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THE CONNECTION BETWEEN GENERAL OBSERVERS AND LANCZOS POTENTIAL

by

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ABSTRACT

We present some algorithms to find the explicit form of Lanczos potential in an arbitrary geometry.

Key-words: Lanczos potential; Gravitation.

1 INTRODUCTION

In the early sixties Cornelius Lanczos[1] made the remark that in any Riemannian geometry the Weyl conformal tensor, $W_{\alpha\beta\mu\nu}$ [that is the traceless part of the curvature tensor $R_{\alpha\beta\mu\nu}$] can be written as first derivatives of a third order potential $L_{\alpha\beta\mu}$. All tentatives to generalize this result to the curvature tensor failed. Although research on the Weyl tensor became very important in gravitational theory, the same did not occur with Lanczos potential. There are two main reasons for this. The first one, of more general character, was just due to the suspicion [linked to the particular demonstration used by Lanczos] of the non-existence of $L_{\alpha\beta\mu}$ in every Riemannian geometry. Lanczos used a variational principle to obtain Bianchi's identities and in this way, the potential $L_{\alpha\beta\mu}$ appeared as Lagrange multipliers. There remained some doubts on the generality of this procedure. Twenty years after the first Lanczos paper on this subject, Bampi and Caviglia[2] gave a completely new proof. However, both demonstrations were not able to provide an algorithm which could be used to obtain the form of $L_{\alpha\beta\mu}$ in a given geometry. This indeed is the second main reason which made Lanczos tensor be so seldom employed until nowadays. The purpose of the present paper is precisely to remedy this situation searching for general methods to obtain Lanczos potential for an arbitrary Riemannian geometry.

2 NOTATION AND SOME USEFUL FORMULAS

We denote by the symbol; the covariant derivative in the four-dimensional riemannian space time (ST).

Antisymmetrization: $A_{[\mu\nu]} \equiv A_{\mu\nu} - A_{\nu\mu}$

Symmetrization : $A_{(\mu\nu)} \equiv A_{\mu\nu} + A_{\nu\mu}$

Weyl conformal tensor ($W_{\alpha\beta\mu\nu}$), the curvature tensor ($R_{\alpha\beta\mu\nu}$) and the contracted tensor ($R_{\mu\nu}$) are related by the formula:

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - M_{\alpha\beta\mu\nu} + \frac{1}{6} R\, g_{\alpha\beta\mu\nu} \tag{2.1}$$

in which

$$g_{\alpha\beta\mu\nu} = g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}$$

$$M_{\alpha\beta\mu\nu} = \frac{1}{2} [R_{\alpha\mu} g_{\beta\nu} + R_{\beta\nu} g_{\alpha\mu} - R_{\alpha\nu} g_{\beta\mu} - R_{\beta\mu} g_{\alpha\nu}]$$

The dual operators, represented by an asterisk * is defined by

$$A^{*}_{\mu\nu} = \frac{1}{2} \eta_{\mu\nu}{}^{\rho\sigma} A_{\rho\sigma}$$

for any anti-symmetric tensor. $\eta_{\mu\nu\rho\sigma}$ is the Levi-Civita completely antisymmetric object.

We have

$$\overset{*}{g}_{\alpha\beta\mu\nu} \equiv g_{\overset{*}{\alpha\beta}\mu\nu} = g_{\alpha\beta\overset{*}{\mu\nu}} = \eta_{\alpha\beta\mu\nu}$$

$$\overset{*}{\eta}_{\alpha\beta\mu\nu} = -g_{\alpha\beta\mu\nu}$$

The symbol $g_{\alpha\beta\mu\nu}$ is a sort of metric for bi-vectors, in the sense that, for any bi-vector $A_{\mu\nu} = -A_{\nu\mu}$ we have

$$g_{\mu\nu\alpha\beta}\, A^{\alpha\beta} = 2A_{\mu\nu} \quad .$$

The electric ($E_{\mu\nu}$) and the magnetic ($H_{\mu\nu}$) parts of the Weyl tensor are defined for an arbitrary time-like vector by

$$E_{\mu\nu} = -W_{\mu\alpha\nu\beta}\, V^\alpha V^\beta \tag{2.2a}$$

$$H_{\mu\nu} = -\overset{*}{W}_{\mu\alpha\nu\beta}\, V^\alpha V^\beta \tag{2.2b}$$

Due to the fact that the Weyl tensor is trace-less we can show that

$$\overset{*}{W}_{\alpha\beta\mu\nu} \equiv W_{\overset{*}{\alpha}\beta\mu\nu} = W_{\alpha\beta\mu\overset{*}{\nu}}$$

The ten degrees of freedom of $W_{\alpha\beta\mu\nu}$ are equally distributed among its electric and magnetic parts. Indeed, (2.2) implies that

$$\begin{aligned} E_{\mu\nu} &= E_{\nu\mu} \\ E_{\mu\nu}V^\mu &= 0 \\ E_\mu{}^\mu &= 0 \\ H_{\mu\nu} &= H_{\nu\mu} \\ H_{\mu\nu}V^\mu &= 0 \\ H_\mu{}^\mu &= 0 \end{aligned} \tag{2.3}$$

We define the projector tensor $h_{\mu\nu}$:

$$h_{\mu\nu} = g_{\mu\nu} - V_\mu V_\nu$$

which projects any tensor in the 3-dimensional rest-space of the observer V^{μ}, with the properties

$$h_{\mu}{}^{\lambda} h_{\lambda\nu} = h_{\mu\nu}$$
$$h_{\mu\nu} = h_{\nu\mu}$$
$$h_{\mu\nu} g^{\mu\nu} = 3$$

