ISSN 0029-3865

CBPF-NF-012/90

ON THE WEIGHTED APPROXIMATION OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

by

Leopoldo NACHBIN

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

Department of Mathematics
University of Rochester
Rochester, NY 14627 USA

Abstract. This note is an improvement of the available methods for getting results on the weighted approximation of continuously differentiable functions.

Keywords and phrases: weighted approximation, fundamental weight, analytic weight, quasi-analytic weight.

We shall present what we believe to be a simplified version of the reasoning to establish some results concerning weighted approximation of continuously differentiable scalar functions on $\mathbf{R}^n$ (see Zapata [3]). For references to weighted approximation of continuous scalar functions on $\mathbf{R}^n$ see Horvath[1] and Nachbin [3]. Lemma 1 reduces the search for sufficient conditions in order that a weight on $\mathbf{R}$ be C^m-fundamental to the finding of sufficient conditions for a weight on $\mathbf{R}$ to be C- fundamental. Similarly, Lemma 2 reduces the finding of sufficient conditions for a weight on $\mathbf{R}^n$ to be C^m-fundamental to the search of sufficient conditions for a weight on $\mathbf{R}$ to be C^m-fundamental. We then apply Lemmas 1 and 2 together to obtain Propositions 2, 4 and 5.

Fix integers $n \in \mathbf{N}$, $n \geq 1$ and $m \in \mathbf{N}$ (the case $m = \infty$ will be excluded since it follows easily from all $m \in \mathbf{N}$). Let $\mathbf{K}$ denote either $\mathbf{R}$ or $\mathbf{C}$. Consider the algebras $\mathcal{P}(\mathbf{R}^n)$ of all $\mathbf{K}$-valued polynomials on $\mathbf{R}^n$ and $C^m(\mathbf{R}^n)$ of all continuously m-differentiable $\mathbf{K}$-valued functions on $\mathbf{R}^n$. Write $\mathbf{N}^n_m = \{\alpha \in \mathbf{N}^n \; ; |\alpha| \leq m\}$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$. Let $D^\alpha f$ be the α-th partial derivative of $f \in C^m(\mathbf{R}^n)$ for $\alpha \in \mathbf{N}^n_m$. Denote by $\mathcal{D}^m(\mathbf{R}^n)$ the subalgebra of $C^m(\mathbf{R}^n)$ of all functions with compact support.

A C^m-weight on $\mathbf{R}^n$ is a family $v = (v_\alpha; \alpha \in \mathbf{N}^n_m)$ of upper semicontinuous functions $v_\alpha \geq 0$ on $\mathbf{R}^n$. Such a weight v defines the vector space $C^m v_\infty(\mathbf{R}^n)$ of all $f \in C^m(\mathbf{R}^n)$ such that $v_\alpha D^\alpha f$ tends to zero at infinity for every $\alpha \in \mathbf{N}^n_m$. Set $\|f\|_{v_\alpha} = \sup\{v_\alpha(t) \cdot |D^\alpha f(t)|; \, t \in \mathbf{R}^n\}$ to get a seminorm $f \in C^m v_\infty(\mathbf{R}^n) \mapsto \|f\|_{v_\alpha} \in \mathbf{R}_+$ for every $\alpha \in \mathbf{N}^n_m$. The finite family of such seminorms makes $C^m v_\infty(\mathbf{R}^n)$ into a seminormable space, actually a seminormed space if for instance we use the seminorm $f \in C^m v_\infty(\mathbf{R}^n) \mapsto \|f\|_v = \sup\{\|f\|_{v_\alpha}; \alpha \in \mathbf{N}^n_m\} = \sup\{v_\alpha(t) \cdot |D^\alpha f(t)|; t \in \mathbf{R}^n, \alpha \in \mathbf{N}^n_m\} \in \mathbf{R}_+$. We now have that $\mathcal{D}^m(\mathbf{R}^n) \subset C^m v_\infty(\mathbf{R}^n)$. It is known that $\mathcal{D}^m(\mathbf{R}^n)$ is dense in $C^m v_\infty(\mathbf{R}^n)$ if the weight v is decreasing in the sense that for every α, $\beta \in \mathbf{N}^n_m$ with $\beta \leq \alpha$ there exists $C_{\alpha\beta} \geq 0$ such that $v_\alpha \leq C_{\alpha\beta} v_\beta$ (see Lemma 1 in Zapata [3]). $\mathcal{D}^m(\mathbf{R}^n)$ may be dense in $C^m v_\infty(\mathbf{R}^n)$ even if v is not decreasing, and it may fail to be dense as well. The weight v is said to be rapidly decreasing if $\mathcal{P}(\mathbf{R}^n) \subset C^m v_\infty(\mathbf{R}^n)$. If, moreover, $\mathcal{P}(\mathbf{R}^n)$ is dense in $C^m v_\infty(\mathbf{R}^n)$ then v is called C^m- fundamental.

