integer. Such potentials have both, discrete and continuous modes with the advantageous property of being reflectionless. The first quantum corrections to the mass of the kinks are given by eq.(8). To sum the continuous modes, we have to know the density of states. For the case of reflectionless potentials this can be given in terms of the phase shifts of the one dimensional scattering problem. In general this sum is logarithmically divergent and we need to renormalize the theory. In two-dimensional scalar field theories such divergences can be eliminated using a normal ordering prescription. These properties was used by Cahill *et al* [16] to find a finite result for the quantum corrections to the mass of the static solitons. Moreover they also obtained the quantum mass corrections only in terms of the discrete eigenvalues of the associated Schrodinger equation. Using the former result Boya *et al* [17] obtained a closed expression for the first quantum corrections for the mass of the static kinks given by eq.(21) for any integer  $A > 0$  without explicit knowledgement of the field theory model that support such kinks. But here a question arises. Are these expressions valid even we have showed that there is not possible to construct a well behaved field theory model for  $A > 2$ ? It

is possible to show that these expressions are valid since the kinks only see the parts of the field theory models that lies between the degenerate vacua. In such domain, that is, between the extremum values of  $\phi_c$  it is always possible to invert  $y$  as function of  $\phi$ , and then it will be possible to reconstruct the field theory model for those values of the field that lies between the degenerate vacua. Out of this interval, the theory can be anything. In other words the complete field theory model is ambiguous, but in general not well behaved. We can say that the masses of the kinks (for  $A > 2$ ) that was computed in [17] are the masses of an infinite class of not well behaved field theory models.

**3.1.b**  $\alpha \neq \beta$  : In this case  $B \neq 0$ . We can rewrite eq.(19) as

$$\phi_c(x) = \int^{\tanh(x)} (1-y)^{(\alpha-2)/2} (1+y)^{(\beta-2)/2} dy . \quad (22)$$

Choosing  $(\alpha - 2)/2 = m$  and  $(\beta - 2)/2 = n$  we have

$$\phi_c(x) = \int^{\tanh(x)} (1-y)^n (1+y)^m dy . \quad (23)$$

Let us consider the case in which  $n = 0$ . Then, from eq.(23) we obtain

$$\phi_c(x) = (1 + \tanh(x))^{m+1} \quad (24)$$

from which we can solve for  $y = \tanh(x)$  and then substituting in eq.(20) we obtain

$$U(\phi) = \phi^2 \left( 2 - \phi^{1/(m+1)} \right)^2 . \quad (25)$$

We see that for the values of  $m$  such that  $1/(m + 1)$  is fractionary we will have in some cases (for example when  $1/(m + 1) = 1/2$ ) an imaginary value for  $U(\phi)$ . For such values of  $m$  we can redefine  $\phi^{1/(m+1)}$  as  $(\phi^2)^{1/2(m+1)}$  to make  $U(\phi)$  a real valued funtion, but in these cases we obtain a discontinuity in the derivative for  $U(\phi)$ , making the theory not well behaved. If we take  $m$  such that  $1/(m + 1) = 2l$  with  $l$  integer we will obtain

$$U(\phi) = \phi^2 \left( 2 - \phi^{2l} \right)^2 , \quad (26)$$

that is, we obtain a well behaved polinomial like field theory models with three degenerate vacua. The case  $l = 1$  is the  $\phi^6$  model with three degenerate vacuum that was considered in ref. [18]. In this paper Lohe obtained the expression of the renormalized mass of the soliton. It is interesting to point out that Flores *et al* [19] studying the vacuum decay rate in the the massive  $\phi_{3D}^6$  model in the thin wall approximation obtained the same kink solution associated with the model given by the  $l = 1$  case. Note that in this case  $V(x)$  given by eq.(15) is not reflectionless making the computations of the quantum mass corrections very hard. The eq.(26) also was analyzed in [5]. If we put  $1/(m + 1) = 2l + 1$  with  $l$  integer we will obtain

$$U(\phi) = \phi^2 \left( 2 - \phi^{2l+1} \right)^2 , \quad (27)$$

that is, well behaved field theory models with two degenerate vacua. The case  $l = 1$  is the case of the  $\phi^8$  theory with two degenerate vacua. The above field theory models was



also obtained in ref. [5]. If we consider the case in which  $m = 0$  we obtain the same configurations that in the  $n = 0$  case. If we consider the cases in which both  $n$  and  $m$  are integers, we can still integrate the eq. (23) but in this case it is not possible to obtain a well behaved field theory model for the same reason stressed in the  $B = 0$  case. From the above discussion we conclude that these are the only cases in which we can reconstruct well behaved stable field theoretic models. In the next sub-section we will analyze the unstable models (lumps).

