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THE POTTS MODEL AND FLOWS. III, STANDARD AND SUBGRAPH BREAK-COLLAPSE METHODS

by

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ABSTRACT

An algorithm is developed for the exact calculation of the many spin correlation functions of Potts model clusters which is more efficient than the standard break-collapse method traditionally used in real space renormalisation group calculations. The improved performance is based on a relationship which, at any stage of the calculation, allows the replacement of certain subgraphs by single effective edges. Our method avoids, as in the standard one, the time consuming summation over spin states and can be very useful in series expansion and real space renormalisation group calculations on crystal lattices.

Key-words: Potts model; Break-collapse method; Graph theory; Statistical mechanics.

1 INTRODUCTION

The performance of sums over configurations plays a fundamental role in problems of statistical physics. The task of examining and counting configurations is very time consuming and alternative ways of performing such sums have been proposed (Kasteleyn and Fortuin 1969, Tsallis and Levy 1981, Mariz et al 1985, Tsallis 1986) for some problems in statistical mechanics which can be formulated in a graph theoretic way. In particular, for the model we shall consider in the present paper, namely, for the λ-state Potts model (Potts 1952, see Wu 1982 for a review), a simple method of performing configurational sums has been conjectured (Tsallis and Levy 1981, Tsallis 1986). This method is called the break-collapse method (BCM) and involves series and parallel relations as well as an equation which we will call the break-collapse equation (BCE).

The BCE as given in Tsallis and Levy (1981) refers to the equivalent transmissivity $t_{12}^{eq}(t,G)$ between the roots 1 and 2 of a graph G, where to each of its edges *e* is associated a variable t_e (called by them the "thermal transmissivity"). This equation relates $t_{12}^{eq}(t,G)$ to the same quantity calculated for two simpler graphs (with one edge less) obtained from G by deleting and contracting one edge*. Applying the BCE, for example, to the edge between unrooted vertices of the Wheatstone bridge (see graph L of Fig. 5a) they obtained deleted and contracted graphs (see graphs G_1 and G_2 of Fig. 5a respectively) whose equivalent transmissivities were easily computed by the series and parallel equations. They calculated, by this simple procedure, $t_{12}^{eq}(t,G)$ for this five-edge graph G without examining the λ^4 spin configurations or alternatively the 2^5=32 bond percolation configurations represented by the partial graphs G' of G. They also mentioned that the equivalent transmissivity $t_{12\ldots m}^{eq}(t,G)$ among the roots 1,2,...,m of a graph G could also be evaluated by the BCM.

*These graphs were called by Tsallis and Levy (1981) the "broken and collapsed clusters", but we prefer to follow the nomenclature of graph theory and call them the deleted and contracted graphs respectively.

Later on, the BCE for $t^{eq}_{12}(t,G)$ referred to above was proved (Essam and Tsallis 1986 – which we shall henceforth refer to as PF1 from the title "the Potts Model and Flows I") and then extended to the partitioned m–rooted equivalent transmissivity $t^{eq}_{P}(t,G)$, where P refers to any partition of the roots 1,2,...,m of G (de Magalhães and Essam 1986– which we will henceforth refer to as PF2). Similar BCE's were also derived in PF2 for $t^{eq}_{P}(p,G)$ where to each edge e of G was associated the variable p_e used by Kasteleyn and Fortuin (1969).

It has been proved (PF1) that the correlation function between spins s_1 and s_2 is proportional to $t^{eq}_{12}(t,G)$ and that the partition function Z(t,G) is proportional to its denominator. More generally, it has been shown (PF2) that the correlation function $\Gamma_{12..m}(G)$ among the components of m spins can be expressed, in the t or p–variable, as a linear combination of the $t^{eq}_{P}(G)$ corresponding to partitions of the m roots of G into blocks. Thus the BCM can be applied to the calculation of the partition function and all correlation functions. But the BCM for $t^{eq}_{P}(G)$, where to each edge e of G is associated a variable t_e or p_e has, in fact, a restricted use since it does not apply to a graph G which results from a previous use of series and/or parallel equations. In order to remove this restriction, Tsallis (1986) conjectured a BCM for $t^{eq}_{12..m}(t,G)$ which involves equivalent (or effective) edges whose thermal transmissivities are ratios of multi–linear functions of the t_e's, thus extending Tsallis and Levy's conjecture (1981). It allowed the exact calculation of $t^{eq}_{12..m}(t,G)$ for complex graphs such as, for example, the two–rooted graph shown in Fig.1h of da Silva et al (1984) which has 35 edges and 20 independent cycles. The computing time of $t^{eq}_{12}(t_x,t_y,t_z,G)$ for this graph calculated by the BCM was, for example, 200 minutes for $\lambda=3$ (da Silva, private communication) on the IBM–370 (model 158; 4Mb memory) computer. Notice that it would be practically impossible to calculate it from its definition as spin trace which would involve the examination of λ^{16} configurations. This is just one example, among many others, of graphs with many independent cycles which appear frequently in real space renormalisation group calculations. In fact the BCM for a general graph has been successfully applied (Chaves et al 1979, de Oliveira et al 1980, Tsallis and Levy 1981 and references therein, Chao 1981, de Magalhães et al 1982, de Oliveira and Tsallis 1982, Tsallis and dos Santos 1983, Lam and Zhang 1983, da Silva et al 1984, Costa and Tsallis 1984, Tsallis 1986) to the calculation of critical frontiers and critical exponents by the renormalisation group procedure. It has been applied to the pure as well as the randomly bond-diluted (isotropic or anisotropic) Potts model for arbitrary and specific values of λ.

In the present paper we prove a BCE for $t_p^{sq}(G)$ which applies to any graph G. We also derive a method for calculating $t_p^{sq}(G)$ which differs from the BCM in that we replace not only edges in series and in parallel by single effective edges but also any subgraph L of G which has only two vertices in common with the remainder of G and in which the internal vertices (i.e. vertices other than the two intersection vertices) are not rooted. We call it the subgraph break-collapse method (SBCM) and, with few exceptions, less iterations will be needed in the SBCM than in the BCM.

It has been proved (PF1, PF2) that the multi-linear forms of the denominator and numerator of $t_p^{sq}(t,G)$ are the respective generating functions for the flow polynomials $F(\lambda,G')$ (see PF1 and references therein) and the partitioned m-rooted flow polynomials $F_p(\lambda,G')$ (PF2). Similarly the multilinear forms of the denominator and numerator of $t_p^{sq}(p,G)$ are respectively the generating functions for the chromatic polynomials $P(\lambda,G')$ (see for example Tutte (1984)) and the partitioned m-rooted chromatic polynomials $P_p(\lambda,G')$ (PF2). These polynomials can also be expressed as sums over configurations (PF1,PF2) which can be evaluated in terms of graphs obtained by deleting and contracting edges (Forster, see note in Whitney 1932; Tutte 1954; PF2). In this paper we extend this deletion-contraction technique from edges to subgraphs and this forms the basis of the SBCM for the above polynomials. This is also the starting point for our derivation of the SBCM for $t_{\mathbf{p}}^{q}(G)$.

In summary, we present a powerful algorithm (the SBCM) for computing the expressions for the partition function and m-spin correlation functions of the Potts model derived in PF1 and PF2. The SBCM is considerably more efficient than the previously used BCM since it allows for the replacement of subgraphs by effective edges at any stage of the calculation. Both the SBCM and BCM yield exact expressions for the above functions for finite graphs which may be used either in the derivation of series expansions or for real space renormalisation group calculations on crystal lattices.

This paper is divided into five sections and one appendix. First (section 2) we summarise results from PF2 concerning Z(G) and $\Gamma_{12..m}(G)$ expressed in the t-variable. In section 3, we derive expressions for $F_{\mathbf{p}}(\lambda,G')$ and $F(\lambda,G')$ from which we obtain (section 4) the SBCM and the BCM for $t_{\mathbf{p}}^{q}(t,G)$; explicit illustrations of both algorithms are given. In section 5, we study the $\lambda\to 1$ limit of our results. Consideration of this limit enables the SBCM to be extended to the partitioned m-rooted connectedness function of percolation theory. Finally, in the appendix, we quote the SBCM formulae in terms of the p-variable and also give its extension to the Whitney rank function and to its generalisation, the partitioned m-rooted rank function.

2 MAIN RESULTS OF PAPERS I AND II.

In this section we quote the expressions obtained for the partition function (PF1) and the correlation function $\Gamma_{12..m}(G)$ among the components of m spins along one of the λ special directions $e_1, e_2, \ldots, e_\lambda$ in which the spins are allowed to point (PF2). The extension of the latter results to the case of correlation functions $\Gamma_P(G)$ among the components of the m spins along several *different* directions involves more definitions which can be found in Section 3.3 of PF2. Since the t-variable has been shown to be more convenient than the p-variable (see PF2), we shall restrict ourselves, throughout the main body of this paper, to the t-variable. The corresponding results in the p-variable will be quoted in the appendix.

2.1 The Partition Function Z(t,G)

Let us consider a finite graph G with vertex-set V and edge-set E. The partition function Z(t,G) of the λ-state Potts model on a graph G, whose Hamiltonian is given by eq (2.1) of PF2 with $s_1^2=\lambda-1$, can be expressed as (PF1):

$$Z(t,G) = \lambda^{|V|-|E|} \{ \prod_{e \in E} [\exp[(\lambda-1)K_e] + (\lambda-1)\exp(-K_e)] \} D(t,G) \qquad (2.1a)$$

with

$$D(t,G) \equiv \langle \lambda^c \rangle_{G,t} \qquad (2.1b)$$

where c(G') is the cyclomatic number of the subgraph G' and the percolation average denoted by $\langle .. \rangle_{G,t}$, defined in eq 2.3 of PF2, is relative to the t-variable. The "thermal transmissivity" t_e (see Tsallis and Levy 1981) of the edge *e* is given by:

$$t_e = [1 - \exp(-\lambda K_e)]/[1 +(\lambda-1)\exp(-\lambda K_e)] \tag{2.1c}$$

where

$$K_e = J_e/k_B T \tag{2.1d}$$

The multi-linear form of D(t,G) in the t_e variables is (PF1):

$$D(t,G) = \sum_{G'\subseteq G} F(\lambda,G') \prod_{e\in E'} t_e \tag{2.2a}$$

where E' is the edge set of the partial graph G' of G (i.e. V'=V and E'⊆E), F(λ,G) is the flow polynomial of G given by

$$F(\lambda,G) = \sum_{G'\subseteq G} (-1)^{|E\backslash E'|} \lambda^{c(G')} \tag{2.2b}$$

where $|E\backslash E'|$ is the number of edges in the complement of G' with respect to G. Notice that if t_e=t for all edges e of E then eq. (2.2a) becomes a polynomial in t.

2.2 The Correlation Function $\Gamma_{12..m}(t,G)$

The correlation function $\Gamma_{12..m}(t,G)$ among the components $s_{11},s_{21},\ldots,s_{m1}$ of the m spins $s_1,s_2,\ldots s_m$ along one of the λ special directions, say e_1, is related to the partitioned equivalent transmissivities $t_{\beta}^{eq}(t,G)$ through:

$$\Gamma_{12\ldots m}(t,G) = \langle s_{11}s_{21}\ldots s_{m1}\rangle_G^T$$
$$= (\lambda-1)^{-m/2} \sum_{P\in\tilde{P}(M)} t_{\beta}^{eq}(t,G)\; F(\lambda,I_P) \tag{2.3a}$$

where

$$s_{i1} = s_i.e_1/|e_1| \qquad (|e_1|^2 = |s_i|^2 = \lambda-1) \tag{2.3b}$$

and $<\ldots>^T_G$ means a thermal average. $t^{eq}_P(t,G)$ is an extension of $t^{eq}_{12..m}(t,G)$ to the case where not all the roots are connected among themselves i.e.:

$$t^{eq}_P(t,G) = N_P(t,G)/D(t,G) \tag{2.3c}$$

with $D(t,G)$ given by eq. (2.1b) and $N_P(t,G)$ defined by:

$$N_P(t,G) = <\lambda^c \gamma_P>_{G,t} \tag{2.3d}$$

where

$$\gamma_P(G') = \begin{cases} 1 & \text{if the roots in the same block of the the partition P are connected among themselves in G' and if roots of different blocks are not connected.} \\ 0 & \text{otherwise} \end{cases} \tag{2.3e}$$

When $\gamma_P(G')=1$ the roots are said to be P–partitioned by G'. We write $P=\{B_1,B_2,..,B_b\}$ and the block B_i will be said to have ℓ_i roots of type i. For example, in Fig. 4c where $P=\{\{1,2\},\{3,4\},\{5,6,7,8,9\}\}$, the roots of type 1,2 and 3 are represented by squares, triangles and circles respectively. In eq. (2.3a) the sum is over the set $\tilde{P}(M)$ of all partitions P of the set $M = \{1,2,\ldots,m\}$ of roots of G into blocks which contain at least two roots. $F(\lambda,I_P)$ is the flow polynomial of the "interface graph" I_P constructed from the partition P as follows: for $i=1,..,b$ associate with the block B_i a vertex v_i and connect it to an "external" vertex u by an edge of multiplicity ℓ_i (hence I_P has b+1 vertices and m edges). $F(\lambda,I_P)$ is given by:

$$F(\lambda,I_P) = \prod_{B_i \in P} \{(\lambda-1)[(\lambda-1)^{\ell_i-1} + (-1)^{\ell_i}]/\lambda\} \tag{2.3f}$$