The covariant derivative of V^{μ} is separated in its irreducible parts by the expression

$$V_{\mu;\nu} = \sigma_{\mu\nu} + \frac{\theta}{3} h_{\mu\nu} + \omega_{\mu\nu} + a_{\mu} V_{\nu} \tag{2.4}$$

in which the symmetric shear $\sigma_{\mu\nu}$ is given by

$$\sigma_{\mu\nu} = \frac{1}{2} h_{(\mu}{}^{\alpha} h_{\nu)}{}^{\beta} V_{\alpha;\beta} - \frac{1}{3} \theta h_{\mu\nu} \tag{2.5a}$$

the antisymmetric vorticity $\omega_{\mu\nu}$ by:

$$\omega_{\mu\nu} = \frac{1}{2} h_{[\mu}{}^{\alpha} h_{\nu]}{}^{\beta} V_{\alpha;\beta} \tag{2.5b}$$

the expansion θ:

$$\theta = V^{\mu}{}_{;\mu} \tag{2.5c}$$

and the acceleration a_{μ} :

$$a_{\mu} = V_{\mu;\lambda} V^{\lambda} \tag{2.5d}$$

with the properties:

$$\begin{aligned} \sigma_{\mu\nu}V^{\mu} &= 0 \\ \omega_{\mu\nu}V^{\mu} &= 0 \\ a_{\mu}V^{\mu} &= 0 \\ \sigma_{\mu\nu}g^{\mu\nu} &= 0 \end{aligned} \tag{2.6}$$

From $\omega_{\mu\nu}$ we can define the corresponding vector ω_{μ}

$$\omega^{\tau} = \frac{1}{2}\, \eta^{\alpha\beta\rho\tau}\, \omega_{\alpha\beta}V_{\rho} \tag{2.7a}$$

or, inversely,

$$\omega_{\alpha\beta} = \eta_{\alpha\beta\mu\nu}\,\omega^{\mu}V^{\nu} \tag{2.7b}$$

In any riemannian ST these quantities obeys three constraints and three evolution equations:

Equations of Constraint:

$$\frac{2}{3}\,\theta_{,\mu}h^{\mu}{}_{\lambda} - (\sigma^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta})_{;\alpha}\, h^{\beta}{}_{\lambda} - a^{\alpha}(\sigma_{\lambda\alpha} + \omega_{\lambda\alpha}) = R_{\mu\alpha}V^{\mu}h^{\alpha}{}_{\lambda} \tag{2.8a}$$

$$\omega^{\alpha}{}_{;\alpha} + 2\,\omega^{\alpha}a_{\alpha} = 0 \tag{2.8b}$$

$$\frac{1}{2}\, h^{\varepsilon}{}_{(\rho}h^{\alpha}{}_{\sigma)}\, \eta_{\varepsilon}{}^{\beta\gamma\nu}V_{\nu}(\omega_{\alpha\beta}+\sigma_{\alpha\beta})_{;\gamma} - a_{(\rho}\omega_{\sigma)} = -H_{\rho\sigma} \tag{2.8c}$$

Dynamical Equations:

$$\dot{\theta} + \frac{\theta^2}{3} + \sigma^2 - \omega^2 - a^{\mu}{}_{;\mu} = R_{\mu\nu}V^{\mu}V^{\nu} \tag{2.9a}$$

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{\sigma}_{\mu\nu} + \frac{1}{3} h_{\alpha\beta}[\frac{1}{2}\omega^2 - \sigma^2 + a^\lambda{}_{;\lambda}] + a_\alpha a_\beta - \frac{1}{2} h_\alpha{}^\mu h_\beta{}^\nu a_{(\mu;\nu)} +$$

(2.9b)

$$+ \frac{2}{3}\theta\,\sigma_{\alpha\beta} + \sigma_{\alpha\mu}\sigma^\mu{}_\beta - \omega_\alpha\omega_\beta = R_{\alpha\varepsilon\beta\nu}V^\varepsilon V^\nu - \frac{1}{3} R_{\mu\nu}V^\mu V^\nu h_{\alpha\beta}$$

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{\omega}_{\mu\nu} - \frac{1}{2} h_\alpha{}^\mu h_\beta{}^\nu a_{[\mu;\nu]} + \frac{2}{3}\theta\,\omega_{\alpha\beta} + \sigma_{\alpha\mu}\omega^\mu{}_\beta - \sigma_{\beta\mu}\omega^\mu{}_\alpha = 0 \qquad (2.9c)$$

in which $\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu}$ and $\omega^2 = \omega_{\alpha\beta}\omega^{\alpha\beta} = -2\omega_\mu\omega^\mu$.
A dot means derivative projected in the V^μ-direction, that is, $\dot{\theta} = \theta_{,\mu}V^\mu$.

<u>Einstein's Equations</u>:

$$R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} = -k\, T_{\mu\nu} - \Lambda\, g_{\mu\nu} \qquad (2.10)$$

We use in this paper k = 1.
The right hand side will be represented by a perfect fluid:

$$T_{\mu\nu} = \rho V_\mu V_\nu - p\, h_{\mu\nu} \qquad (2.11)$$

The generalization of the subsequent results for more general kind of fluids is straightforward, although rather tediously long.