### 3.2 Unstable models

In this case the lumps are obtained integrating the  $n = 1$  case of eq.(16)

$$\psi_1(x) = (1 - y)^{\alpha/2}(1 + y)^{\beta/2} ((\alpha + \beta + 2)y + \alpha - \beta) , \quad (28)$$

that is we have for the bounces

$$\phi_c(x) = \int^{\tanh(x)} \frac{(1 - y)^{\alpha/2}(1 + y)^{\beta/2} ((\alpha + \beta + 2)y + \alpha - \beta)}{1 - y^2} dy . \quad (29)$$

The field theoretic model are given by

$$U(\phi) = (1 - y)^\alpha(1 + y)^\beta ((\alpha + \beta + 2)y + \alpha - \beta)^2 . \quad (30)$$

It is possible to analytically solve the integral given by eq.(29) in the following cases

**3.2.a**  $\alpha = \beta = A$  : In this case  $B = 0$ . Consequently we have for  $\psi_1(x)$

$$\psi_1(x) = 2A \frac{\sinh(x)}{\cosh^A(x)} , \quad (31)$$

and the integral in eq.(29) can be easily performed. We obtain

$$\phi_c(x) = \frac{1}{\cosh^{A-1}(x)} . \quad (32)$$

Substituting the eq.(32) in eq.(30) we obtain the field theory models

$$U(\phi) = (A - 1)\phi^2(1 - \phi^{2/(A-1)}) . \quad (33)$$

For  $A = 2$  we have the unstable  $\phi^4$  theory. This model was used by Langer [20] as a field theoretic model for the study of the kinematics of first order phase transitions. Further, Coleman and Callan [21] extended the Langer's work to the related issue of the false vacuum decay in field theory [21]. For  $A = 3$  we obtain the unstable  $\phi^3$  model. This last model was used as a laboratory for computing the decay rate of a system trapped in a false vacuum [10]. Also, recently the  $\phi^3$  model was used as an exactly solvable toy model for tachyon condensation in string field theory [11]. For some values of  $A$  such that  $2/(A - 1)$  is fractionary the eq.(33) not describe a well behaved field theoretic model. Also we have to stress that we must assume  $A > 0$  to guarantee the normalizability of

$\psi_1(x)$ . In ref. [5] the authors considered the case  $A = 1$  and obtained the Liouville field theory model [22]. But in such case is easy to see that the classical solutions that meet or leave the unique asymptotic vacuum have an infinite energy, that is, such solutions are not lumps. Finally that we would stress that the density potential given by eq.(33) was also obtained in references [5] and [11].

**3.2.b  $\alpha \neq \beta$**  : In this case  $B \neq 0$  and the integral given by eq.(29) splited in two parts,

$$\phi_c(x) = (\beta+\alpha+2) \int^{\tanh(x)} \frac{(1-y)^{\alpha/2}(1+y)^{\beta/2}y}{1-y^2} dy + (\alpha-\beta) \int^{\tanh(x)} \frac{(1-y)^{\alpha/2}(1+y)^{\beta/2}}{1-y^2} dy . \quad (34)$$

The above integral can be performed for some particular values of  $\alpha, \beta$  but it is not possible to obtain a well behaved field theoretic model.

### 3.3 The $A \rightarrow \infty$ Limit

In ref. [23] the limit  $A \rightarrow \infty$  of eq.(33) was studied. It was obtained the unstable field theoretic model

$$U(\phi) = -\phi^2 \ln \phi^2 \quad (35)$$

with lumplike solution given by

$$\phi_c(x) = x \exp(-x^2/2) . \quad (36)$$

It is easy to see that in this case the Rosen-Morse potential given by eq. (15) reduces to one of the harmonic oscillator type. To perform the limit first one rescale the  $x$  coordinate that appear in eq.(15) as  $x/\sqrt{2(A-1)}$ . The authors considered first the case in which  $A$  is integer and then performed the  $A \rightarrow \infty$  limit. Recently it was considered [24] the case in which  $A$  is arbitrary, and was showed that the  $A$  infinite limit can be taken continuously. But as we showed in last subsection the eq.(33) does not define a well behaved theory for those values of  $A$  as for example  $2/(A-1) = 1/2$ . In other words one can not take the limit  $A \rightarrow \infty$  continuously.