The multi-linear form of $N_P(t,G)$ is:

$$N_P(t,G) = \sum_{G'\subseteq G} F_P(\lambda,G') \prod_{e\in E'} t_e \tag{2.4a}$$

where the partitioned m-rooted flow polynomial is:

$$F_P(\lambda,G) = \sum_{G'\subseteq G} (-1)^{|E\setminus E'|} \lambda^{c(G')} \gamma_P(G') \quad . \tag{2.4b}$$

Using eqs. (2.3a), (2.3c), (2.4a) and (2.2a), $\Gamma_{12..m}(t,G)$ may be expressed in the form

$$\Gamma_{12\ldots m}(t,G) = (\lambda-1)^{-m/2} \frac{\sum_{G'\subseteq G} \left(\sum_{P\in \tilde{P}(M)} F(\lambda,I_P) F_P(\lambda,G') \prod_{e\in E'} t_e \right)}{\sum_{G'\subseteq G} F(\lambda,G') \prod_{e\in E'} t_e} \tag{2.5}$$

An alternative expression for $\Gamma_{12..m}(t,G)$ is

$$\Gamma_{12..m}(t,G) = (\lambda-1)^{-m/2} \left. \frac{\frac{\partial}{\partial t_m} \cdots \frac{\partial}{\partial t_1} D(t,G^+)}{D(t,G^+)} \right|_{t_1=t_2=\ldots=t_m=0} \tag{2.6}$$

where $G^+=G\cup K_{1,m}$ and $K_{1,m}$ is the complete bipartite graph formed by linking a vertex v_g (which does not belong to G) to the roots 1,2,...,m of G. To each edge g_i, (i = 1,2,...,m), of $K_{1,m}$ is associated a thermal transmissivity t_i. From eqs. (2.2a) and (2.6) an alternative form for $\Gamma_{12..m}(t,G)$ may be obtained:

$$\Gamma_{12\ldots m}(t,G) = (\lambda-1)^{-m/2} \frac{\sum_{G'\subseteq G} \left(F(\lambda,G'\cup K_{1,m}) \prod_{e\in E'} t_e \right)}{\sum_{G'\subseteq G} \left(F(\lambda,G') \prod_{e\in E'} t_e \right)} . \tag{2.7}$$

We shall develop in the subsequent sections SBCM's for D(t,G), $N_P(t,G)$ and $t_{\mathfrak{P}}^{q}(t,G)$ which provide a powerful technique for evaluating the above expressions for Z(t,G) and $\Gamma_{12..m}(t,G)$. We shall also discuss the BCM for $t_{\mathfrak{P}}^{q}(t,G)$.

3 EXTENSIONS OF THE DELETION-CONTRACTION RULES FOR $F_P(\lambda,G)$ and $F(\lambda,G)$

We now extend the deletion-contraction rules for $F_P(\lambda,G)$ (see eq.(4.14) of PF2) and $F(\lambda,G)$ (see eq. (4.8) of PF1) to the cases where, instead of a single edge *e*, we consider a subgraph L of G which has only two vertices in common with the remainder of G. As we will see in the next section, these extensions form the heart of the proof of the SBCM's for $N_P(t,G)$ and $D(t,G)$.

In this section we shall assume, unless otherwise stated to the contrary, that G is a two-reducible m-rooted graph (Essam 1970) which is the union of two subgraphs L and H subject to the following conditions: (i) they intersect only at the vertices i and j (there are no edges in common); (ii) one of them, say H, contains all of the m roots of G. The vertices i and/or j may be rooted or not (see Fig 1).

3.1 <u>The Subgraph Break-Collapse Equations (SBCE's)</u>

In order to derive the extended deletion-contraction rule for $F_P(\lambda,G)$, which we shall call the subgraph break-collapse equation (SBCE), we shall examine how the quantities $\gamma_P(G')$, $c(G')$ and $|E'|$ which appear in the definition of $F_P(\lambda,G)$ (see eq. (2.4b)) relate to the corresponding ones in the partial graphs L' of L and H' of H. We shall suppose for the moment that i and j are not rooted, as shown in Fig. 1a.

First let us see what is the relationship between $\gamma_P(G')$ and $\gamma_P(H')$. If *there is no path from i to j on L'* (i.e. $\gamma_{i,j}(L')=1$) then in order that the m roots are P-partitioned by G' (i.e. $\gamma_P(G')=1$) they must also be P-partitioned by H' (i.e. $\gamma_P(H')=1$) since no root of H' can be connected to another root of H' via a path on L' (see the first parenthesis on the right hand side of the equality sign of Fig.

2). If *there is a path between i and j on L'* (i.e. $\gamma_{ij}(L')=1$; see the last parenthesis in Fig. 2) then the condition $\gamma_P(G')=1$ can be satisfied in the following cases: (i) $\gamma_P(H')=0$ or (ii) $\gamma_P(H')=1$. In case (i) some roots of H' must be connected to other roots of H' via paths on L' (see the penultimate graph of Fig. 2). In both cases (i) and (ii) the m roots are P-partitioned by the *bicollapsed-graph* graph $H'_{i=j}$ obtained from H' by identifying the vertices i and j (without deleting any edges). Hence, for every partial graph G' of G the following identity holds:

$$\gamma_P(G') = \gamma_{i,j}(L')\gamma_P(H') + \gamma_{ij}(L')\gamma_P(H'_{i=j}) \qquad (\forall G'=H'\cup L'\subseteq G) \qquad (3.1)$$

Now let us examine how c(G') relates to c(H') and c(L'). If $\gamma_{i,j}(L')=1$ then no new cycle can be formed when we consider the union of L' with H', i.e. c(G') is just the sum of c(L') and c(H') (see the first parenthesis on the right hand side of the equality sign of Fig. 2). If $\gamma_{ij}(L')=1$ then we have to consider two cases, namely (a) $\gamma_{i,j}(H')=1$ and (b) $\gamma_{ij}(H')=1$. In case (b), the union of paths on L' and H' between i and j gives rise to an extra cycle in $G' = L'\cup H'$, i.e. $c(G')=c(L')+c(H')+1$ (see the last graph of Fig. 2). In case (a) no new cycles appear and c(G') is just the sum of c(L') and c(H') (see the penultimate graph of Fig. 2). It is easy to verify that in both cases (a) and (b), c(G') is equal to the sum of c(L') and $c(H'_{i=j})$. It follows therefore that:

$$c(G') = \gamma_{i,j}(L')[c(H') + c(L')] + \gamma_{ij}(L')[c(H'_{i=j}) + c(L')] \qquad (\forall G'=H'\cup L'\subseteq G) \quad (3.2)$$

Concerning the number of edges $|E'|$, since H' and L' have no edges in common and since by definition $|E(H'_{i=j})| = |E(H')|$ it follows trivially that

$$|E'| = |E(H')| + |E(L')| = |E(H'_{i-j})| + |E(L')| \quad (\forall G'=H'\cup L'\subseteq G) \qquad (3.3)$$

Combining eqs. (3.1), (3.2) and (3.3) we get that:

$$\gamma_p(G')\lambda^{c(G')}(-1)^{|E\setminus E'|} =$$
$$\gamma_{i,j}(L')\lambda^{c(L')}(-1)^{|E(L)\setminus E(L')|}\gamma_p(H')\lambda^{c(H')}(-1)^{|E(H)\setminus E(H')|}+$$

$$+\gamma_{i,j}(L')\lambda^{c(L')}(-1)^{|E(L)\setminus E(L')|}\gamma_p(H'_{i-j})\lambda^{c(H'_{i-j})}(-1)^{|E(H_{i-j})\setminus E(H'_{i-j})|}$$

$$(\forall G'=H'\cup L'\subseteq G) \qquad (3.4)$$

Summing over all partial graphs G' of G we get (cf. eq. (2.4b)) the following SBCE for $F_p(\lambda,G)$:

$$F_p(\lambda,G) = F_{i,j}(\lambda,L)F_p(\lambda,H) + F_{ij}(\lambda,L)F_p(\lambda,H_{i-j}) \qquad (3.5)$$

Notice that the arguments used to derive the above SBCE apply also to the cases where i and/or j are roots except if i and j are roots of different types. In this latter situation, if $\gamma_{ij}(L') = 1$ the condition $\gamma_p(G')=1$ cannot be satisfied. Nevertheless eq. (3.1) continues to be valid since

$$\gamma_p(H'_{i-j})=0 \quad \text{for i and j are roots of different types} \qquad (3.6)$$

and the right-hand-side of eq. (3.1) correctly reduces to the first product. Furthermore, from eqs. (3.6) and (2.4b) it follows that

$$F_p(\lambda,H_{i-j})=0 \text{ for i and j are roots of different types} \qquad (3.7)$$

and the right hand side of eq. (3.5) correctly reduces to the first product. Consequently eq. (3.5) applies to all the four situations in Fig. 1. In the construction of $H_{i=j}$, the collapse of i and j leads to: (i) a root if at least one of them is a root, (ii) an unrooted vertex if both i and j are unrooted. Observe that when i and j are the only roots of G and P contains a single block then:

$$F_{ij}(\lambda, H_{i=j}) = F(\lambda, H_{i=j}) \tag{3.8}$$

Since a partial graph $G''=L''\cup H''$ of $G'=H'\cup L'$ is also a partial graph of $G=H\cup L$, it follows that eq.(3.4) holds also for all $G''\subseteq G'$. Consequently eq.(3.5) remains true for any partial graph G' of G, namely

$$F_P(\lambda, G') = F_{i,j}(\lambda, L')F_P(\lambda, H') + F_{ij}(\lambda, L')F_P(\lambda, H'_{i=j}) \quad (\forall G'\subseteq G) \tag{3.9}$$

When L is a single edge *e* between i and j eq.(3.5) becomes

$$F_P(\lambda, H\cup e) = F_P(\lambda, H_{i=j}) - F_P(\lambda, H) \tag{3.10}$$

which can be written equivalently in terms of $G=H\cup e$ as

$$F_P(\lambda, G) = F_P(\lambda, G_e^{\gamma}) - F_P(\lambda, G_e^{\delta}) \tag{3.11}$$

where G_e^{γ} and G_e^{δ} are the respective graphs obtained from G by contracting and deleting the edge *e* of G. Eq.(3.11) is the deletion-contraction rule obtained previously (see eq.(4.14) of PF2).

Eq. (3.5) can be written in terms of $F(\lambda,L)$ and $F_{ij}(\lambda,L)$ by noticing that:

$$F_{i,j}(\lambda, G) = F(\lambda, G) - F_{ij}(\lambda, G) \qquad \forall G \tag{3.12}$$

which is a consequence of the identity $\gamma_{i,j}(G')+\gamma_{ij}(G')=1$ and the definitions (2.2b) and (2.4b). Combining eqs (3.5), (3.10) and (3.12) we get the following alternative SBCE for $F_P(\lambda,G)$:

$$F_P(\lambda,G) = F(\lambda,G)F_P(\lambda,H) + F_{ij}(\lambda,L)F_P(\lambda,H\cup e) \qquad (3.13)$$

In the particular case of P={ij} (see Fig. 4e) we can write the SBCE for $F_{ij}(\lambda,H\cup L)$ in the following form symmetric in H and L

$$F_{ij}(\lambda,H\cup L) = F_{ij}(\lambda,L)F(\lambda,H) + F_{ij}(\lambda,H)F(\lambda,L) + \\ +(\lambda-2)F_{ij}(\lambda,L)F_{ij}(\lambda,H) \qquad (3.14)$$

where we have used eqs. (3.5) and (3.8), the unrooted version of eq. (3.10) and the following relation (see eq. (4.6) of PF1):

$$F_{ij}(\lambda,L) = F(\lambda,G\cup e)/(\lambda-1) \qquad (3.15)$$

The SBCE for $F(\lambda,G)$ can be deduced from eq. (3.4) by imposing that $\gamma_P(G')=1$ (and consequently $\gamma_P(H')=\gamma_P(H'_{i=j})=1$) for every graph $G'\subseteq G$. We get thus:

$$F(\lambda,G) = F_{i,j}(\lambda,L)F(\lambda,H) + F_{ij}(\lambda,L)F(\lambda,H_{i=j}) \qquad (3.16)$$

or alternatively

$$F(\lambda,G) = F(\lambda,L)\ F(\lambda,H) + F_{ij}(\lambda,L)F(\lambda,H\cup e) \qquad (3.17)$$

If we use eq. (3.15) and the unrooted version of eq. (3.10) we can rewrite eq. (3.17) in the following forms symmetric in H and L:

$$F(\lambda,G) = F(\lambda,L)F(\lambda,H) + (\lambda-1)F_{ij}(\lambda,L)F_{ij}(\lambda,H) \tag{3.18}$$

or in terms of only unrooted flow polynomials

$$F(\lambda,G) = [F(\lambda,L)-F(\lambda,L_{i=j})][F(\lambda,H)-F(\lambda,H_{i=j})]/(\lambda-1) + F(\lambda,L)F(\lambda,H). \tag{3.19}$$

Notice that eqs. (3.13) – (3.19) are also valid for any partial graph G' of G.