Two important consequences of this hypothesis in the above equations. Using (2.10), (2.11) and (2.9a) this later equation takes the form

$$\dot{\theta} + \frac{\theta^2}{3} + \sigma^2 - \omega^2 - a^\mu{}_{,\mu} = -\frac{1}{2}(\rho+3p) + \Lambda \qquad (2.9a)'$$

and for the evolution of the shear:

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{\sigma}_{\mu\nu} + \frac{1}{3} h_{\alpha\beta} \left[-\frac{1}{2}\omega^2 - \sigma^2 + a^\lambda{}_{;\lambda}\right] + \qquad (2.9b)'$$

$$+ a_\alpha a^\beta - \frac{1}{2} h_\alpha{}^\mu h_\beta{}^\nu a_{(\mu;\nu)} + \frac{2}{3}\theta\,\sigma_{\alpha\beta} + \sigma_\alpha{}^\mu \sigma_{\mu\beta} - \omega_\alpha\omega_\beta = -E_{\alpha\beta}$$

and for (2.8a):

$$\frac{2}{3}\theta_{,\mu} h^\mu{}_\lambda - (\sigma^\alpha{}_\beta + \omega^\alpha{}_\beta)_{;\alpha} h^\beta{}_\lambda - a^\alpha(\sigma_{\lambda\alpha} + \omega_{\lambda\alpha}) = 0 \quad . \qquad (2.8a)'$$

3 LANCZOS POTENTIAL

Definition:

A tensor $L_{\mu\nu\rho}$ which has the symmetries

$$L_{\alpha\beta\mu} + L_{\beta\alpha\mu} = 0 \qquad (3.1a)$$

$$L_{\alpha\beta\mu} + L_{\beta\mu\alpha} + L_{\mu\alpha\beta} = 0 \qquad (3.1.b)$$

and from which Weyl conformal tensor $W_{\alpha\beta\mu\nu}$ can be obtained by the formula

$$W_{\alpha\beta\mu\nu} = L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]} + \qquad (3.2)$$

$$+ \frac{1}{2}\left\{L_{(\alpha\nu)} g_{\beta\mu} + L_{(\beta\mu)} g_{\alpha\nu} - L_{(\alpha\mu)} g_{\beta\nu} - L_{(\beta\nu)} g_{\alpha\mu}\right\} +$$

$$+ \frac{2}{3} L^{\sigma\lambda}{}_{\sigma;\lambda}\, g_{\alpha\beta\mu\nu} \quad ,$$

is called a Lanczos potential. In this expression

$$L_{\alpha\mu} \equiv L_{\alpha\ \ \mu;\sigma}^{\ \sigma} - L_{\alpha\ \ \sigma;\mu}^{\ \sigma} \quad . \tag{3.3}$$

Remark that although the Weyl tensor has only 10 degrees of freedom, a tensor $L_{\alpha\beta\mu}$ which obeys relations (3.1) has 20 independent components. This means that there are 10 degrees of freedom. Lanczos choose to fix this arbitrariness by imposing extra conditions, e.g., $L_{\alpha\beta}{}^{\beta} = 0$ and $L_{\alpha\beta}{}^{\mu}{}_{;\mu} = 0$, which gives precisely 10 more equations to eliminate the freedom. The trace-free condition comes from the invariance of $W_{\alpha\beta\mu\nu}$ under the map

$$L_{\alpha\beta\mu} \to \tilde{L}_{\alpha\beta\mu} = L_{\alpha\beta\mu} + M_{\alpha}g_{\beta\mu} - M_{\beta}g_{\alpha\mu} \tag{3.4}$$

for an arbitrary vector M_{α}. The second (divergence-free) condition comes from the observation that in expression (3.2) this divergence is completely absent. It is clear however, that there is no sound argument to impose such gauge, and it remains as arbitrary as any other.

From (3.2) it follows that there is no local relationship between $L_{\alpha\beta\mu}$ and the metric $g_{\mu\nu}$ (although such relation can be exhibited in the quasi-Minkowskian ST in the approximation $g_{\mu\nu} \simeq \eta_{\mu\nu} + \varepsilon\psi_{\mu\nu}$ for $\varepsilon^2 << \varepsilon$. In this case it was shown that $L_{\alpha\beta\mu} = \frac{1}{4}\,[\psi_{\alpha\mu,\beta} - \psi_{\beta\mu,\alpha} + \frac{1}{6}\,\psi_{,\alpha}\eta_{\mu\beta} - \frac{1}{6}\,\psi_{,\beta}\eta_{\mu\alpha}]$) and this seems to turn the role of $L_{\alpha\beta\mu}$ somehow mysterious[3][4]. This situation can be overcome by exhibiting $L_{\alpha\mu\beta}$ for any geometry. We will investigate this approach and prove some useful lemmas. Let us remind the reader that all along this paper we call space time (ST) any four-dimensional Riemannian geometry which satisfies Einstein's equations with a perfect fluid as its source.

<u>Lemma 1.</u> If in a given ST there is a field of observers V^{μ} which is shear free and irrotational, then the magnetic part of Weyl tensor vanishes for the observers ($H_{\mu\nu} = 0$)

<u>Proof</u> Trivial (use eq. (2.8c)).

<u>Lemma 2.</u> If in a given ST there is a field of observers V^{μ} which is shear-free and irrotational, then the Lanczos potential is given by

$$L_{\alpha\beta\mu} = a_{\alpha}V_{\beta}V_{\mu} - a_{\beta}V_{\alpha}V_{\mu} \quad , \tag{3.5}$$

up to a gauge.