Using the harmonic oscillator potential the authors of ref. [25] also considered the reconstruction of the stable field theoretic model. In this case it is not possible to obtain a closed expression of  $U(\phi)$  since to obtain the kink we have to integrate  $\exp(-x^2/2)$ . Unfortunately one can not express the result in terms of elementary funtions. Indeed it is given by a error funtion, that can not be inverted in terms of  $\phi_c$ . In order to circumvent this problem the authors of ref. [25] redefined the field to write an explicit form for the lagrangean of the field theoretic model. At this point an interesting question arises. We showed in the last section that it is not possible to reconstruct stable field theory models, with exception of the  $A = 1$  and  $A = 2$  cases. Is the model considered in ref. [25] a well behaved field theoretic model? Note that we analyzed only the cases of finite values of  $A$ . For  $A \rightarrow \infty$  our analysis is incomplete. The answer to the above question is afirmative. The argument for it is the following: in the models that we studied the singularities always appear in the perturbative vacua,  $\phi = \phi_0$ . We can solve for  $U(\phi)$  for any value of  $A$  around one of the trivial vacua as has been done in ref. [3]. The result is

$$U(\phi) \approx (\phi - \phi_0)^2 + \mathcal{O}\left((\phi - \phi_0)^{2+2/A}\right) . \quad (37)$$

In the limit in which  $A \rightarrow \infty$  we see that the singularity in  $\phi = \phi_0$  disappear, showing that in this limit a well behaved field theory model is obtained. It is important to point out the following. Redefining the field, the authors of ref. [25] obtained the following density lagrangean

$$\mathcal{L} \sim \left( \frac{1}{2} e^{-2T^2} \partial_\mu T \partial^\mu T - \frac{1}{8} e^{-2T^2} \right), \quad (38)$$

where  $T$  is the redefined field given by

$$T = \text{erf}^{-1}(\phi), \quad (39)$$

and the kink is given by

$$\phi_c(x) = \text{erf}(x/2). \quad (40)$$

Note that the lagrangean density written in terms of the new field  $T$  does not cover all the values of the field  $\phi$ , since the error function is a function with finite range. The inverse functions will be defined only for a finite domain, *i.e* for those values of  $\phi$  that lies between the trivial vacua. Consequently, although the  $A$  infinite limit defines a well behaved field theoretic model the density lagrangean given by eq.(38) does not describe the complete theory. For example it is not valid to perform the perturbative analysis of the model, although for the solitonic sector it is still useful.

## 4 The Morse potential: the $\frac{m^2}{8} \phi^2 \ln^2 \left( \frac{\alpha^2 \phi^2}{9m^4} \right)$ model.

The Morse potential is given by

$$V(x) = A^2 + B^2 \exp(-2x) - 2B(A + 1/2) \exp(-x). \quad (41)$$

In this case the eigenfunctions and eigenvalues of eq.(41) are given by

$$\psi_n(x) = y^{A-n} e^{-y/2} L_n^{2A-2n}(y), \quad y = 2Be^{-x}, \quad (42)$$

and

$$\omega_n^2 = A^2 - (A - n)^2, \quad n = 0, 1, 2, \dots \quad (43)$$

From eq.(42) we obtain the ground state eigenfunction  $\psi_0$

$$\psi_0(x) = y^A e^{-y/2}, \quad (44)$$

from which we can obtain the kink

$$\phi_c(x) = \int^{2Be^{-x}} y^{A-1} e^{-y/2} dy. \quad (45)$$

The stable field theory model is given by

$$U(\phi) = y^{2A} e^{-y}. \quad (46)$$

The integral given by eq.(45) can be performed only when  $A = 1, 2, \dots$ . For  $A = 1$  we obtain

$$\phi_c(x) = \exp(-Be^{-x}), \quad (47)$$

from which, solving for  $x$  and replacing in eq.(46) we obtain

$$U(\phi) = \phi^2 (\ln(\phi^2))^2 . \quad (48)$$

Although the integral given by eq.(45) can be done for  $A = 2, 3..$  it is not possible to invert it to obtain a well behaved field theory model. In this case it is not possible to reconstruct well behaved unstable field theory models.