Now let us interpret eq. (3.17) which is illustrated schematically in Fig. 3. Let us call Φ_{net} the value of the net flow from L to H at the intersection vertex i (which, by conservation of "fluid" mod-λ, must be equal to the value of the net flow mod-λ from H to L at j). The proper mod-λ flows in $G = H \cup L$ may be partitioned into two sets: (a) proper mod-λ flows in which $\Phi_{net}=0$ and (b) proper mod-λ flows in which $\Phi_{net}\neq 0$. The proper flows on G which satisfy condition (a) may be counted by combining any proper flow in H with any proper flow in L and hence the total number of such flows will simply be $F(\lambda,H)F(\lambda,L)$ (this is illustrated by the pair of graphs just after the equality sign in Fig.3). The proper flows in G subject to condition (b) may be counted by considering the proper flows on the graph $\overline{G}$ obtained from G by replacing L by a single edge *e*. Any proper mod-λ flow on $\overline{G}$ can be combined with any proper mod-λ flow on L which is subject to a non-zero external flow in at j and out at i. In PF2 (eq. (2.24)) we showed that $F_{ij}(\lambda,L)$ could be interpreted as just this number of flows. Consequently the total number of proper mod-λ flows on G with $\Phi_{net}\neq 0$ is given by $F(\lambda,\overline{G})F_{ij}(\lambda,L)$ (see the last pair of graphs of Fig.3).

3.2 Series combination of graphs.

When two graphs G_1 and G_2 intersect at only one vertex i, i.e. when G_1 and G_2 are in series, then (cf. property (ii) of $F(\lambda,G)$ in PF1):

$$F(\lambda,G_1 \cup G_2) = F(\lambda,G_1)\ F(\lambda,G_2) \tag{3.20}$$

The series equation for $F_P(\lambda,G_1 \cup G_2)$ depends on the distribution of the roots of $G_1 \cup G_2$. There are four cases to consider which are pictorially illustrated in Figs. (4a), (4b), (4c) and (4d).

(a) There are no roots in G_2 except possibly i. This case is covered by property (v) of PF2 and

$$F_P(\lambda,G_1 \cup G_2) = F_P(\lambda,G_1)\ F(\lambda,G_2). \tag{3.21a}$$

(b)There are roots in both G_1 and G_2 but each block of **P** is contained within either G_1 or G_2. It is again possible to factorise F_P and

$$F_P(\lambda,G_1 \cup G_2) = F_{P'}(\lambda,G_1)\ F_{P''}(\lambda,G_2). \tag{3.21b}$$

where P' and P" are the restrictions of P to G_1 and G_2 respectively.

(c) Exactly one block of P contains roots in both G_1 and G_2. The result is the same as (b) except that if i is not a root then it must be converted into a root of the same type as the common block before calculating P' and P".

(d) If more than one block of P contains roots in both G_1 and G_2 then $\gamma_P(G')$ is zero for all partial graphs of $G_1 \cup G_2$ and so $F_P(\lambda,G_1 \cup G_2)=0$.

4 THE SBCM and the BCM FOR $t_{\beta}^{q}(t,G)$.

Having derived the SBCE's for $F(\lambda,G')$ and $F_p(\lambda,G')$, we now deduce the SBCE's for their respective generating functions, i.e., the denominator $D(t,G)$ and the numerator $N_p(t,G)$ of $t_{\beta}^{q}(t,G)$. A comparison of the formulae which we obtain for $t_{\beta}^{q}(t,G)$ with those of Tsallis and Levy's paper (1981) leads to the SBCM and BCM. Both methods allow the calculation of $\Gamma_{12,m}(t,G)$ (eq. 2.3) and $Z(t,G)$ (eq. 2.1a) without having to examine all the subgraphs G' of G which contribute to $D(t,G)$ (see eq. 2.2a) or to $N_p(t,G)$ (see eq. 2.4a).

4.1 The SBCE and the BCE for $t_{\beta}^{q}(t,G)$.

Combining eqs. (2.4a) and (3.9) we get the following SBCE for $N_p(t,G)$:

$$N_p(t,G) = N_{i,j}(t,L)N_p(t,H) + N_{ij}(t,L)N_p(t,H_{i=j}) \tag{4.1}$$

Similarly we get (cf. eqs. (2.2a) and (3.16) applied to G') for $D(t,G)$:

$$D(t,G) = N_{i,j}(t,L)D(t,H) + N_{ij}(t,L)D(t,H_{i=j}) \tag{4.2}$$

where $N_{i,j}(t,L)$ relates to $N_{ij}(t,L)$ (cf. eqs. (3.12), (2.4a), (2.2a)) through

$$N_{i,j}(t,L) = D(t,L) - N_{ij}(t,L) \tag{4.3}$$

Notice that when i and j are roots of different types then eq. (3.7) holds for all $H'_{i=j} \subseteq H_{i=j}$. Therefore

$$N_p(t,H_{i=j}) = 0 \text{ if i and j are roots of different types.} \tag{4.4}$$

If i and j are roots of the same type and G has no other roots then eq. (3.8) applies for all $H'_{i=j} \subseteq H_{i=j}$ and so

$$N_{ij}(t,H_{i=j}) = D(t,H_{i=j}). \qquad (4.5)$$

As a consequence of eqs. (3.10) and (3.15) it turns out that $N_{ij}(t,G)$ (see eq. (2.4a)) is related to D(t,G) through:

$$N_{ij}(t,G) = [D(t,G_{i=j}) - D(t,G)]/(\lambda-1) \qquad (4.6)$$

$$= [D(t,G\cup e)-D(t,G)]/[(\lambda-1)t_e] \qquad (4.6')$$

where e is an extra edge between i and j whose thermal transmissivity is t_e.

When **P**={ij} (see Fig. 4e) we can write the SBCE for $N_{ij}(t,G_1\cup G_2)$ (eq.4.1) in the form symmetric in G_1 and G_2:

$$N_{ij}(t,G_1\cup G_2) = N_{ij}(t,G_1)D(t,G_2) + N_{ij}(t,G_2)D(t,G_1) + (\lambda-2)N_{ij}(t,G_1)N_{ij}(t,G_2) \qquad (4.7)$$

where we have used eqs. (4.3), (4.5) and (4.6). Similarly, combining eqs. (4.2), (4.3) and (4.6) we obtain the following symmetric form of the SBCE for $D(t,G_1\cup G_2)$:

$$D(t,G_1\cup G_2) = D(t,G_1)D(t,G_2) + (\lambda-1)N_{ij}(t,G_1)N_{ij}(t,G_2) \qquad (4.8)$$

From the last two equations and from eq. (2.3c) we get the following formula for two graphs in parallel as shown in Fig.4e:

$$t_{ij}^{eq}(t,G_1 \cup G_2) = \frac{N_{ij}(t,G_1)D(t,G_2) + N_{ij}(t,G_2)D(t,G_1) + (\lambda-2)N_{ij}(t,G_1)N_{ij}(t,G_2)}{D(t,G_1)\ D(t,G_2) + (\lambda-1)N_{ij}(t,G_1)N_{ij}(t,G_2)} \quad (4.9)$$

or in terms of equivalent transmissivities:

$$t_{ij}^{eq}(t,G_1 \cup G_2) = \frac{t_{ij}^{eq}(t,G_1) + t_{ij}^{eq}(t,G_2) + (\lambda-2)t_{ij}^{eq}(t,G_1)t_{ij}^{eq}(t,G_2)}{1 + (\lambda-1)t_{ij}^{eq}(t,G_1)t_{ij}^{eq}(t,G_2)} \quad (4.9')$$

which reduces, for two *edges* e_1 and e_2 in parallel (i.e. $G_1=e_1, G_2=e_2$) to eq. (23) of Domb (1974), namely:

$$t_{12}^{eq}(t,e_1 \cup e_2) = [t_1 + t_2 + (\lambda-2)t_1t_2]/[1 + (\lambda-1)t_1t_2] \quad (4.10)$$

In the general case of n graphs G_α ($\alpha=1,2,\ldots,n$) in parallel with equivalent transmissivities N_α/D_α between i and j, the successive application of eqs. (4.8) and (4.7) leads to:

$$D(t, \bigcup_{\alpha=1}^{n} G_\alpha) = \lambda^{-1}[X(t) + (\lambda-1)Y(t)] \quad (4.11a)$$

and

$$N(t, \bigcup_{\alpha=1}^{n} G_\alpha) = D(t, \bigcup_{\alpha=1}^{n} G_\alpha) - Y(t) \quad (4.11b)$$

where

$$X(t) = \prod_{\alpha=1}^{n} [D_\alpha + (\lambda-1)N_\alpha] \quad (4.11c)$$

and

$$Y(t) = \prod_{\alpha=1}^{n} [D_\alpha - N_\alpha] \quad (4.11d)$$

We see from eqs. (4.9') and (4.10) that the parallel equation for graphs has the same functional form as the corresponding one for edges.

From eqs. (4.1), (4.2), (4.3) and the definition of $t_P^{eq}(t,G)$ (eq. 2.3c) we obtain the following SBCE:

$$t_P^{eq}(t,G) = \frac{[D(t,L) - N_{ij}(t,L)]N_P(t,H) + N_{ij}(t,L)N_P(t,H_{i=j})}{[D(t,L) - N_{ij}(t,L)]D(t,H) + N_{ij}(t,L)D(t,H_{i=j})} \qquad (4.12)$$

or equivalently

$$t_P^{eq}(t,G) = \frac{[1 - t_{ij}^{eq}(t,L)]N_P(t,H) + t_{ij}^{eq}(t,L)N_P(t,H_{i=j})}{[1 - t_{ij}^{eq}(t,L)]D(t,H) + t_{ij}^{eq}(t,L)D(t,H_{i=j})} \qquad (4.12')$$

When L is a single edge *e* eq. (4.12') reduces to:

$$t_P^{eq}(t,H\cup e) = \frac{(1-t_e)\ N_P(t,H) + t_e N_P(t,H_{i=j})}{(1 - t_e)\ D(t,H) + t_e D(t,H_{i=j})} \qquad (4.13)$$

or in terms of $G = H\cup e$:

$$t_P^{eq}(t,G) = \frac{(1 - t_e)\ N_P(t,G_e^{\delta}) + t_e N_P(t,G_e^{\gamma})}{(1 - t_e)\ D(t,G_e^{\delta}) + t_e D(t,G_e^{\gamma})} \qquad (4.13')$$

which recovers our previous result (eq. (5.2) of PF2) and extends the BCE for $t_{12..m}^{eq}(t,G)$ of Tsallis and Levy (1981).

4.2 Series equations.

The equation for $D(t,G_1 \cup G_2)$ where G_1 and G_2 are in series is given by (cf. property (ii) of $D(G)$ in PF1):

$$D(t,G_1 \cup G_2) = D(t,G_1)D(t,G_2) \qquad (4.14)$$

The series equation for $N_p(t,G_1 \cup G_2)$ when all roots belong to G_1 as in Fig. 4a is (cf. property (v) of $N_p(t,G)$ in PF2):

$$N_p(t,G_1 \cup G_2) = N_p(t,G_1)\ D(t,G_2) \qquad (4.15a)$$

It is easy to show, using the results of section 3.2, that the series relation for $N_p(t,G)$ corresponding to the cases (b) and (c) is:

$$N_p(t,G_1 \cup G_2) = N_{p'}(t,G_1)\ N_{p''}(t,G_2) \qquad (4.15b)$$

and in case (d)

$$N_p(t,G_1 \cup G_2) = 0. \qquad (4.15c)$$

The series relations for $t_p^{eq}(t,G_1 \cup G_2)$ are given by the ratios of eqs. (4.15) and (4.14). Notice that in case (c) particularised for two edges e_1 and e_2 in series (i.e. m=2, $G_1=e_1$ and $G_2=e_1$) we recover a known result (Domb 1974, Yeomans ad Stinchcombe 1980, Tsallis and Levy 1981), namely:

$$t_{12}^{eq}(t,e_1 \cup e_2) = t_1 t_2 \qquad (4.16)$$

where t_1 and t_2 are the thermal transmissivities of the edges e_1 and e_2 respectively.