<u>Proof</u> A direct manipulation of the above kinematical equations will be used. Suppose formula (3.5) applies.Then we have

$$L_{\alpha\beta\mu;\nu} = a_{\alpha;\nu}V_{\beta}V_{\mu} + a_{\alpha}(\frac{\theta}{3}\, h_{\beta\nu}+a_{\beta}V_{\nu})V_{\mu} + a_{\alpha}V_{\beta}(\frac{\theta}{3}\, h_{\mu\nu}+a_{\mu}V_{\nu}) - \tag{3.6}$$

$$- a_{\beta}V_{\alpha}(\frac{\theta}{3}\, h_{\mu\nu}+a_{\mu}V_{\nu}) - a_{\beta}(\frac{\theta}{3}\, h_{\alpha\nu}+a_{\alpha}V_{\nu})V_{\mu}$$

Contracting with $V^{\beta}V^{\nu}$

$$L_{\alpha\beta\mu;\nu}V^{\beta}V^{\nu} = \dot{a}_{\alpha}V_{\mu} + a_{\alpha}a_{\mu} + a^{2}V_{\alpha}V_{\mu}$$

Then, in an analogous way

$$L_{\alpha\beta\nu;\mu}V^{\beta}V^{\nu} = a_{\alpha;\mu} + \frac{\theta}{3}\, a_{\mu}V_{\alpha} + a^{2}V_{\alpha}V_{\mu}$$

$$L_{\mu\nu\alpha;\beta}V^{\beta}V^{\nu} = \dot{a}_{\mu}V_{\alpha} + a_{\mu}a_{\alpha} + a^{2}V_{\mu}V_{\alpha}$$

$$L_{\mu\nu\beta;\alpha}\, V^{\beta}V^{\nu} = a_{\mu;\alpha} + \frac{\theta}{3}\, a_{\alpha}V_{\mu} + a^{2}\, V_{\alpha}V_{\mu}$$

Then,

$$\left(L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]}\right) V^\beta V^\nu = \dot{a}_{(\alpha}V_{\mu)} + 2a_\alpha a_\mu - a_{(\alpha;\mu)} - \qquad (3.7)$$

$$- \frac{\theta}{3}\, a_{(\mu}V_{\alpha)}$$

The contracted tensor is

$$L_{\alpha\mu} = L_\alpha{}^\sigma{}_{\mu;\sigma} - L_\alpha{}^\sigma{}_{\sigma;\mu}$$

In this case, from (3.5)

$$L_\alpha{}^\sigma{}_\sigma = a_\alpha \quad ,$$

then

$$L_{\alpha\mu} = \dot{a}_\alpha V_\mu + \frac{2}{3}\,\theta\, a_\alpha V_\mu + a_\alpha a_\mu - a^\varepsilon{}_{;\varepsilon}\, V_\alpha V_\mu - a_{\alpha;\mu}$$

Then, we have

$$\frac{1}{2}\, L_{(\alpha\mu)} g_{\beta\nu} V^\beta V^\nu = \frac{1}{2}\, L_{(\alpha\mu)}$$

$$\frac{1}{2}\, L_{(\beta\nu)} g_{\alpha\mu} V^\beta V^\nu = -\, a^\varepsilon{}_{;\varepsilon}\, g_{\alpha\mu}$$

$$\frac{1}{2}\, L_{(\alpha\nu)} g_{\beta\mu} V^\beta V^\nu = \frac{\theta}{3}\, a_\alpha V_\mu - a^\varepsilon{}_{;\varepsilon} V_\alpha V_\mu$$

$$\frac{1}{2}\, L_{(\beta\mu)} g_{\alpha\nu} V^\beta V^\nu = \frac{\theta}{3}\, a_\mu V_\alpha - a^\varepsilon{}_{;\varepsilon} V_\alpha V_\mu$$

$$\frac{1}{2}\left\{L_{(\alpha\nu)}g_{\beta\mu} + L_{(\beta\mu)}g_{\alpha\nu} - L_{(\alpha\mu)}g_{\beta\nu} - L_{(\beta\nu)}g_{\alpha\mu}\right\}V^\beta V^\nu = \tag{3.8}$$
$$= -\frac{1}{2}\dot{a}_{(\alpha}V_{\mu)} + \frac{1}{6}\theta\, a_{(\alpha}V_{\mu)} - a_\alpha a_\mu + a^\lambda{}_{;\lambda}h_{\alpha\mu} + \frac{1}{2}a_{(\mu;\alpha)}$$

And,

$$\frac{2}{3}L^{\sigma\lambda}{}_{\sigma;\lambda}\, g_{\alpha\beta\mu\nu}V^\beta V^\nu = -\frac{2}{3}a^\lambda{}_{;\lambda}\, h_{\alpha\mu} \tag{3.9}$$

Then, collecting all terms (3.7), (3.8) and (3.9) we obtain

$$-E_{\mu\nu} = \frac{1}{2}\dot{a}_{(\mu}V_{\nu)} + a_\mu a_\nu - \frac{1}{2}a_{(\mu;\nu)} \mp \frac{1}{6}\theta\, a_{(\mu}V_{\alpha)} + \frac{1}{3}a^\varepsilon{}_{;\varepsilon}h_{\mu\nu}$$

Comparing this expression with (2.9b) we see that formula (3.5) yields the correct electric part of the Weyl tensor. It remains to show (by Lemma 1) that (3.5) takes to the vanishing of the magnetic part.We have

$$\left(L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]}\right)V_\lambda V^\nu \eta^{\sigma\lambda\alpha\beta} = -\frac{2}{3}\theta\, a_\alpha V_\lambda\, \eta^{\sigma\lambda\alpha}{}_\mu$$

and

$$\frac{1}{2}\left[L_{(\nu\alpha)}g_{\beta\mu} + L_{(\beta\mu)}g_{\alpha\nu} - L_{(\alpha\mu)}g_{\beta\nu} - L_{(\beta\nu)}g_{\alpha\mu}\right]V_\lambda V^\nu \eta^{\sigma\lambda\alpha\beta} = +\frac{2}{3}\theta a_\alpha V_\lambda \eta^{\sigma\lambda\alpha}{}_\mu$$

and

$$L^{\sigma\varepsilon}{}_{\sigma;\varepsilon}g_{\alpha\beta\mu\nu}V_\lambda V^\nu \eta^{\sigma\lambda\alpha\beta} = 0$$

Adding those terms, we obtain $H_{\mu\nu} = 0$, which ends the proof.