The density potential given by eq.(48) can be redefined in such a way that will appear coupling constants in the model. Since we constructed this field theory model starting from a Schrodinger equation with one free parameter ( $A = 1$  and  $B$  arbitrary) and since we can rescale field and coordinates in the lagrangean (thus eliminating two coupling constants) we conclude that the density potential given by eq.(48) can be redefined with no more than three coupling constants. We redefine eq.(48) with two coupling constants as

$$U(\phi) = \frac{m^2}{8} \phi^2 \ln^2 \left( \frac{\alpha^2 \phi^2}{9m^4} \right) , \quad (49)$$

where we have choose the numerical factors adequately. The model given by eq.(49) have two degenerate vacua as can be see in fig. (2) at the points given by  $\pm\phi_0$ , where

$$\phi_0 = \frac{3m^2}{\alpha} . \quad (50)$$

The kinks and antikinks can be obtained easily, they are given by

$$\phi_c(x) = \pm \frac{3m^2}{\alpha} \exp(-e^{\pm m(x-x_0)}) . \quad (51)$$

We have two pair of kink antikink solutions that can link  $\phi = -\phi_0$  and  $\phi = 0$  or  $\phi = 0$  and  $\phi = \phi_0$ . The masses of these kinks (antikinks) solutions are the same and are given (classically) by

$$\begin{aligned} H[\phi_c] &= \int_{-\infty}^{\infty} dx \left( \frac{1}{2} \left( \frac{d\phi_c}{dx} \right)^2 + \frac{m^2}{8} \phi_c^2 \left[ \ln \left( \frac{\alpha^2 \phi_c^2}{9m^4} \right) \right]^2 \right) \\ &= \frac{9m^6}{2\alpha^2} \int_{-\infty}^{\infty} dx e^{\pm mx} \exp(-2e^{\pm mx}) \\ &= \frac{9m^5}{4\alpha^2} . \end{aligned} \quad (52)$$

Let us briefly develop how to find the quantum corrections for the soliton mass. As we discussed this quantity is given by eq.(8). To define the soliton mass we have to substract from eq.(8) the energy fluctuations around the asymptotic limits of the solitonic solution, that is, around the perturbative vacua  $\phi = \pm\phi_0$ . This subtraction only modify the quantum corrections since in the perturbative vacuum  $H[\pm\phi_0]$  vanishes. With this modification we have for the mass of the kinks

$$M = H[\phi_c] + \sum_n \omega_n - \sum_q \omega_0(q) . \quad (53)$$

In eq.(53) the  $\omega_n$ 's satisfy eq.(6), that in the present case is

$$\left[ -\frac{d^2}{dx^2} + m^2(e^{\pm 2mx} - 3e^{\pm mx} + 1) \right] \psi_n(x) = \omega_n^2 \psi_n(x), \quad (54)$$

where  $\pm$  signs correspond respectively to the kink and antikink configurations. On the other hand  $\omega_0(q)$  satisfy

$$\left[ -\frac{d^2}{dx^2} + V(x) \right] \psi_n(x) = \omega_n^2 \psi_n(x) \quad (55)$$

with  $V(x)$  given by the asymptotic behavior of the potential in eq. (54) at  $x \rightarrow \pm\infty$ . If we kept the  $(-)$  sign in eq.(54) then  $V(x)$  is given by

$$V(x) = \begin{cases} \infty & x \leq 0 \\ m^2 & x > 0 \end{cases}$$

If we kept the  $(+)$  sign the form of  $V(x)$  is reverted. But in any case the quantum mass correction for the kinks or antikinks will be the same. If in eq.(54) we make  $z = mx$  we obtain

$$\left[ -\frac{d^2}{dz^2} + e^{\pm 2z} - 3e^{\pm z} + 1 \right] \psi_n(z) = \delta_n \psi_n(z), \quad (56)$$

with  $\delta_n = \omega_n/m^2$ . Note that eq.(56) is the  $A = B = 1$  case of the Morse potential as expected by construction.