4.3 Effective edges.

A comparison between eqs. (4.12') and (4.13) shows that the SBCE for $t_P^{eq}(t,G=H\cup L)$ is equal to the BCE for $t_P^{eq}(t,\bar{G}=H\cup e_{eff})$ where $\bar{G}$ is the graph obtained from $G\equiv H\cup L$ by replacing L by an effective edge e_{eff} whose effective thermal transmissivity $t_{eff}= t_{ij}^{eq}(t,e_{eff})$ between i and j is equal to $t_{ij}^{eq}(t,L)$. In other words, eq. (4.12) is equivalent to the following effective break-collapse equation

$$t_P^{eq}(t,\bar{G}) = \frac{[D_{eff}-N_{eff}]\; N(t,\bar{G}^{\delta}_{e_{eff}}) + N_{eff} N(t,\bar{G}^{\gamma}_{e_{eff}})}{[D_{eff}-N_{eff}]\; D(t,\bar{G}^{\delta}_{e_{eff}}) + N_{eff} D(t,\bar{G}^{\gamma}_{e_{eff}})} \qquad (4.17a)$$

where

$$N_{eff} = N_{ij}(t,e_{eff}) = N_{ij}(t,L) \qquad (4.17b)$$

and

$$D_{eff} = D(t,e_{eff}) = D(t,L) \qquad (4.17c)$$

When P has just one block, eq. (4.17a) reduces to the BCE conjectured by Tsallis (1986).

Furthermore the series equation for $t_{12}^{eq}(t,G_1\cup G_2)$ when the roots 1 and 2 belong respectively to G_1 and G_2 (see eqs. 4.15c and 4.14) is the same as the one for $t_{12}^{eq}(t,e'_{eff}\cup e''_{eff})$ (see eq. 4.16). Here e'_{eff} is the effective edge corresponding to G_1 with an effective thermal transmissivity $t'_{eff}\equiv N'_{eff}/D'_{eff} = t_{1i}^{eq}(t,e'_{eff})$ given by $t_{1i}^{eq}(t,G_1)$ with a corresponding definition for e'_{eff}. Similarly the parallel equation for $t_{ij}^{eq}(t,G_1\cup G_2)$ (eq. 4.9') is the same as the one for $t_{ij}^{eq}(t,e'_{eff}\cup e''_{eff})$ where $t_{ij}^{eq}(t,e'_{eff})=t_{ij}^{eq}(t,G_1)$ and $t_{ij}^{eq}(t,e''_{eff})=t_{ij}^{eq}(t,G_2)$ (see eq. 4.10). Therefore the series and parallel equations for effective edges e'_{eff} and e''_{eff}, whose effective thermal transmissivities are ratios of multilinear functions in t's (which we shall represent respectively by N'_{eff}/D'_{eff} and N''_{eff}/D''_{eff}), are given respectively by:

$$t^{(s)}_{eff} = \frac{N'_{eff}\, N''_{eff}}{D'_{eff}\, D''_{eff}} \tag{4.18a}$$

and

$$t^{(p)}_{eff} = \frac{N'_{eff}\, D''_{eff} + N''_{eff}\, D'_{eff} + (\lambda-2) N'_{eff}\, N''_{eff}}{D'_{eff}\, D''_{eff} + (\lambda-1) N'_{eff}\, N''_{eff}} \tag{4.18b}$$

If we apply eq. (4.17a) recursively then effective edges will appear also in $\bar{G}^{\delta}_{e}$ and $\bar{G}^{\gamma}_{e}$. Therefore the functions N_p and D which appear in eq. (4.17a) can be respectively defined by eqs. (2.4a) and (2.2a) with t_e replaced by $t^{(s)}_{eff}$.

We conclude therefore that in the calculation of $t_p(t,G)$ (and hence of the correlation functions) *we can always replace a subgraph L of G of the type shown in Fig. 1 by a single effective edge e_{eff} with an effective thermal transmissivity* $t_{eff}=N_{eff}/D_{eff}$ where N_{eff} and D_{eff} are given by eqs. (4.17b) and (4.17c) respectively. Notice that when L=G this result reduces to a previous one (PF1) showing thus that eq. (3.16) of PF1 remains valid even in the presence of other spins which interact with s_i and s_j. When L has neither edges in series nor in parallel we will call the operation of replacing L by an effective edge "subgraph replacement".

The results derived in this section lead to two methods for computing $t_p^{eq}(t,G)$ which we shall describe below namely: the subgraph break-collapse method (SBCM), and the break-collapse method (BCM).

4.4 The SBCM for $t_P^{eq}(t,G)$

The SBCM for $t_P^{eq}(t,G)$ consists essentially in applying successively a combination of: (i) the relation for graphs in series (ii) the relations for effective edges in series (eq. 4.18a) and in parallel (eq. 4.18b), (iii) the subgraph replacement, (iv) the effective BCE (eq. 4.17a). We can compute $t_P^{eq}(t,G)$ for a given m-rooted connected graph G and a given partition $P=\{B_1,B_2,\ldots,B_b\}$ of the roots using a recursive language (e.g. PL1 or PASCAL) and a recursive procedure T(G,P,N,D). In this procedure we assume that all the graphs are decorated, i.e. to each edge $e=[i,j]$ we associate a pair (N_e,D_e) where $N_e \equiv N_{ij}(t,e)$ and $D_e \equiv D(t,e)$ are the numerator and denominator of the effective thermal transmissivity $t_{eff}= t_{ij}^{eq}(t,e)$. For a "non-effective" edge *e* this pair is just $(t_e,1)$, but for an effective edge both N_e and D_e are multilinear functions of the t_e's whose coefficients are polynomials in λ.

The inputs of T(G,P,N,D) are the above "decorated" graph G and the partition P and the outputs are the numerator N and denominator D of $t_P^{eq}(t,G)$. When P is the null partition P_0 (i.e. when there are no roots at all) this procedure calculates only D and makes N = D for reasons which we shall see later. The main steps of T(G,P,N,D) are the following:

I) Split into pieces

Find the ℓ articulation points $a_1,a_2,\ldots,a_\ell$ of G, and split G at these points into r pieces $G_1,G_2,\ldots,G_r$ (using, for example, the algorithm described by Tucker(1980)). In order to find the partitions P_k of the roots of G_k we proceed in the following way. First, whenever an articulation point a_j $(j=1,\ldots,\ell)$ belongs to any path connecting roots of a type i $(i=1,2,\ldots,b)$ we transform a_j into a root of type i. If any a_j becomes a root of two or more types then eq. (4.15c) holds and we need to calculate only D(t,G). In this case we set KEY=0, ignore all roots of G and make $P_k=P_0$ for all values of k (k = 1,2,...,r). Otherwise the partition P_k is defined by the blocks of roots of the same type which belong to G_k. If P_k has one unique block with a single root then make $P_k=P_0$. For example, the respective partitions P_1 and P_2 of the roots of G_1 and G_2 in the cases shown in Figs. (4a), (4b), (4c) and (4d) are respectively:

$P_1 = \{\{1,2\},\{3,4\},\{5\}\}$; $P_2 = P_0$

$P_1 = \{\{1,2\}, \{5,6,7\}\}$; $P_2 = \{\{3,4\}, \{8,9\}\}$

$P_1 = \{\{1,2\}, \{5,6,7,i\}\}$; $P_2 = \{\{3,4\}, \{i,8,9\}\}$

$P_1 = P_0$; $P_2 = P_0$.

II) Calculation of N_k and D_k.

For each piece G_k do the following:

While $|V(G_k)|>2$ and (a), (b) or (c) is possible do the first one which is possible.

(a) Edges in series.

Replace two edges e'_{eff} and e''_{eff} in series with respective effective transmissivities N'_{eff}/D'_{eff} and N''_{eff}/D''_{eff} by a single edge with effective transmissivity given by eq. (4.18a).

(b) Edges in parallel.

Replace two edges e'_{eff} and e''_{eff} in parallel with respective effective transmissivities N'_{eff}/D'_{eff} and N''_{eff}/D''_{eff} by a single edge with effective transmissivity given by eq. (4.18b)

(c) Subgraph replacement.

(c1) Look for an articulation pair {i,j}, breaking G_k into the subgraphs L_k and H_k such that $H_k \cap L_k = \{i,j\}$, $H_k \cup L_k = G_k$, all the roots belong to H_k (except possibly i and j) and the number of edges in H_k and L_k are as nearly equal as possible.

(c2) Call $T(L_k, \{ij\}, NL_k, DL_k)$.

(c3) Construct $\bar{G}_k$ obtained from G_k by replacing L_k by a single edge e_{eff} with effective transmissivity NL_k/DL_k. Replace G_k by $\bar{G}_k$.

(d) Break–Collapse.

If $|V(G_k)| > 2$ then do the following, else do (e)

(d1) Look for one edge $e=[i,j]$ of $E(G_k)$ where the sum of the number of edges incident with i and those incident with j is the maximum possible. The thermal transmissivity of e is N_e/D_e.

(d2) Construct G_k^δ from G_k by deleting the edge e and call $T(G_k^\delta, P_k, N_k^\delta D_k^\delta)$.

(d3) Construct G_k^γ and P_k^γ from G_k^δ and P_k respectively by identifying the vertices i and j. If P_k^γ has a single root or if i and j are roots of different types then set $P_k^\gamma = P_0$. Call $T(G_k^\gamma, P_k^\gamma, N_k^\gamma, D_k^\gamma)$.

(d4) Check if i and j are roots of different types. If so set $N_k^\gamma = 0$ (cf. eq.4.4).

(d5) Set (cf. eq. 4.17a)

$$D_k = (D_e - N_e) * D_k^\delta + N_e * D_k^\gamma$$

and

$$N_k = (D_e - N_e) * N_k^\delta + N_e * N_k^\gamma \qquad \text{if } P_k \neq P_0$$

else

$$N_k = D_k.$$

(e) Terminal Conditions.

Check if G_k is in a terminal condition, i.e. we know N_k and D_k explicitly. This happens when G_k consists of n ($n \geq 1$) edges e_α (α=1,2,...,n) in parallel with effective thermal transmissivities N_α / D_α. Set D_k equal to the right hand side of eq. (4.11a) and N_k equal to the right hand side of eq. (4.11b) or (4.11d) according to the respective cases where the two vertices of G_k are roots of the same type or not.

III) Calculation of N and D.

After computing N_k and D_k for all values of k (k=1,2,...,r), set:

$$D = \prod_{k=1}^{n} D_k \tag{4.19a}$$

and

$$N = \prod_{k=1}^{n} N_k \qquad \text{if } P \neq P_0 \tag{4.19b}$$

else

$$N = 0 \qquad \text{if} \quad KEY=0 \tag{4.19c}$$

or

$$N = D \qquad \text{if} \quad KEY=1. \tag{4.19d}$$

Notice that eq. (4.19b) is true only because the procedure sets $N_k = D_k$ when $P_k = P_0$; otherwise we would have (cf. eq. 4.15a) to replace N_k by D_k whenever P_k was equal to the null partition.

It is worth emphasizing that a similar SBCM holds for the calculation of $F_p(\lambda,G)$ and $F(\lambda,G)$. In this case $F_p(\lambda,G)$ and $F(\lambda,G)$ play the same role as $N_p(t,G)$ and $D(t,G)$ and to each edge e we associate the pair $(F_{ij}(\lambda,e),F(\lambda,e))$ instead of $(N_{ij}(t,e),D(t,e))$. For a non-effective edge this pair is equal to (1,0). All the formulae remain valid provided we replace N_α and D_α by $F_{ij}^{(\alpha)}$ and $F^{(\alpha)}$ respectively.

4.5 An illustration of the SBCM for $t_{ij}^{eq}(t,G)$

Figure 5 illustrates the SBCM described above by the calculation of $t_{12}^{eq}(t,G)$ for a three-rooted graph G whose edges have the same thermal transmissivity $t_e=t$ $\forall e \in E$. The equivalent transmissivities which appear there are defined as follows

$$t_r = [2t^2 + (\lambda-2)t^4]/[1 + (\lambda-1)t^4] \tag{4.20a}$$

$$t_p = [2t + (\lambda-2)t^2]/[1 + (\lambda-1)t^2] \tag{4.20b}$$

$$t_v = \frac{4t + 6(\lambda-2)t^2 + 4(\lambda^2-3\lambda+3)t^3 + (\lambda-2)(\lambda^2-2\lambda+2)t^4}{1 + 6(\lambda-1)t^2 + 4(\lambda-1)(\lambda-2)t^3 + (\lambda-1)(\lambda^2-3\lambda+3)t^4} \tag{4.20c}$$

$$t_1 \equiv t_{ij}^{eq}(t,L) = \frac{2t^2 + 2t^3 + 5(\lambda-2)t^4 + (\lambda-2)(\lambda-3)t^5}{1 + 2(\lambda-1)t^3 + (\lambda-1)t^4 + (\lambda-1)(\lambda-2)t^5} \tag{4.20d}$$

$$t_q = \frac{[3t^2 + 2t^3 + 7(\lambda-2)t^4 + \lambda(\lambda-1)t^5 + (5\lambda^2-19\lambda+19)t^6 + (\lambda-2)(\lambda^2-4\lambda+5)t^7]}{[1 + 2(\lambda-1)t^3 + 3(\lambda-1)t^4 + \lambda(\lambda-1)t^5 + 5(\lambda-1)(\lambda-2)t^6 + (\lambda-1)(\lambda-2)(\lambda-3)t^7]} \tag{4.20e}$$

The effective thermal transmissivity t_1 was calculated first in order to replace L by a single edge, generating thus the graph $\bar{G}$ which appears on the left-hand-side of figure 5a.