<u>Example</u>. Lanczos potential for Schwarzschild geometry.

Consider the metric

$$ds^2 = (1 - \frac{2M}{r})dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 \ (d\theta^2 + \sin^2\theta d\phi^2) \tag{3.10}$$

Let us choose a frame defined by the observer V^μ which, in the system (t,r,θ,ϕ), has components

$$V^\mu = \delta^\mu{}_0 \ (g_{00})^{-1/2} = \delta^\mu{}_0 \ (1 - \frac{2M}{r})^{-1/2} \tag{3.11}$$

This observer is shear free and irrotational and has a non-vanishing acceleration given by

$$a_\mu = (0, - \frac{-M}{r^2(1 - \frac{2M}{r})}, 0, 0) \tag{3.12}$$

A direct application of lemma 2 implies that the Lanczos potential has the form $L_{\alpha\beta\mu} = a_{[\alpha}V_{\beta]}V_\mu$, which yields for the only non-vanishing term:

$$L_{010} = \frac{M}{r^2} \ . \tag{3.13}$$

If we want to exhibit $L_{\alpha\beta\mu}$ in Lanczos gauge we have just to subtract the trace, or to set

$$\tilde{L}_{\alpha\beta\mu} = (a_\alpha V_\beta V_\mu - a_\beta V_\alpha V_\mu) - \frac{1}{3} \ (a_\alpha g_{\beta\mu} - a_\beta g_{\alpha\mu}) \tag{3.14}$$

Remark that in this form we have incidently both Lanczos conditions satisfied

$$\tilde{L}_{\alpha\beta}{}^{\beta} = 0$$
$$\tilde{L}_{\alpha\beta}{}^{\mu}{}_{;\mu} = 0$$

In this gauge

$$\tilde{L}_{010} = \frac{2}{3}\frac{M}{r^2}$$

$$\tilde{L}_{122} = -\frac{1}{3}\frac{M}{1-\frac{2M}{r}}$$

$$\tilde{L}_{133} = -\frac{1}{3}\frac{M\sin^2\theta}{1-\frac{2M}{r}}$$

<u>Lemma 3</u>. If in a given ST there is a field of free observers (geodetic) V^μ which is irrotational and such that either $H_{\mu\nu} \neq 0$ and

$$\text{(i)}\quad \sigma_\mu{}^\varepsilon\sigma_{\varepsilon\nu} - \frac{1}{3}\sigma^2 h_{\mu\nu} - \frac{1}{3}\theta\,\sigma_{\mu\nu} = 0 \qquad (3.15)$$

or $H_{\mu\nu} = 0$ and

$$\text{(ii)}\quad \dot{\sigma}_{\mu\nu} + \theta\,\sigma_{\mu\nu} = 0 \qquad (3.16)$$

then the Lanczos potential is given respectively by

$$L_{\alpha\beta\mu} = \sigma_{\mu\alpha}V_\beta - \sigma_{\mu\beta}V_\alpha \quad \text{in case (i), and by} \qquad (3.17)$$

$$L_{\alpha\beta\mu} = \frac{1}{3}(\sigma_{\mu\alpha}V_\beta - \sigma_{\beta\mu}V_\alpha) \quad \text{in case (ii), up to a gauge.} \qquad (3.18)$$

<u>Proof</u>. The procedure is the same as in the precedent Lemma.

Suppose

$$L_{\alpha\beta\mu} = \sigma_{\mu\alpha}V_\beta - \sigma_{\mu\beta}V_\alpha \quad ,$$

then

$$\left[L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]}\right]\eta_\sigma{}^{\lambda\alpha\beta}V_\lambda V^\nu = \eta_\sigma{}^{\lambda\alpha\beta}\sigma_{\mu\alpha;\beta}V_\lambda \quad ;$$

all the others terms vanish identically when symmetrized in σ,μ. Then we obtain

$$H_{\sigma\mu} = -\frac{1}{2}\,\eta_{(\sigma}{}^{\beta\lambda\nu}\sigma_{\mu)\beta;\lambda}V_\nu \quad ,$$

which is the value of the magnetic part of the Weyl tensor in the case of absence of vorticity (cf. eq. (2.8c)).

From (3.17) we obtain the electric part as given by

$$E_{\mu\nu} = -\dot{\sigma}_{\mu\nu} - 3\sigma_\mu{}^\varepsilon\sigma_{\varepsilon\nu} + \sigma^2 h_{\mu\nu}$$

which will be compatible with expression (2.9b)' of the evolution of shear only if condition (3.15) applies. This ends the proof. In a similar way we show that in the case of vanishing $H_{\mu\nu}$ the form of Lanczos tensor is given by (3.18) under condition (3.16).