LEMMA 1. *Let $m \in \mathbf{N}$, $v_i \geq 0 (i = 0, \ldots, m)$ and $u \geq 0$ be upper semicontinuous on $\mathbf{R}$. Consider the C^m-weight $v = (v_0, \ldots, v_m)$ on $\mathbf{R}$ which is assumed to be decreasing. Let u be C-fundamental on $\mathbf{R}$, such that $s, t \in \mathbf{R}$, $|s| \leq |t|$ imply $u(t) \leq u(s)$, and that $|t|^{m-i} v_i(t) \leq u(t)$, $v_i(t) \leq u(t)$ for $t \in \mathbf{R}$, $i = 0, \ldots, m$. Then v is a C^m-fundamental weight on $\mathbf{R}$.*

PROOF: The lemma is true if $m = 0$ because then $v_0 \leq u$ and hence v_0 is C-fundamental along with u. Assume now that $m \geq 1$ and that the lemma is true for $m - 1$. Notice that $\mathcal{P}(\mathbf{R}) \subset C^m v_\infty(\mathbf{R})$ because $\mathcal{P}(\mathbf{R}) \subset C u_\infty(\mathbf{R})$ and $v_i \leq u (i = 0, \ldots, m)$. Hence v is rapidly decreasing. Fix any $f \in \mathcal{D}^m(\mathbf{R})$ and $\epsilon > 0$. Then $f^{(m)} \in \mathcal{D}(\mathbf{R}) \subset C u_\infty(\mathbf{R})$ and there is $P^{(m)} \in \mathcal{P}(\mathbf{R})$ so that

$$(1) \qquad u(t) \cdot |P^{(m)}(t) - f^{(m)}(t)| \leq \epsilon \quad (t \in \mathbf{R}) \, .$$

Choose $P \in \mathcal{P}(\mathbf{R})$ whose m-th derivative is $P^{(m)}$ such that $P^{(i)}(0) = f^{(i)}(0) (i = 0, \ldots, m - 1)$. We claim that

$$(2) \qquad v_i(t) \cdot |P^{(i)}(t) - f^{(i)}(t)| \leq \epsilon \quad (t \in \mathbf{R}, i = 0, \ldots, m) \, .$$

In fact, for $i = m$ this follows from (1) and $v_m \leq u$. For every $t \in \mathbf{R}$, $i = 0, \ldots, m - 1$ there is $s \in \mathbf{R}$, $|s| \leq |t|$ such that $|P^{(i)}(t) - f^{(i)}(t)| \leq |t|^{m-i} \cdot |P^{(m)}(s) - f^{(m)}(s)|$ by the mean value theorem applied $m - i$ times. Notice that $|t|^{m-i} v_i(t) \leq u(t) \leq u(s)$. Hence (1) with s in place of t shows that (2) is true for $i = 0, \ldots, m - 1$ as well as for $i = m$. It follows that $\mathcal{D}^m(\mathbf{R})$ is contained in the closure of $\mathcal{P}(\mathbf{R})$ in $C^m v_\infty(\mathbf{R})$. Therefore, $\mathcal{P}(\mathbf{R})$ is dense in $C^m v_\infty(\mathbf{R})$ along with $\mathcal{D}^m(\mathbf{R})$. Thus v is C^m-fundamental. ∎

Let $v \geq 0$ be upper semicontinuous on $\mathbf{R}$. We say that v is an analytic weight on $\mathbf{R}$ when there exist $C > 0$, $c > 0$ such that $v(t) \leq Ce^{-c|t|} (t \in \mathbf{R})$. It is then known that v is a C- fundamental weight on $\mathbf{R}$ (see Lemma 2, §28 in Nachbin [2]). More generally, if $v \geq 0$ is upper semicontinuous on $\mathbf{R}^n$, we say that v is an analytic weight on $\mathbf{R}^n$ when there exist $C > 0, c > 0$ such that $v(t) \leq Ce^{-c(|t_1|) + \cdots + |t_n|)} (t \in \mathbf{R}^n)$. It is then known that v is C-fundamental on $\mathbf{R}^n$.