Combining the results of Fig. 5b (where $\tau_1 = t_1$ and $\tau_2 = t_q$) with the BCE (eq 4.17a) we get the following equivalent transmissivities for the graphs generated by the application of the SBCM to $t^{eq}_{1\,2,3}(t,\bar{G})$:

$$t^{eq}_{12,3}(t,G_8) = \frac{\begin{array}{l}[t - t^2 - 3t^3 + (2\lambda-1)t^4 - (6\lambda-15)t^5 \\ \quad + (4\lambda-11)t^6 + (4\lambda-9)t^7 - (6\lambda-13)t^8 + 2(\lambda-2)t^9]\end{array}}{\begin{array}{l}[1 + 2(\lambda-1)t^3 + 6(\lambda-1)t^4 + (\lambda-1)(\lambda+2)t^5 \\ \quad + 12(\lambda-1)(\lambda-2)t^6 + 2(\lambda-1)(\lambda^2-3\lambda+3)t^7 \\ \quad + (\lambda-1)(5\lambda^2-19\lambda+19)t^8 + (\lambda-1)(\lambda-2)(\lambda^2-4\lambda+5)t^9]\end{array}} \qquad (4.21)$$

$$t^{eq}_{12,3}(t,G_5) = \frac{\begin{array}{l}[2t + (\lambda-6)t^2 - 2(\lambda-1)t^3 + (3\lambda+2)t^4 + 2(\lambda^2-10\lambda+15)t^5 \\ \quad + (-8\lambda^2+55\lambda-82)t^6 + 2(6\lambda^2-33\lambda+43)t^7 \\ \quad + (\lambda-2)(-8\lambda+21)t^8 + 2(\lambda-2)^2 t^9]\end{array}}{\begin{array}{l}[1 + 2(\lambda-1)t^2 + 2(\lambda-1)t^3 + (\lambda^2+7\lambda-8)t^4 \\ \quad + (\lambda-1)(13\lambda-14)t^5 + 2(\lambda-1)(\lambda^2+11\lambda-25)t^6 \\ \quad + 2(\lambda-1)(15\lambda^2-59\lambda+59)t^7 + (\lambda-1)(9\lambda^3-57\lambda^2+122\lambda-87)t^8 \\ \quad + (\lambda-1)(\lambda-2)(\lambda^3-6\lambda^2+14\lambda-11)t^9]\end{array}} \qquad (4.22)$$

Combining eqs. (4.21), (4.22) with eq. (4.17a) we finally get that:

$$\begin{array}{rl} N_{1,23}(t,G) = & t + (\lambda-8)t^3 + 4t^4 - (5\lambda-18)t^5 + 2(\lambda^2-5\lambda+2)t^6 \\ & - (8\lambda^2-55\lambda+80)t^7 + 4(3\lambda^2-19\lambda+27)t^3 \\ & - (8\lambda^2-45\lambda+59)t^9 + 2(\lambda-2)(\lambda-3)t^{10} \end{array} \qquad (4.23a)$$

and

$$\begin{array}{rl} D(t,G) = & 1 + 4(\lambda-1)t^3 + 6(\lambda-1)t^4 + 2(\lambda^2+\lambda-2)t^5 + 8(\lambda-1)(3\lambda-5)t^6 \\ & + 4(\lambda-1)(\lambda^2+\lambda-5)t^7 + (\lambda-1)(33\lambda^2-131\lambda+131)t^8 \\ & + (\lambda-1)(10\lambda^3-68\lambda^2+154\lambda-116)t^9 \\ & + (\lambda-1)(\lambda-2)^2(\lambda^2-5\lambda+8)t^{10} \end{array} \qquad (4.23b)$$

Notice that the coefficient of $t^{|E'|}$ in D(t,G) (cf eq. 2.2a) is given by the sum of the $F(\lambda,G')$ corresponding to all the subgraphs G' with $|E'|$ edges, each of which belongs to a cycle (see section 6 of PF2). For example, the coefficient of t^3 in eq. (4.23b) is the sum of the flow polynomials corresponding to the four subgraphs shown in Fig. 6. In the case of $N_{12,3}(t,G)$, the coefficient of $t^{|E'|}$ (cf eq. 2.4a) is the sum of the $F_{12,3}(\lambda,G')$ corresponding to all the subgraphs G' with $|E'|$ edges which have no "dangling end" and in which 1 and 2 are connected but not via 3 (otherwise $\gamma_{12,3}(G")$ would vanish for all $G"\subseteq G'$). For example, there are only seven subgraphs G' (see Fig. 7) which contribute to the coefficient of t^3 in eq. (4.23a).

4.6 The BCM for $t_p^{eq}(t,G)$

The BCM for $t_p^{eq}(t,G)$ consists in combining: (i) the series equation for graphs; (ii) the equations for effective edges in series and in parallel; (iii) the effective BCE (eq. 4.17a). Unlike the SBCM, it does not search for the mentioned pair of vertices {ij} which appears in the "subgraph replacement". Fig. 8 shows schematically the application of the BCM to the calculation of $t^{eq}_{12,3}(t,G)$ corresponding to the same graph used in the illustration of the SBCM. In this figure, t_r and t_p are defined by eqs. (4.20a) and (4.20b) while t_w and t_s are respectively

$$t_w = \frac{3t^2 +3(\lambda-2)t^4 +(\lambda^2-3\lambda+3)t^6}{1 +3(\lambda-1)t^4 +(\lambda-1)(\lambda-2)t^6} \tag{4.24a}$$

and

$$t_s = \frac{5t^2 +4(\lambda-2)t^3 +(\lambda^2+2\lambda-6)t^4 +4(\lambda-2)^2t^5+(\lambda^3-5\lambda^2+10\lambda-7)t^6}{1 +2(\lambda-1)t^2+(\lambda-1)(\lambda+3)t^4 +4(\lambda-1)(\lambda-2)t^5 +(\lambda-1)(\lambda-2)^2t^6} \tag{4.24b}$$

Combining the results of Fig. 8 with the effective BCE we obtain the expected expressions (eqs. 4.23) for $N_{12,3}(t,G)$ and $D(t,G)$.

Comparing figs. 5 and 8 we see that for this two-reducible graph the BCM necessitates the calculation of the equivalent transmissivities of more graphs (23) than does the SBCM (17 graphs). Fig. 8 shows the application of the BCE 7 times while in fig. 5 one "subgraph replacement" is made and the BCE is used four times.

When the partition P of the m roots has only one block, it is possible to construct a BCM for $t^{eq}_{12..m}(t,G)$ without using the split procedure described in step I of the last subsection (Tsallis 1986).

5 THE SBCM FOR BOND PERCOLATION.

In this section we consider the $\lambda \to 1$ limit of our formulae in order to obtain results for the connectedness functions of bond percolation theory. As we have seen in Section 7 of PF2, the flow polynomial vanishes for $\lambda=1$ (except for the null graph) and $F_P(1,G)$ is the partitioned d-weight $d_P(G)$ (cf. eq. (7.6) of PF2) which is a generalisation of the ordinary d-weight which occurs in the expansion of the pair connectedness (see, for example, Essam 1971b). In this limit, the *t* and *p*-variables become equal, and $t^{eq}_P(t,G)$ reduces to the partitioned m-rooted connectedness $C_P(p,G)$ which generalises the pair connectedness $C_{12}(p,G)$ which appears in bond percolation.

5.1 Main formulae $d_P(G)$.

From eq. (3.13) we get the following SBCE for $d_P(G)$:

$$d_P(G) = d_{ij}(L)\,d_P(H \cup e) \tag{5.1}$$

The series equations corresponding to the cases illustrated in Figs. (4a); (4b) and (4c); and (4d) are respectively (cf. eq. (3.21)):

$$d_P(G_1 \cup G_2) = 0 \tag{5.2a}$$

$$d_P(G_1 \cup G_2) = d_{P'}(G_1)\; d_{P''}(G_2) \tag{5.2b}$$

and

$$d_P(G_1 \cup G_2) = 0 \tag{5.2c}$$

From eq. (3.14) we get the following equation for parallel combination

$$d_{ij}(G_1 \cup G_2) = -d_{ij}(G_1)\,d_{ij}(G_2) \tag{5.3}$$

Eqs. (5.1), (5.2c) and (5.3) agree, when m= 2 and **P** has a single block, with known results (Essam 1971b).

5.2 The SBCM and BCM for $C_P(p,G)$.

Considering now the probability $C_P(p,G)$ that the m roots of G are connected in blocks according to the partition **P**, we can see from eqs. (4.1), (4.3) and eqs. (7.2) and (7.3) of PF2 that it satisfies the following SBCE:

$$C_P(p,G) = [1 - C_{ij}(p,L)]C_P(p,H) + C_{ij}(p,L)C_P(p,H_{i=j}) \qquad (5.4)$$

which, for P={12}, recovers eq. (3.11) of Essam (1971b, referred to as the edge substitution equation for the pair-connectedness).

Eq. (5.4) can be interpreted as follows. $C_P(p,G)$ can be written as the sum of the probabilities of two disjoint events: (a) the probability P_a that the roots of G are **P**-partitioned and that i and j are not connected in L; (b) the probability P_b that the roots of G are **P**-partitioned *and* that i and j are connected in L.

According to probability theory, the probability $P(\alpha_1 \cap \alpha_2)$ that two events α_1 and α_2 occur simultaneously is given by

$$P(\alpha_2 \cap \alpha_1) = P(\alpha_2 | \alpha_1)\ P(\alpha_1) \qquad (5.5)$$

where $P(\alpha_1)$ is the probability that event α_1 occurs and $P(\alpha_2|\alpha_1)$ is the conditional probability that event α_2 occurs given that α_1 occurs. In case (a), $P_a(\alpha_1)$ represents the probability that i and j are not connected in L (hence $P_a(\alpha_1)=1-C_{ij}(p,L)$), and $P_a(\alpha_2|\alpha_1)$ is the probability that the roots of G are **P**-partitioned given that i and j are not connected on L ($P_a(\alpha_2|\alpha_1)=C_P(p,H)$ since in this case the roots of G must be **P**-partitioned on H itself). In case (b), $P_b(\alpha_1)$ is the probability that i and j are connected on L (hence $P_b(\alpha_1)=C_{ij}(p,L)$), and $P_b(\alpha_2|\alpha_1)$ is the probability that the roots of G are **P**-partitioned given that i and j are connected on L. This conditional probability is equal to $C_P(p,H_{i=j})$ since as we have seen in Section 3.1, when $\gamma_{ij}(L')=1$ we need to consider the connections among the roots of $H'_{i=j}$

The equations for series combination corresponding to Figs. (4a); (4b) and (4c);

and (4d) are given respectively by (cf. eq. (4.15) and eqs. (7.2) and (7.3) of **PF2**):

$$C_P(p,G_1 \cup G_2) = C_P(p,G_1) \tag{5.6a}$$

$$C_P(p,G_1 \cup G_2) = C_{P'}(G_1)\, C_{P''}(G_2) \tag{5.6b}$$

$$C_P(G_1 \cup G_2) = 0 \tag{5.6c}$$

The parallel equation for $C_{ij}(p,G)$ can be written as (cf. eq. (4.7) and eqs. (7.2) and (7.3) of PF2):

$$1 - C_{ij}(G_1 \cup G_2) = [1 - C_{ij}(G_1)][1 - C_{ij}(G_2)] \tag{5.7}$$

Eqs. (5.6c) particularised for P={ij} and eq. (5.7) are respectively the same as eqs. (3.3) and (3.1) of Essam (1971b).

The above formulae may be used, instead of the corresponding transmissivity equations, in the algorithm of section 4 to define the SBCM and BCM for $C_P(p,G)$. The recursive procedure T(G,P,N,D) is replaced by CO(G,P,C) which has only one output C, the partitioned connectedness, instead of the pair N,D. Also only one multilinear function, the equivalent pair connectedness is associated with each edge of G.

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APPENDIX. THE SBCM FOR OTHER QUANTITIES.

Let us first quote very briefly previous results (see PF2) expressed in the p variable of Kasteleyn and Fortuin (1969).

The partition function Z(p,G) is:

$$Z(p,G) = \left(\prod_{e \in E} \exp[(\lambda-1)K_e] \right) D(p,G) \tag{A.1a}$$

with

$$D(p,G) \equiv \langle \lambda^{\omega} \rangle_{G,p} \tag{A.1b}$$

where $\omega(G')$ is the number of components of G' and p_e is given by

$$p_e = 1 - \exp(-\lambda K_e). \tag{A.1c}$$

The multilinear form of D(p,G) is

$$D(p,G) = \sum_{G' \subseteq G} (-1)^{|E'|} P(\lambda,G') \prod_{e \in E'} p_e \tag{A.2a}$$

where $P(\lambda,G)$ is the chromatic polynomial of G with λ colours given by

$$P(\lambda,G) = \sum_{G' \subseteq G} (-1)^{|E'|} \lambda^{\omega(G')} \tag{A.2b}$$

$\Gamma_{12..m}(p,G)$ is given by eq. (2.3a) where now:

$$t_P^{eq}(p,G) = N_P(p,G)/D(p,G) \tag{A.3a}$$

with

$$N_P(p,G) \equiv \langle \lambda^{\omega} \gamma_P \rangle_{G,p} \tag{A.3b}$$

The multilinear form of $N_P(p,G)$ is

$$N_P(p,G) = \sum_{G' \subseteq G} (-1)^{|E'|} P_P(\lambda,G') \prod_{e \in E'} p_e \qquad (A.4a)$$

where the partitioned m-rooted chromatic polynomial is

$$P_P(\lambda,G) = \sum_{G' \subseteq G} (-1)^{|E'|} \lambda^{\omega(G')} \gamma_P(G') \qquad (A.4b)$$

The other expression for $\Gamma_{12..m}(p,G)$ in terms of unrooted functions is rather complicated (see eq. (3.7a) of PF2).