Example 1: In order to exhibit more directly the interdependence of Lanczos potential and an observer let us analyze an example to ilustrate the present case in the same geometry as above (Schwarzschild) as viewed by a distinct observer. We choose here the path of an observer to be geodetic and irrotational. For instance, take

$$V_\mu = \left(1, \frac{\sqrt{\frac{2M}{r}}}{1-\frac{2M}{r}}, 0, 0\right) \tag{3.19}$$

The shear is given by (the system of coordinates is the same as in (3.10))

$$\sigma_{00} = -\frac{1}{r}\left(\frac{2M}{r}\right)^{3/2}$$

$$\sigma_{01} = -\frac{2M}{r^2(1-\frac{2M}{r})}$$

$$\sigma_{11} = -(\frac{2M}{r})^{1/2}\frac{1}{r(1-\frac{2M}{r})^2} \tag{3.20}$$

$$\sigma_{22} = (\frac{Mr}{2})^{1/2}$$

$$\sigma_{33} = (\frac{Mr}{2})^{1/2}\sin^2\theta$$

and, consequently, applying our Lemma 3 the non-null components of Lanczos potential are

$$L_{010} = \frac{2}{3}\frac{M}{r^2}$$

$$L_{011} = \frac{1}{3}(\frac{2M}{r})^{1/2}\frac{1}{r(1-\frac{2M}{r})}$$

$$L_{022} = -\frac{1}{6}(2Mr)^{1/2} \tag{2.21}$$

$$L_{033} = -\frac{1}{6}(2Mr)^{1/2}\sin^2\theta$$

$$L_{122} = -\frac{1}{3}\frac{M}{(1-\frac{2M}{r})}$$

$$L_{133} = -\frac{1}{3}\frac{M}{(1-\frac{2M}{r})}\sin^2\theta$$

Compare this expression of Lanczos potential with the one obtained in (3.13) by choosing an accelerated observer (without shear and vorticity). As the geometry is the same and the system of coordinates is also the same, the components of Weyl tensor must be the same. What is then the difference between the very simple expression (3.13) and the rather long one (3.21) for Lanczos potential ? The answer is simple: it is due to the different gauge choices made in these cases.

Example 2: Kasner geometry. A rather simple case is Kasner geometry, where a geodetic irrotational observer is well-known:

$$V^{\mu} = \delta^{\mu}{}_{0} \quad . \tag{3.22}$$

We write the metric in the standard gaussian system of coordinates

$$ds^2 = dt^2 - t^{2p_1}\, dx^2 - t^{2p_2}\, dy^2 - t^{2p_3}\, dz^2 \tag{3.23}$$

with

$$p_1+p_2+p_3 = 1 \quad \text{and} \quad (p_1)^2+(p_2)^2+(p_3)^2 = 1$$

The non-vanishing components of the shear for (3.22) are

$$\begin{aligned} \sigma_{11} &= \frac{1-3p_1}{3}\, t^{2p_1-1} \\ \sigma_{22} &= \frac{1-3p_2}{3}\, t^{2p_2-1} \\ \sigma_{33} &= \frac{1-3p_3}{3}\, t^{2p_3-1} \end{aligned} \tag{3.24}$$

In this case, a direct calculation shows that

$$H_{\mu\nu}[V] = 0 \tag{3.25a}$$

and

$$\dot{\sigma}_{\mu\nu} + \theta\sigma_{\mu\nu} = 0 \ . \tag{3.25b}$$

We can then apply Lemma 3 for this observer to obtain

$$L_{011} = \frac{1}{3}\,(p_1 - \frac{1}{3})\ t^{2p_1-1}$$

$$L_{022} = \frac{1}{3}\,(p_2 - \frac{1}{3})\ t^{2p_2-1} \tag{3.26}$$

$$L_{033} = \frac{1}{3}\,(p_3 - \frac{1}{3})\ t^{2p_3-1}$$

Incidently, in this case we can easily see that this form is in Lanczos gauge ($L_{\alpha}{}^{\mu}{}_{\mu} = 0$ and $L_{\alpha\beta}{}^{\mu}{}_{;\mu} = 0$).

<u>Lemma 4</u>. If in a given ST there is a field of free observers (geodetic) V^{μ} which is non-expanding and shear free, then the magnetic part of Weyl tensor vanishes for this observer ($H_{\mu\nu} = 0$).

<u>**Proof**</u>. Using (2.8c) and the hypothesis that

$$V_{\mu;\nu} = \omega_{\mu\nu} \tag{3.27}$$

we have

$$H_{\rho\sigma} = -\eta_{\rho}{}^{\beta\gamma\nu} V_{\nu}\,\omega_{\sigma\beta;\gamma} = 2V^{\nu}\,\omega^{*}_{\sigma\nu;\rho} \tag{3.28}$$

But

$$\omega_{\mu\alpha;\beta}\eta^{\sigma\varepsilon\alpha\beta}V_\varepsilon = (\eta_{\mu\alpha}{}^{\rho\lambda}\omega_\rho V_\lambda)_{;\beta}\eta^{\sigma\varepsilon\alpha\beta}\ V_\varepsilon =$$

$$= \eta_{\mu\alpha\rho\lambda}\eta^{\sigma\varepsilon\alpha\beta}(\omega^\rho V^\lambda)_{;\beta}V_\varepsilon$$

$$= \delta^{\sigma\varepsilon\beta}_{\mu\rho\lambda}\ (\omega^\rho V^\lambda)_{;\beta}\ V_\varepsilon$$

$$= (\delta^\sigma_\mu\delta^\varepsilon_\rho\delta^\beta_\lambda - \delta^\sigma_\mu\delta^\varepsilon_\lambda\delta^\beta_\rho + \delta^\sigma_\lambda\delta^\varepsilon_\mu\delta^\beta_\rho - \delta^\sigma_\rho\delta^\varepsilon_\mu\delta^\beta_\lambda + \delta^\sigma_\rho\delta^\varepsilon_\lambda\delta^\beta_\mu -$$

$$- \delta^\sigma_\lambda\delta^\varepsilon_\rho\delta^\beta_\mu)(\omega^\rho V^\lambda)_{;\beta}V_\varepsilon$$

$$= \delta^\sigma_\mu\left[(\omega^\varepsilon V^\beta)_{;\beta}V_\varepsilon - (\omega^\beta V^\varepsilon)_{;\beta}V_\varepsilon\right] + (\omega^\beta V^\sigma)_{;\beta}\ V_\mu -$$

$$- (\omega^\sigma V^\beta)_{;\beta}V_\mu + (\omega^\sigma V^\varepsilon)_{;\mu}V_\varepsilon - (\omega^\varepsilon V^\sigma)_{;\mu}V_\varepsilon$$