Proposition 2. *Let $m \in \mathbb{N}$, $v_i \geq 0$ $(i = 0, \ldots, m)$ be upper semicontinuous on $\mathbb{R}$. Consider the C^m-weight $v = (v_0, \ldots, v_m)$ on $\mathbb{R}$ which is assumed to be decreasing. Let each $v_i (i = 0, \ldots, m)$ be an analytic weight. Then v is a C^m-fundamental weight on $\mathbb{R}$.*

Proof: Assume that $v_i(t) \leq Ce^{-c|t|}(t \in \mathbb{R}, i = 0, \ldots, m)$ for some $C > 0, c > 0$. Choose $D > 0$, $0 < d < c$ so that, if $u(t) = De^{-d|t|}(t \in \mathbb{R})$, all assumptions in Lemma 1 are satisfied. ∎

Lemma 3. *Let $n \in \mathbb{N}$, $n \geq 1, m \in \mathbb{N}, v_\alpha \geq 0(\alpha \in \mathbb{N}^n_m)$ be upper semicontinuous on $\mathbb{R}$. Consider the C^m- weight $v = (v_\alpha; \alpha \in \mathbb{N}^n_m)$ on $\mathbb{R}^n$ which is assumed to be decreasing. Let $u_{ij} \geq 0 (i = 1, \ldots, n, j = 0, \ldots, m)$ be upper semicontinuous on $\mathbb{R}$. Consider the C^m-weights $u_i = (u_{i0}, \ldots, u_{im})$ $(i = 1, \ldots, n)$ on $\mathbb{R}$ which are supposed to be decreasing and C^m-fundamental. Assume*

$$v_\alpha(t) \leq u_{1\alpha_1}(t_1) \cdots u_{n\alpha_n}(t_n)(t \in \mathbb{R}^n, \alpha \in \mathbb{N}^n_m) .$$

Then v is C^m-fundamental on $\mathbb{R}^n$.

Proof: Consider the n-linear mapping π that with every $(f_1, \cdots, f_n) \in C^m(u_1)_\infty(\mathbb{R}) \times \cdots \times C^m(u_n)_\infty(\mathbb{R})$ associates $f_1 \otimes \cdots \otimes f_n \in C^m v_\infty(\mathbb{R}^n)$ where $(f_1 \otimes \cdots \otimes f_n)(t) = f_1(t_1) \cdots f_n(f_n)$ for $t \in \mathbb{R}^n$. The assumptions make it sure that π is well defined and continuous because

$$\|f_1 \otimes \cdots \otimes f_n\|_{v_\alpha} \leq \|f_1\|_{u_{1\alpha_1}} \cdots \|f_n\|_{u_{n\alpha_n}} .$$