The partitioned m-rooted rank function $W_P(x,y,G)$, which extends the Whitney rank function $W(x,y,G)$ (see Essam 1971a) becomes, for $y = \lambda x$:

$$W_P(x, \lambda x, G) \equiv W_P(G) = \sum_{G' \subseteq G} x^{|E'|} \lambda^{c(G')} \gamma_P(G') \qquad (A.5)$$

$P_P(\lambda,G)$, $N_P(p,G)$ and $W_P(x,y,G)$ are related to $F_P(\lambda,G')$ through eqs. (4.5), (4.6) and (4.13) of PF2 respectively. Using these relations and eqs. (3.5), (3.14) and (3.21) we can define the SBCM's for these functions using the algorithm of section 4:

A.1) SBCM for $P_P(\lambda,G)$.

SBCE:

$$P_P(\lambda,G) = \lambda^{-2}\{P_{i,j}(\lambda,L)P_P(\lambda,H) + \lambda P_{ij}(\lambda,L)P_P(\lambda,H_{i=j})\} \quad (A.6a)$$

$$P(\lambda,G) = \lambda^{-2}\{P_{i,j}(\lambda,L)P(\lambda,H) + \lambda P_{ij}(\lambda,L)P(\lambda,H_{i=j})\} \qquad (A.6b)$$

Series equations:

$$P_P(\lambda,G_1 \cup G_2) = \lambda^{-1}\, P_P(\lambda,G_1)\; P(\lambda,G_2) \tag{A.7a}$$

$$P_P(\lambda,G_1 \cup G_2) = \lambda^{-1} P_{P'}(\lambda,G_1)\; P_{P''}(\lambda,G_2) \tag{A.7b}$$

$$P_P(\lambda,G_1 \cup G_2) = 0 \tag{A.7c}$$

Parallel equations:

$$P(\lambda,G_1 \cup G_2)=\lambda^{-2}\{P(\lambda,G_1)P(\lambda,G_2)+(\lambda-1)P_{ij}(\lambda,G_1)P_{ij}(\lambda,G_2)\} \tag{A.8}$$

$$\begin{aligned} P_{ij}(\lambda,G_1 \cup G_2) = \lambda^{-2}\{&P_{ij}(\lambda,G_1)P(\lambda,G_2) + P_{ij}(\lambda,G_2)P(\lambda,G_1) \\ &+(\lambda-2)P_{ij}(\lambda,G_1)P_{ij}(\lambda,G_2)\} \end{aligned} \tag{A.9}$$

A.2) $N_P(p,G)$

SBCE:

$$N_P(p,G) = \lambda^{-2}\{N_{i,j}(p,L)N_P(p,H) + \lambda N_{ij}(p,L)N_P(p,H_{i=j})\} \tag{A.10a}$$

$$D(p,G) = \lambda^{-2}\{N_{i,j}(p,L)\; D(p,H) + \lambda N_{ij}(p,L)\; D(p,H_{i=j})\} \tag{A.10b}$$

Series equations:

$$N_P(p,G_1 \cup G_2) = \lambda^{-1} N_P(p,G_1)\; D(p,G_2) \tag{A.11a}$$

$$N_P(p,G_1 \cup G_2) = \lambda^{-1} N_{P'}(p,G_1)\; N_{P''}(p,G_2) \tag{A.11b}$$

$$N_P(p,G_1 \cup G_2) = 0 \tag{A.11c}$$

Parallel equations:

$$D(p,G_1 \cup G_2)=\lambda^{-2}\{D(p,G_1)D(p,G_2)+(\lambda-1)N_{ij}(p,G_1)N_{ij}(p,G_2)\} \qquad (A.12)$$

$$N_{ij}(p,G_1 \cup G_2)=\lambda^{-2}\{N_{ij}(p,G_1)D(p,G_2) + N_{ij}(p,G_2)D(p,G_1) + (\lambda-2)N_{ij}(p,G_1)N_{ij}(p,G_2)\} \qquad (A.13)$$

A.3) $W_p(G)$

SBCE:

$$W_p(G) = W_{i,j}(L)W_p(H) + W_{ij}(L)W_p(H_{i=j}) \qquad (A.14a)$$

$$W(G) = W_{i,j}(L)W(H) + W_{ij}(L)W(H_{i=j}) \qquad (A.14b)$$

Series equations:

$$W_p(G_1 \cup G_2) = W_p(G_1)\ W(G_2) \qquad (A.15a)$$

$$W_p(G_1 \cup G_2) = W_{p'}(G_1)\ W_{p''}(G_2) \qquad (A.15b)$$

$$W_p(G_1 \cup G_2) = 0 \qquad (A.15c)$$

Parallel equations:

$$W(G_1 \cup G_2) = W(G_1)W(G_2) + (\lambda-1)W_{ij}(G_1)W_{ij}(G_2) \qquad (A.16)$$

$$W_{ij}(G_1 \cup G_2) = W_{ij}(G_1)W(G_2) + W_{ij}(G_2)W(G_1) + (\lambda-2)W_{ij}(G_1)W_{ij}(G_2) \qquad (A.17)$$

Using the above formulae we can construct further SBCM's and BCM's along similar lines to the algorithm of section 4.

FIGURE CAPTIONS

Figure 1 - Pictorial representations of two-reducible m-rooted graphs $G = L \cup H$, where the intersection vertices i and j can be rooted or not. The roots 1,2,...,m are represented by small circles and unrooted vertices by full dots; each subgraph is represented by a half-moon shape.

Figure 2 - Illustration of eq. (3.1) in the case where 1 must be connected to 2, and 3 must be connected to 4 on G' i.e. $P=\{\{1,2\},\{3,4\}\}$. The roots of type 1 and 2 are respectively represented by small circles and squares. A dashed line between any pair of vertices indicates that there is a path between these vertices.

Figure 3 - Pictorial representation of eq. (3.17). Φ_{net} is the value of the net mod-λ flow from L to H at i which is equal to the net mod-λ flow from H to L at j.

Figure 4 - Pictorial representations of two graphs G_1 and G_2 in series ((a), (b), (c) and (d)) and in parallel (e). Roots of the same type are represented by the same symbol (o or △ or ◇) In case (e), G_1 and G_2 have no roots except possibly i and j. Roots of the same type are connected by a dashed line.

Figure 5 – A schematic representation of the calculation of $t^{eq}_{1;2,3}(t,G)$ for the drawn graph G "decorated" with equal thermal transmissivities t through the SBCM using the algorithm described in Section 4.4. The steps used from one graph to the other are indicated in parenthesis. The effective thermal transmissivities associated with their respective effective edges are also indicated. A squiggle indicates the edge t to be deleted and contracted. The respective polynomials at the top and bottom of a rectangle represent the numerator N and denominator D of the equivalent transmissivity of the preceding graph. t_r, t_p, t_v, t_1 and t_q are defined in eqs. (4.20). The double line (=>=) points to the subgraph replacement. The calculation of $t^{eq}_{1;2,3}(t,G)$, with G drawn in Fig.5a, is represented in Fig.5b where $\tau_1=t_1$ and $\tau_2=t_q$.

Figure 6 – The partial graphs G' of G (drawn at the top of Fig. 5a) with their respective flow polynomials $F(\lambda,G')$ which contribute to the coefficient of t^3 in $D(t,G)$ (eq. 4.23b). The dashed lines indicate missing edges.

Figure 7 – The partial graphs G' of G (drawn in Fig. 5a) with their respective $F_{12,3}(\lambda,G')$ which contribute to the coefficient of t^3 in $N_{12,3}(t,G)$ (eq. 4.23a). The dashed lines indicate missing edges.

Figure 8 – A schematical representation of the calculation of $t^{eq}_{1;2,3}(t,G)$ for the same graph G given in Fig. 5 through the BCM described in Section 4.6. The calculation of $t^{eq}_{1;2,3}(t,G^{\delta}_{e})$ and $t^{eq}_{1;2,3}(t,G^{\gamma}_{e})$ require the calculation of $t^{eq}_{1;2,3}(t,G_9)$ (see Fig.5b) where $(\tau_1=t_r,\tau_2=t_w)$ and $(\tau_1=t_p^2,\tau_2=t_s)$ respectively. The effective transmissivities t_r,t_p,t_w and t_s can be found in eqs. (4.20a), (4.20b), (4.24a) and (4.24b) respectively.