Using equation (2.8b) which implies that $\omega^\mu{}_{;\mu}=0$ and the identity $\omega_{\alpha\mu}\omega^\mu = 0$ we arrive at

$$\omega_{\mu\alpha;\beta}\eta^{\sigma\varepsilon\alpha\beta}V_\varepsilon = 0 \quad , \tag{3.29}$$

or, equivalently

$$H_{\mu\nu} = 0 \quad ,$$

which ends the proof.

Lemma 5. If in a given ST there is a field of free observers (geodetic) V^μ which is shear-free and non-expanding and such that

$$\omega_{\mu;\nu} = 0 \tag{3.30}$$

then the Lanczos potential is given by

$$L_{\alpha\beta\mu} = \frac{2}{9}\left[\omega_{\alpha\beta}V_{\mu} + \frac{1}{2}\,\omega_{\alpha\mu}V_{\beta} - \frac{1}{2}\,\omega_{\beta\mu}V_{\alpha}\right] \tag{3.31}$$

<u>Proof</u>. Let us evaluate the electric part ($E_{\mu\nu}$) from (3.31). We have

$$L_{\alpha\beta\mu;\nu}V^{\beta}V^{\nu} = \frac{1}{9}\,\dot{\omega}_{\alpha\mu} = 0$$

which vanishes by (2.9c)

$$L_{\alpha\beta\nu;\mu}V^{\beta}V^{\nu} = -\frac{1}{3}\,\omega_{\alpha\beta}\omega^{\beta}{}_{\mu}$$

$$L_{\mu\nu\alpha;\beta}V^{\beta}V^{\nu} = \frac{1}{9}\,\dot{\omega}_{\mu\alpha} = 0$$

$$L_{\mu\nu\beta;\alpha}V^{\beta}V^{\nu} = -\frac{1}{3}\,\omega_{\mu\beta}\omega^{\beta}{}_{\alpha}$$

Then

$$(L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]})V^{\beta}V^{\nu} = \frac{2}{3}\,\omega_{\alpha}{}^{\lambda}\omega_{\lambda\mu}$$

From (3.31)

$$L_{(\alpha\mu)} = \frac{1}{3}\,\omega_{(\alpha}{}^{\sigma}{}_{;\sigma}V_{\mu)} - \frac{2}{3}\,\omega_{\mu}{}^{\sigma}\omega_{\sigma\alpha} = -\frac{2}{3}\,\omega_{\mu}{}^{\sigma}\omega_{\sigma\alpha}$$

since from (2.8b), $\omega_{\alpha}{}^{\sigma}{}_{;\sigma} = 0$,

Then

$$L_{(\alpha\mu)}g_{\beta\nu}V^{\beta}V^{\nu} = -\frac{2}{3}\,\omega_{\mu}{}^{\varepsilon}\omega_{\varepsilon\alpha}$$

$$L_{(\beta\nu)}g_{\alpha\mu}V^{\beta}V^{\nu} = -\frac{2}{3}\,\omega^{2}g_{\alpha\mu}$$

$$L_{(\alpha\nu)}g_{\beta\mu}V^{\beta}V^{\nu} = \frac{1}{3}\,\omega_{\alpha}{}^{\sigma}{}_{;\sigma}V_{\mu} - \frac{1}{3}\,\omega^{2}V_{\alpha}V_{\mu} = -\frac{1}{3}\,\omega^{2}V_{\alpha}V_{\mu}$$

$$L_{(\beta\mu)}g_{\alpha\nu}V^{\beta}V^{\nu} = -\frac{1}{3}\,\omega^{2}V_{\alpha}V_{\mu}$$

The trace $L_{\alpha\beta}{}^{\beta} = 0$, then we obtain from Lanczos formula of $W_{\alpha\beta\mu\nu}$:

$$-E_{\mu\alpha} = \omega_{\alpha}{}^{\varepsilon}\omega_{\varepsilon\mu} + \frac{1}{3}\,\omega^{2}h_{\mu\alpha}$$

But from the definition (2.7)

$$\omega_{\alpha}{}^{\varepsilon}\omega_{\varepsilon\mu} = -\frac{1}{2}\,\omega^{2}h_{\alpha\mu} - \omega_{\alpha}\omega_{\mu}$$

Thus we can write

(3.32) $$E_{\mu\alpha} = \frac{1}{6}\,\omega^{2}h_{\alpha\mu} + \omega_{\alpha}\omega_{\mu} \qquad (3.32)$$

which is precisely the expression for the electric part of the Weyl tensor as given by (2.9b)' in the present case.