The quantum mass correction for the soliton as given by eq.(53) is in general divergent. In order to renormalize it we have to make a redefinition of the parameters of the theory. To carry such task first we expand  $U(\phi)$  around one of the perturbative vacua,

$$U(\varphi) = \frac{1}{2}m^2\varphi^2 \pm \frac{\alpha}{6}\varphi^3 - \frac{1}{216}\varphi^4 + \frac{1}{14580}\frac{\alpha^4}{m^6}\varphi^6 + \dots \quad (57)$$

where we have shifted the field as  $\phi = \varphi \pm \phi_0$ . Since we are in the semi-classical approximation we will renormalize the mass  $m$  and the coupling constant to one loop order. Details of the renormalization procedure and the calculation of the quantum correction to the soliton mass in this model will be present elsewhere [26].

## 5 The Scarf II Hyperbolic: the $\frac{m^2}{8}\phi^2 \cos^2 \left[ \ln \left( \frac{\alpha^2 \phi^2}{9m^4} \right) \right]$ model.

The Scarf II hyperbolic potential is given by

$$V(x) = A^2 + (B^2 - A^2 - A)/\cosh^2(x) + B(2A + 1)\tanh(x)/\cosh(x). \quad (58)$$

In this case the eigenfunctions and eigenvalues of this potential are given by

$$\psi_n(x) = (i)^n (1 + y^2)^{-A/2} e^{-B \tan^{-1}(y)} P_n^{(iB-A-1/2, -iB-A-1/2)}(y), \quad y = \sinh(x), \quad (59)$$

and

$$\omega_n^2 = A^2 - (A - n)^2, \quad n = 0, 1, 2, \dots \quad (60)$$

The kink like solutions are obtained from the zero mode given by

$$\psi_0(x) = (1 + y^2)^{-A/2} e^{-B \tan^{-1}(y)}, \quad (61)$$

and the field theory model is given by

$$U(\phi) = (1 + y^2)^{-A} e^{-2B \tan^{-1}(y)}. \quad (62)$$

Integrating the eq.(61) we obtain for  $\phi_c$

$$\phi_c(x) = \int^{\sinh(x)} (1 + y^2)^{-(A+1)/2} e^{-B \tan^{-1}(y)} dy. \quad (63)$$

The above integral can be performed analytically only when  $A = 1$ . We have

$$\phi_c(x) = e^{-B \tan^{-1} \sinh(x)}, \quad (64)$$

then solving for  $y = \sinh(x)$  and replacing in eq.(62) we obtain

$$U(\phi) = \phi^2 \cos^2\left(\frac{1}{2B} \ln \phi^2\right). \quad (65)$$

As in the case of the Morse potential in this case we not have unstable field theory models.

We can redefine the potential given by eq.(65) with adequate coupling constants (we consider the most simple  $B = 1/2$  case)

$$U(\phi) = \frac{m^2}{8} \phi^2 \cos^2 \left[ \ln \left( \frac{\alpha^2 \phi^2}{9m^4} \right) \right]. \quad (66)$$

In fig. (3) we have plotted this density potential for  $\phi > 0$ . It has infinitely degenerate trivial vacua at the points  $\phi = \pm \phi_n$  with  $\phi_n$  given by

$$\phi_n = \frac{3m^2}{\alpha} \exp \left( \frac{2n + 1}{4} \pi \right), \quad n = 0, \pm 1, \pm 2, \dots \quad (67)$$

The kinks and antikinks are obtained using eq.(2)

$$\phi_c(x) = \pm \frac{3m^2}{\alpha} \exp \left( \frac{n\pi}{2} \pm \frac{1}{2} \tan^{-1}(\sinh(mx)) \right), \quad n = 0, \pm 1, \pm 2, \dots \quad (68)$$

where the solutions with  $(\pm)$  signs in the exponents correspond to the kinks anti kinks solutions respectively for each value of  $n$  and for each sign that appear in front. We have an infinite number os kinks and anti-kinks that links the infinite number of trivial vacua. This makes remember in some sense the sine-Gordon model. But contrary to sine-Gordon model where all the solitonic sectors describe the same physics in our present case it is not

the case. For example if we compute the the classical masses for the kinks (or anti-kinks) we obtain

$$\begin{aligned}
 H[\phi_c] &= \int_{-\infty}^{\infty} \left( \frac{1}{2} \left( \frac{d\phi_c}{dx} \right)^2 + \frac{m^2}{8} \cos^2 \left[ \ln \left( \frac{\alpha^2 \phi^2}{9m^4} \right) \right] \right) dx + \\
 &= \frac{9m^5 e^{2n\pi}}{4\alpha^2} \int_{-1}^1 ds e^{\sin^{-1}(s)} \\
 &= 5.65 \frac{m^5 e^{2n\pi}}{\alpha^2}, \tag{69}
 \end{aligned}$$