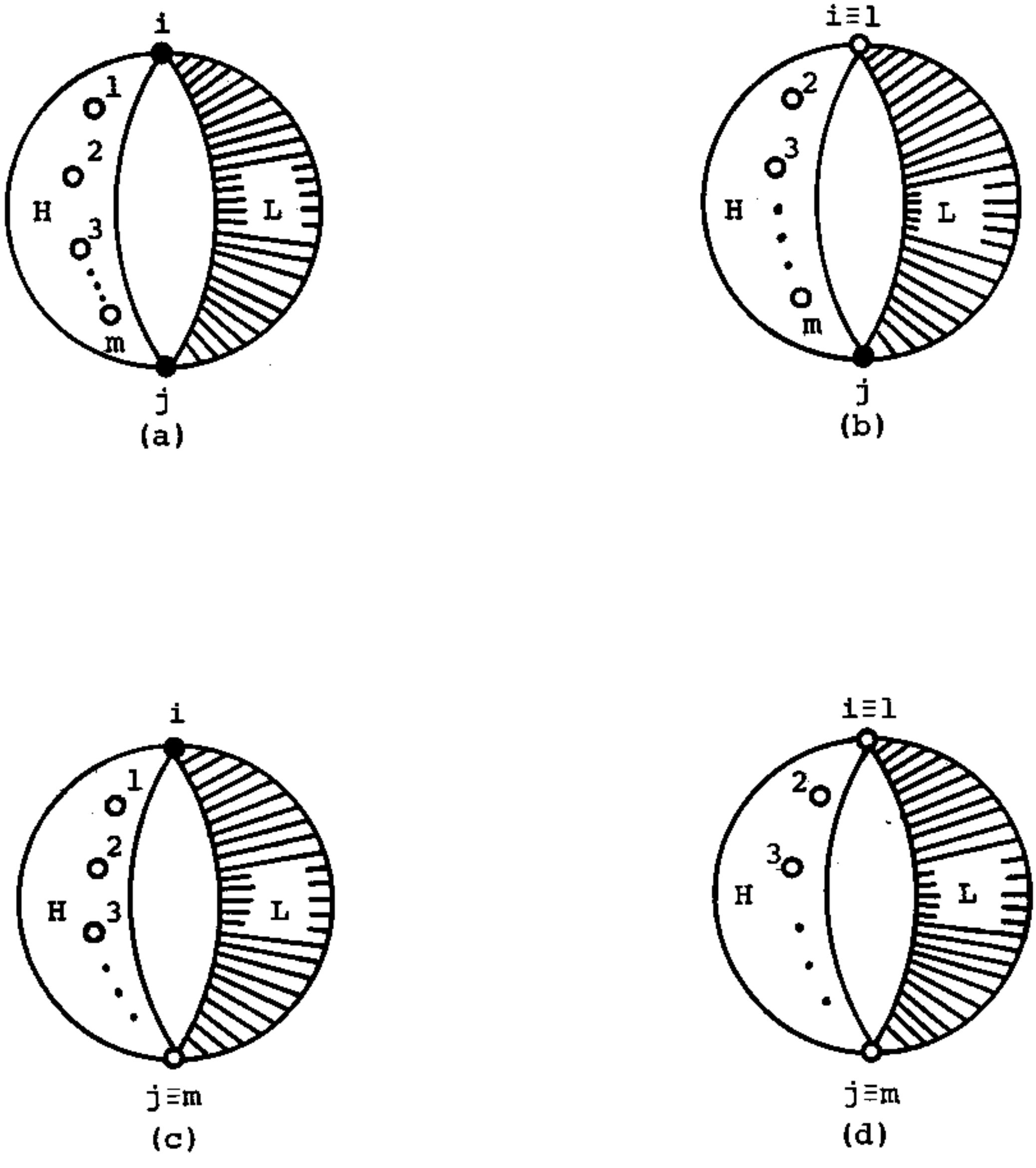


FIG. 1

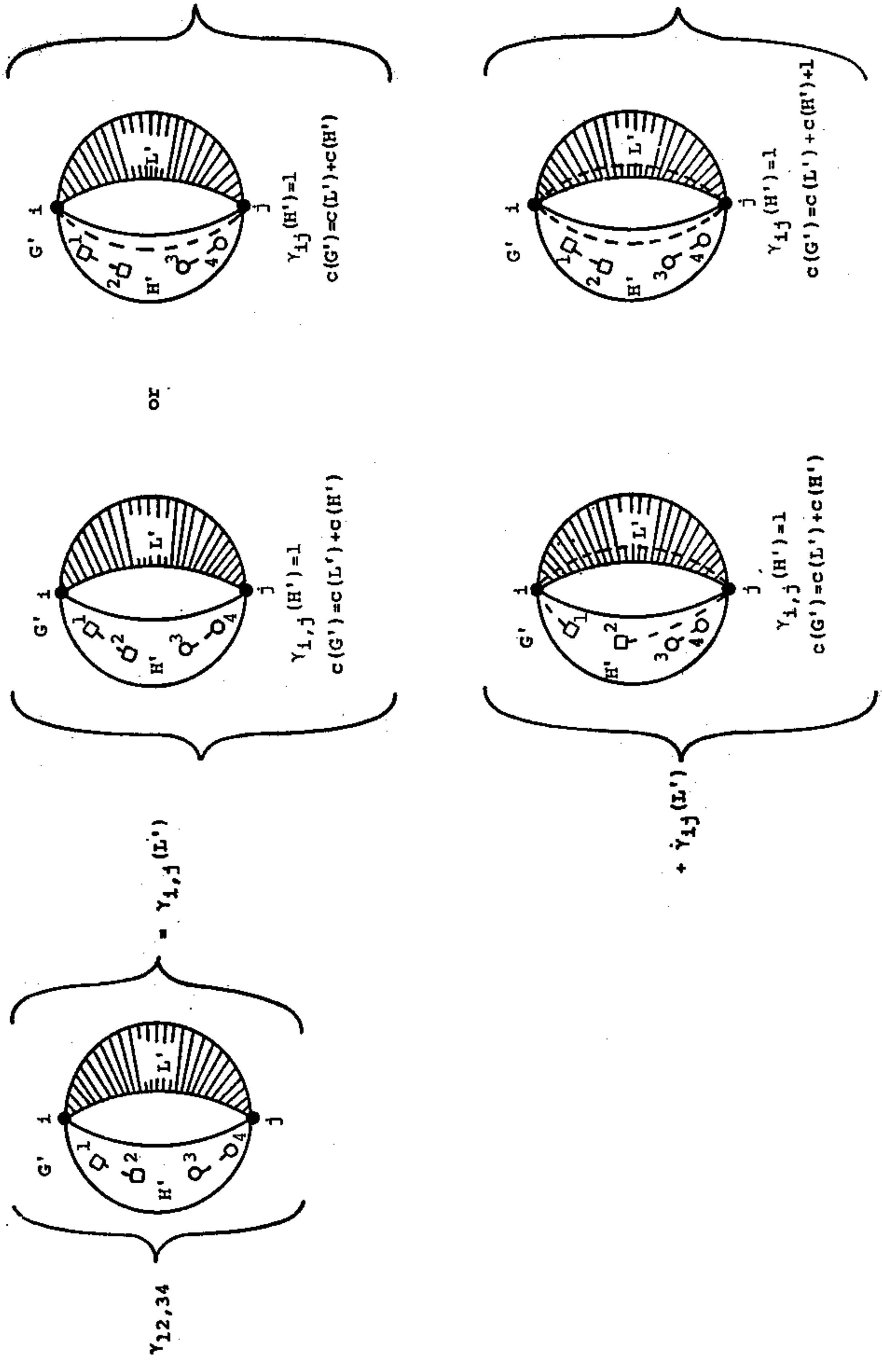


FIG. 2

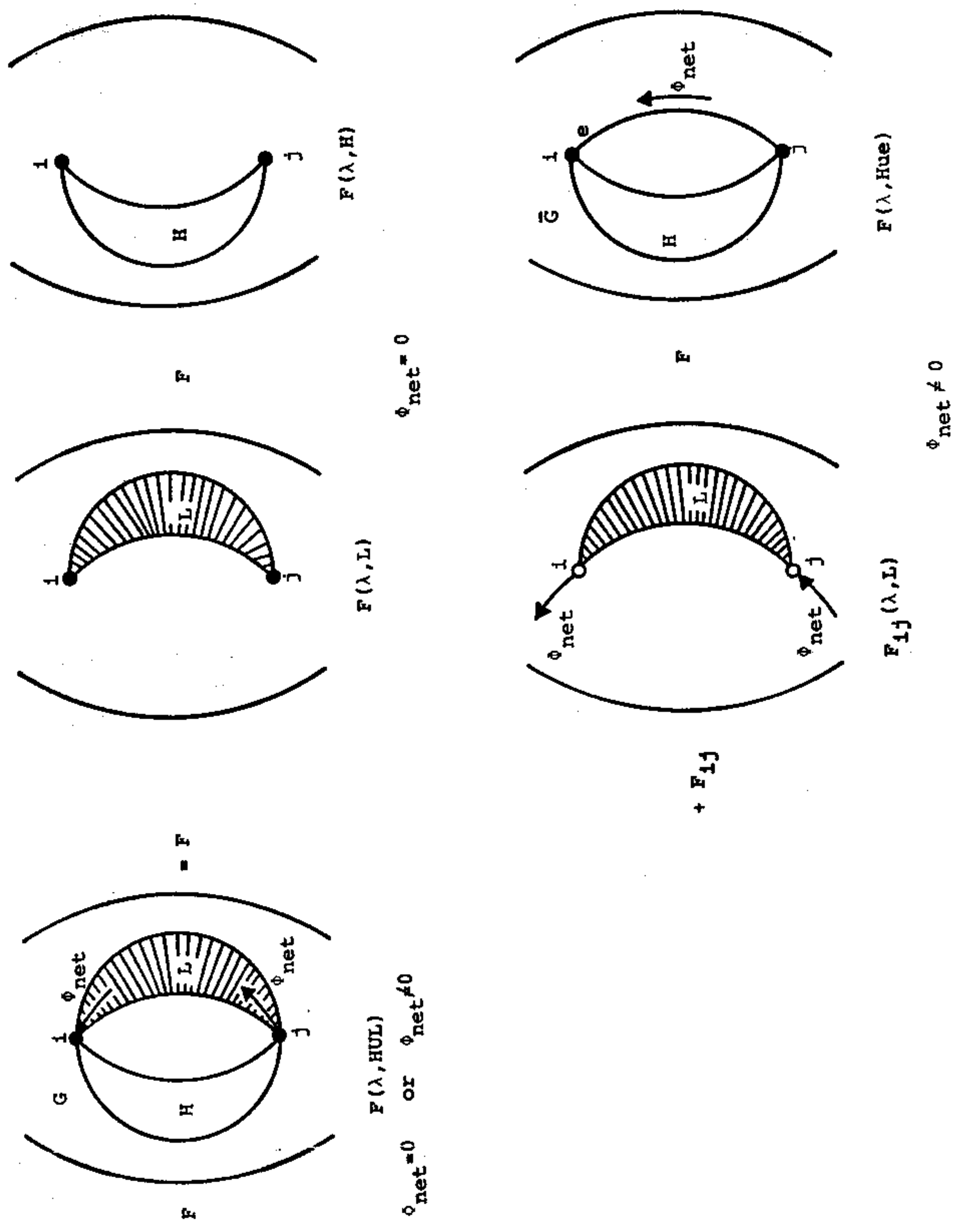


FIG. 3

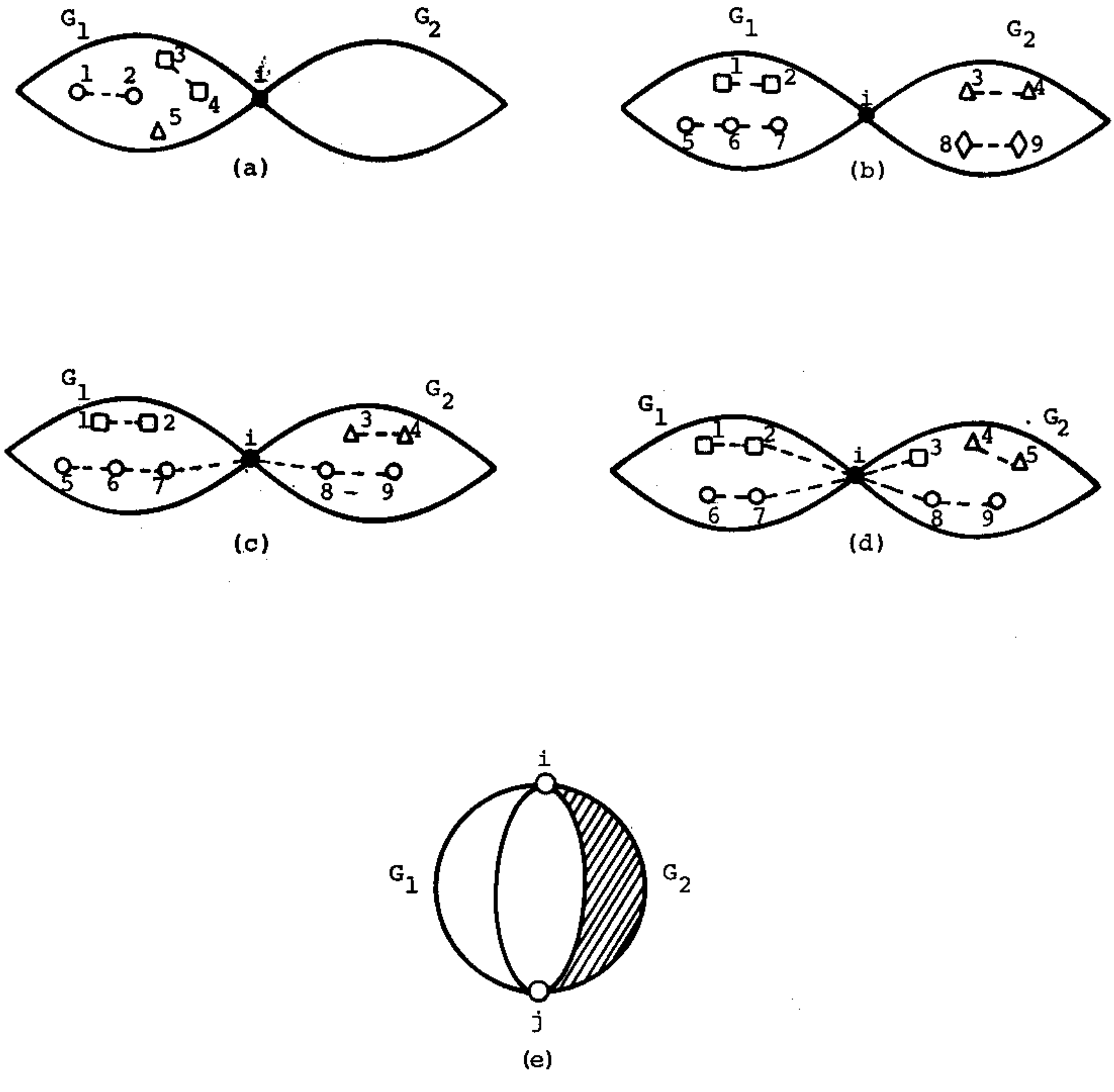

G1
G2
1
2
3
4
5
i
(a)
G1
G2
1
2
3
4
5
6
7
8
9
i
(b)
G1
G2
1
2
3
4
5
6
7
8
9
i
(c)
G1
G2
1
2
3
4
5
6
7
8
9
i
(d)
i
j
G1
G2
(e)