It remains to show (by the previous Lemma 4) that

$$H_{\mu\nu} = 0$$

From (3.31) we have

$$L_{\alpha\beta\mu;\nu}V_{\lambda}V^{\nu}\eta_{\sigma}{}^{\lambda\alpha\beta} = 0$$

$$L_{\alpha\beta\nu;\mu}V_{\lambda}V^{\nu}\eta_{\sigma}{}^{\lambda\alpha\beta} = \frac{4}{3}\,\omega^{*}_{\sigma\lambda;\mu}V^{\lambda}$$

$$L_{\mu\nu\alpha;\beta}V_{\lambda}V^{\nu}\eta_{\sigma}{}^{\lambda\alpha\beta} = \frac{2}{9}\,\omega_{\mu\sigma}{}^{*}_{;\lambda}V^{\lambda} = 0 \text{ (use (3.29))} .$$

Thus

$$\left(L_{\alpha\beta[\mu;\nu]} + L_{\mu\nu[\alpha;\beta]}\right) V_\lambda V^\nu \eta_\sigma{}^{\lambda\alpha\beta} = \frac{4}{9}\, \overset{*}{\omega}_{\sigma\lambda;\mu} V^\lambda$$

The trace $L_{\alpha\mu}{}^{\mu} = 0$, and thus

$$L_{(\alpha\mu)} = -\frac{2}{3}\, \omega_\alpha{}^{\varepsilon} \omega_{\varepsilon\mu}$$

Then, putting all terms together:

$$\frac{1}{2}\left\{L_{(\alpha\nu)} g_{\beta\mu} + L_{(\beta\mu)} g_{\alpha\nu} - L_{(\alpha\mu)} g_{\beta\nu} - L_{(\beta\nu)} g_{\alpha\mu}\right\} V_\lambda V^\nu \eta_\sigma{}^{\lambda\alpha\beta} = 0$$

Finally the magnetic part of Weyl tensor constructed via the Lanczos potential yields

$$H_{\mu\sigma} = -\frac{2}{9}\, \overset{*}{\omega}_{\sigma\lambda;\mu} V^\lambda \tag{3.33}$$

which vanishes by hypothesis (3.30). Indeed, we have

$$\overset{*}{\omega}_{\sigma\lambda;\mu} V^\lambda = \frac{1}{2}\, \eta_{\sigma\lambda}{}^{\alpha\beta} \omega_{\alpha\beta;\mu} V_\lambda = \frac{1}{2}\, (\eta_{\sigma\lambda}{}^{\alpha\beta} \omega_{\alpha\beta})_{;\mu} V^\lambda =$$

$$= \frac{1}{2}\, (\eta_{\sigma\lambda}{}^{\alpha\beta} \eta_{\alpha\beta\varepsilon\tau} \omega^\varepsilon V^\tau)_{;\mu} V^\lambda$$

$$= (\delta^\sigma_\tau \delta^\lambda_\varepsilon - \delta^\sigma_\varepsilon \delta^\lambda_\tau)(\omega^\varepsilon V^\tau)_{;\mu} V^\lambda$$

$$= -\omega_{\sigma;\mu} - \omega_\sigma \omega_{\lambda\mu} V^\lambda + \omega_{\lambda;\mu} V_\sigma V^\lambda$$

$$= -\omega_{\sigma;\mu}$$

This ends the proof of Lemma 5.

Example: Gödel geometry

Let us consider Gödel's metric in the system of coordinates (t,x,y,z):

$$ds^2 = dt^2 - dx^2 + 2e^{ax}dt\, dy + \frac{1}{2}\, e^{2ax}\, dy^2 - dz^2 \quad (3.34)$$

Choose the observers V^μ co-moving with the matter which is the responsible for the curvature of the ST:

$$V^\mu = \delta^\mu{}_0 \quad (3.35)$$

The only no-vanishing component of the vorticity $\omega_{\mu\nu}$ is

$$\omega_{12} = -\omega_{21} = -\frac{a}{2}\, e^{ax} \quad (3.36)$$

and then

$$\omega^\mu = (0,\, 0,\, 0,\, \sqrt{2}\, a)\ . \quad (3.37)$$

We see that for this vector $\omega_{\mu;\nu} = 0$ which is the condition (3.30). Thus, we can obtain Lanczos potential, using Lemma 5, by the formula

$$L_{\alpha\beta\mu} = \frac{2}{9}\left[\omega_{\alpha\beta}V_\mu + \frac{1}{2}\,\omega_{\alpha\mu}V_\beta - \frac{1}{2}\,\omega_{\beta\mu}V_\alpha\right] \quad (3.38)$$

which gives for the non-null components

$$\begin{aligned} L_{012} &= \frac{a}{18}\, e^{ax} \\ L_{021} &= -\, L_{012} \\ L_{120} &= -2\, L_{012} \\ L_{122} &= -3\, L_{012} \end{aligned} \quad (3.39)$$

Incidently, we can see that (3.35) gives $L_{\alpha\beta\mu}$ in Lanczos' gauge ($L_{\alpha\beta}{}^{\beta} = 0$ and $L_{\alpha\beta}{}^{\mu}{}_{;\mu} = 0$).

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2. E. Bampi - G. Caviglia - GRG Journal 15(375)1983.
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4. After this work was completed we received a preprint of José L. Fernandez Chapon, José L.Lopez Bonilla, Gerardo A. Ovando Zuñiga and Marco A. Rosales Medina of Universidad Autonoma Metropolitana (Mexico) in which Lanczos potential for some geometries (Gödel, Minkowski, Schwarzchild, Kasner, Taub & Bertotti) have independently been obtained in a different way.
5. In the calculations of the Lanczos Potential and the correspondent Weyl tensor, using the expression given by the Lemmas above, for the examples, we used the computational resources of REDUCE.