where we see clearly that the masses are different. The quantum mass corrections are given by the last two terms of eq.(53) where now the  $\omega_n$ 's satisfy

$$\left( -\frac{d^2}{dx^2} + m^2 - (7/4)m^2/\cosh^2(mx) + (3/2)\tanh(x)/\cosh(x) \right) \psi_n(x) = \omega_n^2 \psi_n(x) \tag{70}$$

and the  $\omega_0(q)$ 's satisfy the eq.(70) at asymptotic values, that is, an equation with constant potential equal to  $m^2$ . As usual the quantum mass correction for the soliton mass is divergent and in order to renormalize it we have to expand eq.(66) around one of the perturbative vacua

$$U(\varphi) = \frac{m^2}{2} \varphi^2 \pm \frac{\alpha}{6} e^{-(2n+1)\pi/4} \varphi^3 - \frac{17}{9(4!)} \frac{\alpha^2}{m^2} e^{-(2n+1)\pi/2} \varphi^4 + \frac{340}{6!81} \frac{\alpha^4}{m^6} e^{-5(2n+1)\pi/4} \varphi^6 + \dots \tag{71}$$

Where we have redefined  $\phi = (\varphi \pm \phi_n)$  with  $\phi_n$  given by eq.(67). We can see in eq.(71) that when  $n \rightarrow \infty$  the theory becomes free in the perturbative sector, although this fact does not happens in the solitonic sector [26].

## 6 Conclusions

Using the zero mode solution of the Schrodinger type equation for kinks (bounces) we obtain the kinks (bounces) as well the bidimensional scalar field theory models that support such kinks (bounces). Because we start with solvable Schrodinger equations it is automatically guaranteed the computation of the first quantum corrections around the kinks (bounces). We obtained two new models starting from the Morse and Scarf II hyperbolic potentials, *i.e.*, the  $U(\phi) = \phi^2 \ln^2(\phi^2)$  and  $U(\phi) = \phi^2 \cos^2(\ln(\phi^2))$ . The quantum corrections to the solitonic sectors will be presented elsewhere.

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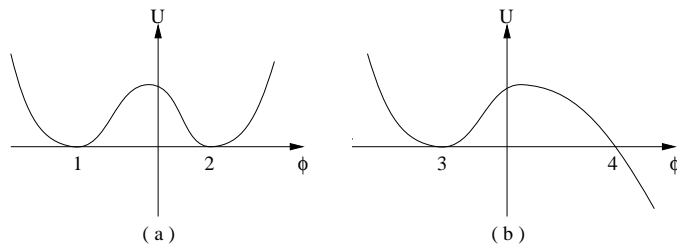


Figure 1: (a)  $U(\phi)$  with two degenerate vacua and (b) with a false vacuum.

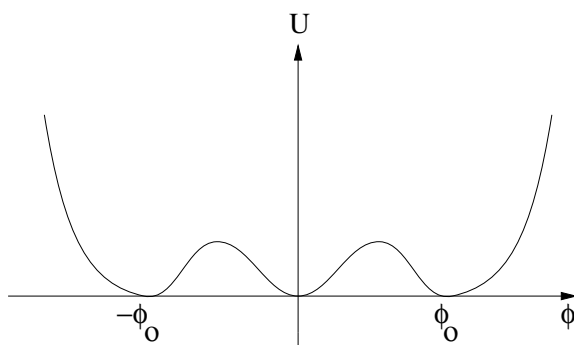


Figure 2: The density potential  $U(\phi)$  given by eq.(49).

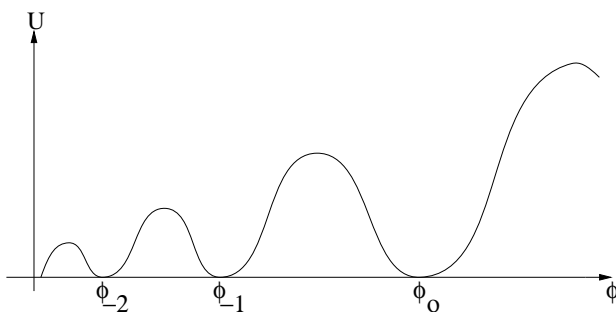


Figure 3: The density potential  $U(\phi)$  given by eq.(66).