Since $\mathcal{P}(\mathbb{R})$ is dense in $C^m(u_i)_\infty(\mathbb{R})$ by hypothesis, then $\mathcal{D}^m(\mathbb{R})$ is contained in the closure of $\mathcal{P}(\mathbb{R})$ in $C^m(u_i)_\infty(\mathbb{R})(i = 1, \ldots, n)$. Therefore $\pi[\mathcal{D}^m(\mathbb{R}) \times \cdots \times \mathcal{D}^m(\mathbb{R})]$ is contained in the closure of $\pi[\mathcal{P}(\mathbb{R}) \times \cdots \times \mathcal{P}(\mathbb{R})]$ in $C^m v_\infty(\mathbb{R}^n)$. Hence the vector subspace $\mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R})$ generated by $\pi[\mathcal{D}^m(\mathbb{R}) \times \cdots \times \mathcal{D}^m(\mathbb{R})]$ is contained in the closure in $C^m v_\infty(\mathbb{R}^n)$ of the vector subspace $\mathcal{P}(\mathbb{R}^n) = \mathcal{P}(\mathbb{R}) \otimes \cdots \otimes \mathcal{P}(\mathbb{R})$ generated by $\pi[\mathcal{P}(\mathbb{R}) \times \cdots \times \mathcal{P}(\mathbb{R})]$. It is known that $\mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R})$ is dense in $\mathcal{D}^m(\mathbb{R}^n)$ in the natural inductive limit topology of $\mathcal{D}^m(\mathbb{R}^n)$, hence in the coarser topology on $D^m(\mathbb{R}^n)$ defined by the norm $f \in D^m(\mathbb{R}^n) \mapsto \sup \{|D^\alpha f(t)|; t \in \mathbb{R}^n, |\alpha| \leq m\} \in \mathbb{R}_+$, hence (because all v_α are upper bounded along with all u_{ij}) in the even coarser topology that the natural topology of $C^m v_\infty(\mathbb{R}^n)$ induces on $\mathcal{D}^m(\mathbb{R}^n)$. Since $\mathcal{D}^m(\mathbb{R}^n)$ is dense in $C^m v_\infty(\mathbb{R}^n)$, then $\mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R})$ is dense in $C^m v_\infty(\mathbb{R}^n)$. The fact that $\mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R})$

is contained in the closure of $\mathcal{P}(\mathbf{R}^n)$ in $C^m v_\infty(\mathbf{R}^n)$ implies then that $\mathcal{P}(\mathbf{R}^n)$ is dense in $C^m v_\infty(\mathbf{R}^n)$. Thus v is C^m-fundamental. ∎

PROPOSITION 4. *Let $n \in \mathbf{N}$, $n \geq 1$, $m \in \mathbf{N}$, $v_\alpha \geq 0(\alpha \in \mathbf{N}^n_m)$ be upper semicontinuous on $\mathbf{R}^n$. Consider the C^m-weight $v = (v_\alpha; \alpha \in \mathbf{N}^n_m)$ on $\mathbf{R}^n$ which is assumed to be decreasing. Assume that each $v_\alpha(\alpha \in \mathbf{N}^n_m)$ is an analytic weight. Then v is C^m-fundamental on $\mathbf{R}^n$.*

PROOF: Let $v_\alpha(t) \leq Ce^{-c(|t_1|+\cdots+|t_n|)}(t \in \mathbf{R}^n, \alpha \in \mathbf{N}^n_m)$ for suitable $C > 0, c > 0$. Choose $u_i j(t_i) = C^{1/n}e^{-c|t_i|}(i = 1, \cdots, n, j = 1, \cdots, m)$ so that all assumptions in Lemma 3 are satisfied. ∎

Let $v \geq 0$ be upper semicontinuous on $\mathbf{R}$ and rapidly decreasing. Put $M_m = \sup\ \{|t^m| \cdot v(t); t \in \mathbf{R}\} \in \mathbf{R}_+(m = 0, 1, \cdots)$. We say that v is a quasi-analytic weight on $\mathbf{R}$ when $\Sigma^\infty_{m=1} 1/\sqrt[m]{M_m} = +\infty$. It is then known that v is a C-fundamental weight on $\mathbf{R}$, and that every analytic weight on $\mathbf{R}$ is quasi-analytic (see Lemma 2, §29 in Nachbin [2]). More generally, if $v \geq 0$ is upper semicontinuous on $\mathbf{R}^n$, we say that v is a quasi-analytic weight on $\mathbf{R}^n$ if there are quasi-analytic weights $v_1, \cdots, v_n$ on $\mathbf{R}$ such that $v(t) \leq v_1(t_1) \cdots v_n(t_n)(t \in \mathbf{R}^n)$. It is then known that v is C-fundamental on $\mathbf{R}^n$, and that every analytic weight on $\mathbf{R}^n$ is quasi-analytic.

PROPOSITION 5. *Let $n \in \mathbf{N}, n \geq 1, m \in \mathbf{N}, v_\alpha \geq 0(\alpha \in \mathbf{N}^n_m)$ be upper semicontinuous on $\mathbf{R}^n$. Consider the C^m-weight $v = (v_\alpha; \alpha \in \mathbf{N}^n_m)$ which is supposed to be decreasing. Assume that there are quasi-analytic weights $v_i(i = 1, \cdots, n)$ on $\mathbf{R}$ such that $v_\alpha(t) \leq v_1(t_1) \cdots v_n(t_n)(t \in \mathbf{R}^n, \alpha \in \mathbf{N}^n_m)$. Then v is a C^m-fundamental weight on $\mathbf{R}^m$.*

PROOF: All assumptions of Lemma 3 apply. ∎

LEMMA 6. *The linear mapping $D : f \in \mathcal{D}^1(\mathbf{R}) \mapsto f' \in \mathcal{K}(\mathbf{R}) = \mathcal{D}^0(\mathbf{R})$ is injective. Its image is $D\mathcal{D}^1(\mathbf{R}) = \{g \in \mathcal{K}(\mathbf{R}); \int g = 0\}$. If $u \geq 0$ is upper semicontinuous on $\mathbf{R}$, this image is dense in $Cu_\infty(\mathbf{R})$ if and only if $\int 1/u = +\infty$.*

PROOF: Only the final part of the lemma requires a proof. Consider the linear form $I : f \in \mathcal{K}(\mathbf{R}) \mapsto \int f \in \mathbf{K}$. Assume $\int 1/u = +\infty$. We claim that I is not continous on $\mathcal{K}(\mathbf{R})$ for the seminorm induced by $Cu_\infty(\mathbf{R})$. In fact given any $c \geq 0$ there is $f \in \mathbf{K}(\mathbf{R})$ such that $0 \leq f \leq 1/u$, $\int f \geq c$. Therefore $\| f \|_u \leq 1$ and $I(f) \geq c$ show that I is not continuous on $\mathcal{K}(\mathbf{R})$ for the seminorm induced by $Cu_\infty(\mathbf{R})$. It follows that $I^{-1}(0) = D\mathcal{D}^1(\mathbf{R})$ is dense in $Cu_\infty(\mathbf{R})$. Conversely let $c = \int 1/u < +\infty$. Then the set where u vanishes has a void interior. It follows that the seminorm on $Cu_\infty(\mathbf{R})$ is actually a norm. We claim that $|I(f)| \leq c.\, \| f \|_u$ for $f \in \mathcal{K}(\mathbf{R})$. This is clear if $\| f \|_u = 0$. If $\| f \|_u > 0$ then $u(t)|f(t)| \leq \| f \|_u$ implies $|f(t)| \leq \| f \|_u /u(t)$ for $t \in \mathbf{R}$, hence $|I(f)| \leq c \| f \|_u$ for $f \in \mathcal{K}(\mathbf{R})$ as asserted. Thus I is continuous on $\mathcal{K}(\mathbf{R})$ for the norm induced by $Cu_\infty(\mathbf{R})$ and I extends uniquely to a continuous linear form I on $Cu_\infty(\mathbf{R})$ since $\mathcal{K}(\mathbf{R})$ is dense in $Cu_\infty(\mathbf{R})$. We know that $I \neq 0$ because I does not vanish on $\mathcal{K}(\mathbf{R})$, but that I vanishes on $D\mathcal{D}^1(\mathbf{R})$. It follows that $D\mathcal{D}^1(\mathbf{R})$ is not dense in $Cu_\infty(\mathbf{R})$. ∎

EXAMPLE 7. *Consider $v_0 = 0, v_1 \geq 0$ upper semicontinuous on $\mathbf{R}$, and the C^1 - weight $v = (v_0, v_1)$ on $\mathbf{R}$. (Notice that v is not decreasing unless $v_1 = 0$.) Then $\mathcal{D}^1(\mathbf{R})$ is dense in $C^1v_\infty(\mathbf{R})$ if and only if $\int 1/v_1 = +\infty$.*

PROOF: Since $v_0 = 0$ we see that $\mathcal{D}^1(\mathbf{R})$ is dense in $C^1v_\infty(\mathbf{R})$ if and only if $D\mathcal{D}^1(\mathbf{R})$ is dense in $C(v_1)_\infty(\mathbf{R})$. There remains to apply Lemma 6 with v_1 in place of u. ∎

REFERENCES

[1] HORVATH, J., "Approximacion y funciones casi- analiticas"' Spain.

[2] NACHBIN, L., "Elements of approximation theory", USA.

[3] ZAPATA, Q., "Bernstein approximation problem for differentiable functions and quasi-analytic weights", Transactions of the American Mathematical Society 182 (1973), 503-509.