FIG. 4

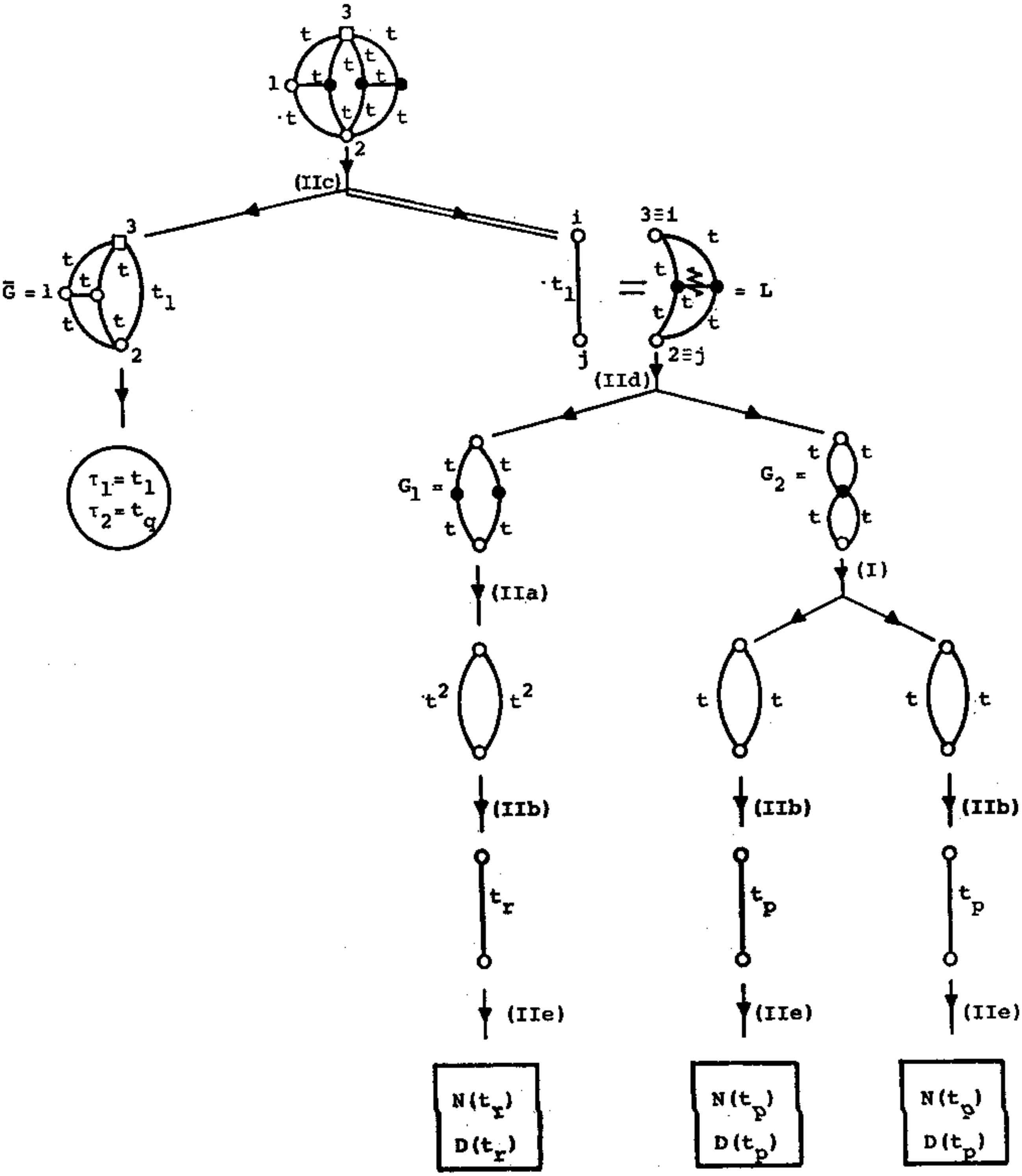

3
t
t
1
t
t
t
t
t
t
t
t
2
(IIc)
3
i
3≡i
t
t
Ḡ = 1
t
t_1
t_1
t
= L
t
t
t
t
t
2
j
2≡j
(IId)
$\tau_1 = t_1$
$\tau_2 = t_q$
G_1 =
G_2 =
(IIa)
(I)
t^2
t^2
(IIb)
(IIb)
(IIb)
t_r
t_p
t_p
(IIe)
(IIe)
(IIe)
$N(t_r)$
$D(t_r)$
$N(t_p)$
$D(t_p)$
$N(t_p)$
$D(t_p)$

FIG. 5a

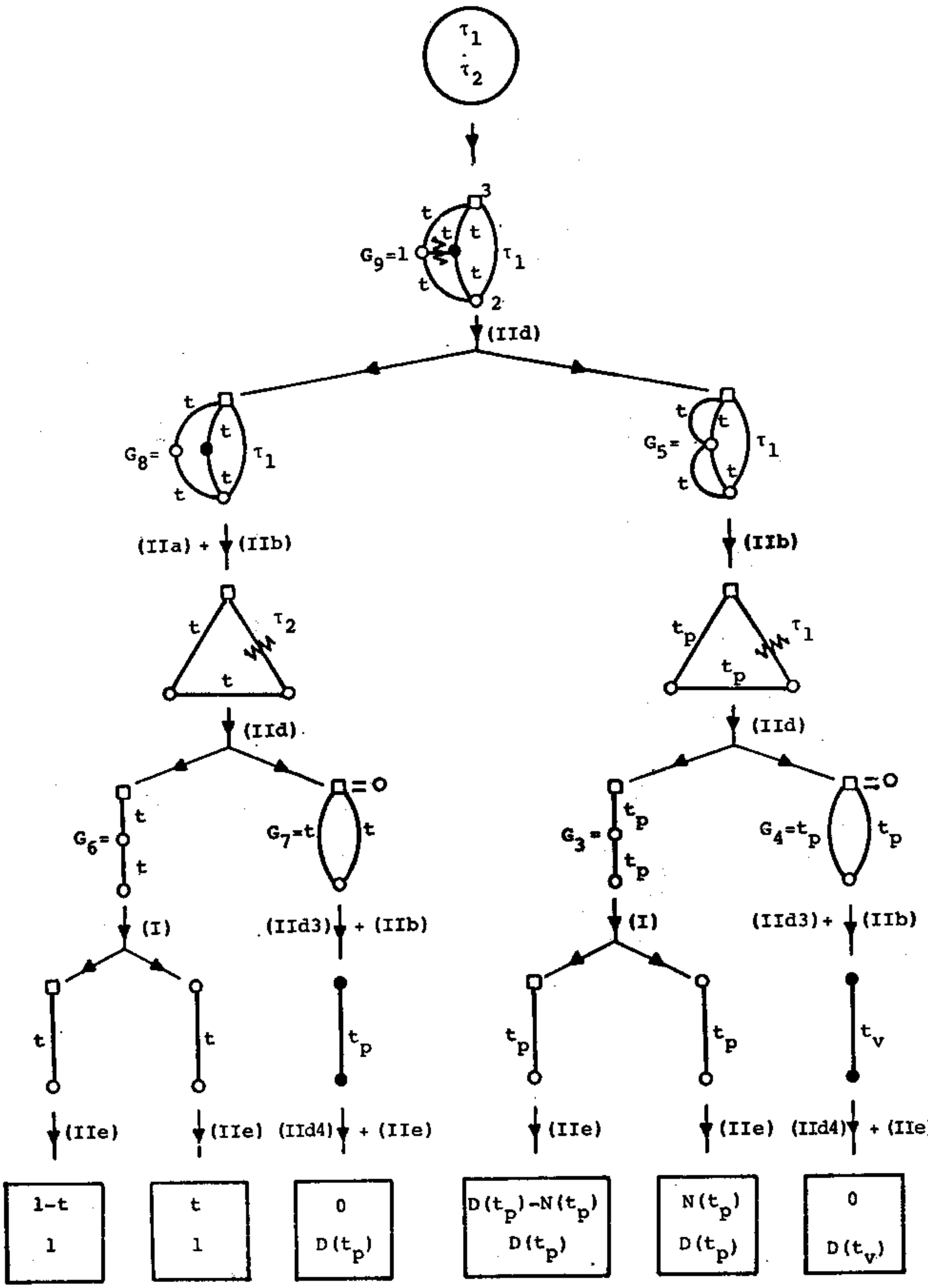

τ1
τ2
(IId)
G9=1
G8=
G5=
(IIa) + (IIb)
(IIb)
(IId)
(IId)
G6=
G7=t
G3=
G4=tp
(I)
(IId3) + (IIb)
(I)
(IId3) + (IIb)
(IIe)
(IIe)
(IId4) + (IIe)
(IIe)
(IIe)
(IId4) + (IIe)
1-t
1
t
1
0
D(tp)
D(tp)-N(tp)
D(tp)
N(tp)
D(tp)
0
D(tv)

FIG. 5b

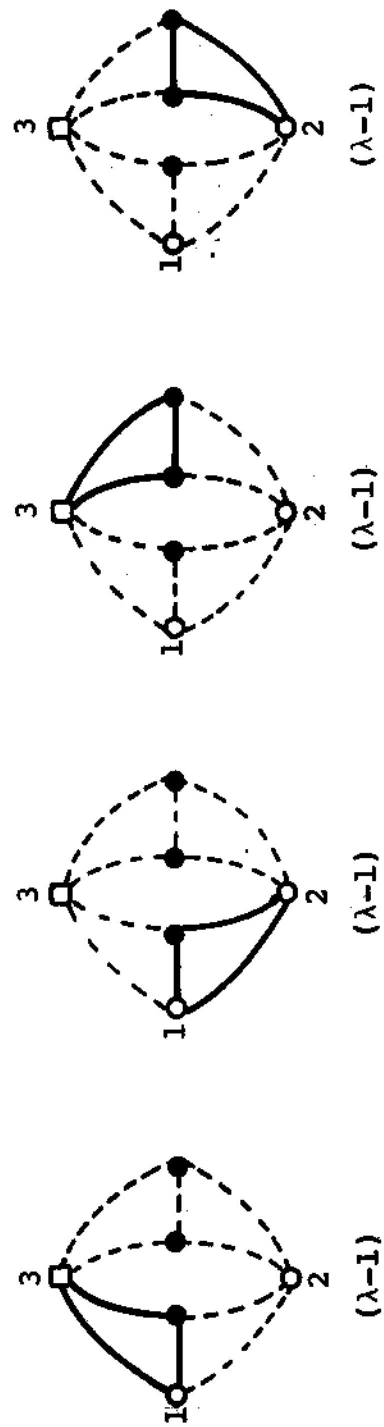
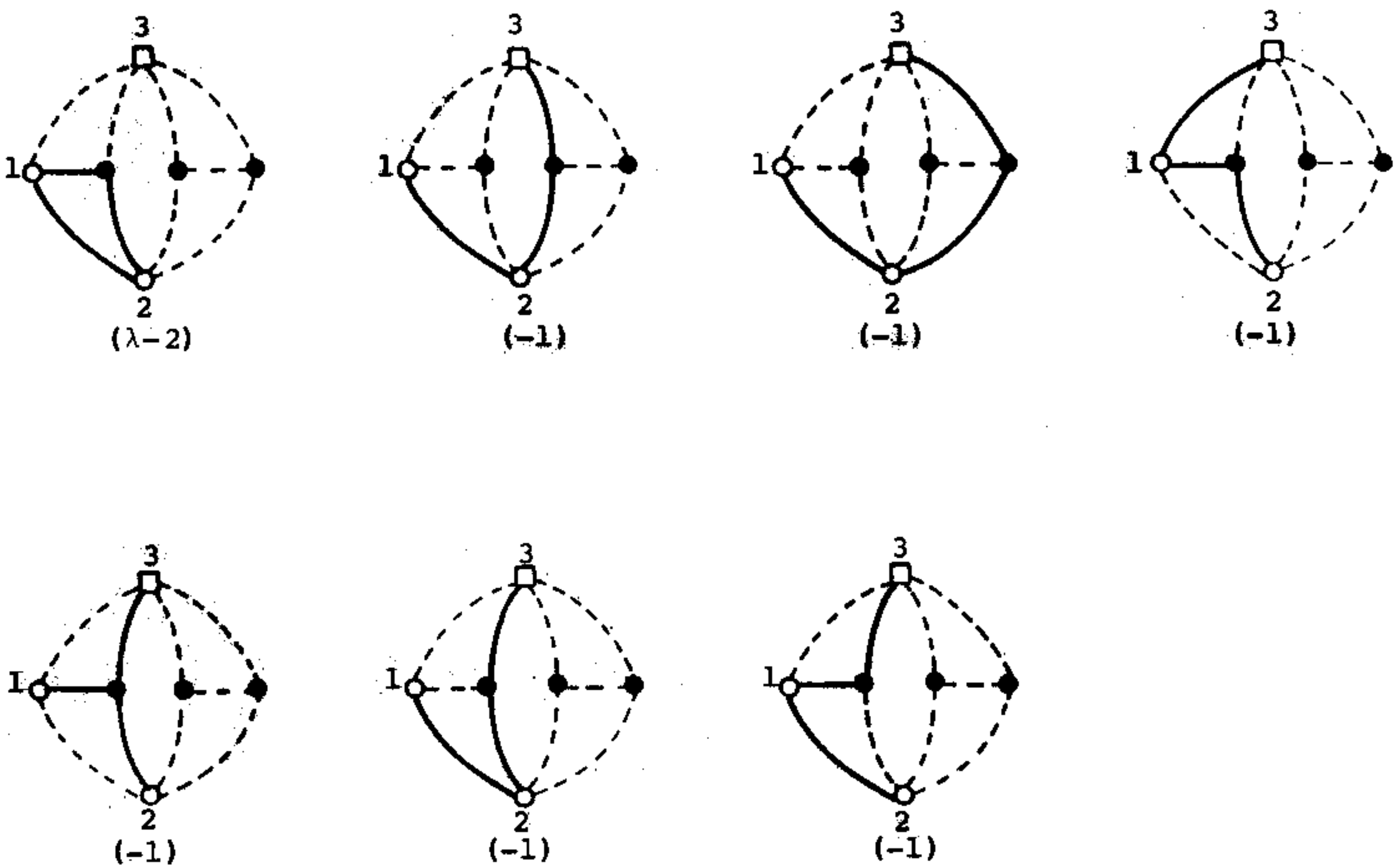


FIG. 6



1
2
3
(λ-2)
(-1)
(-1)
(-1)
(-1)
(-1)
(-1)

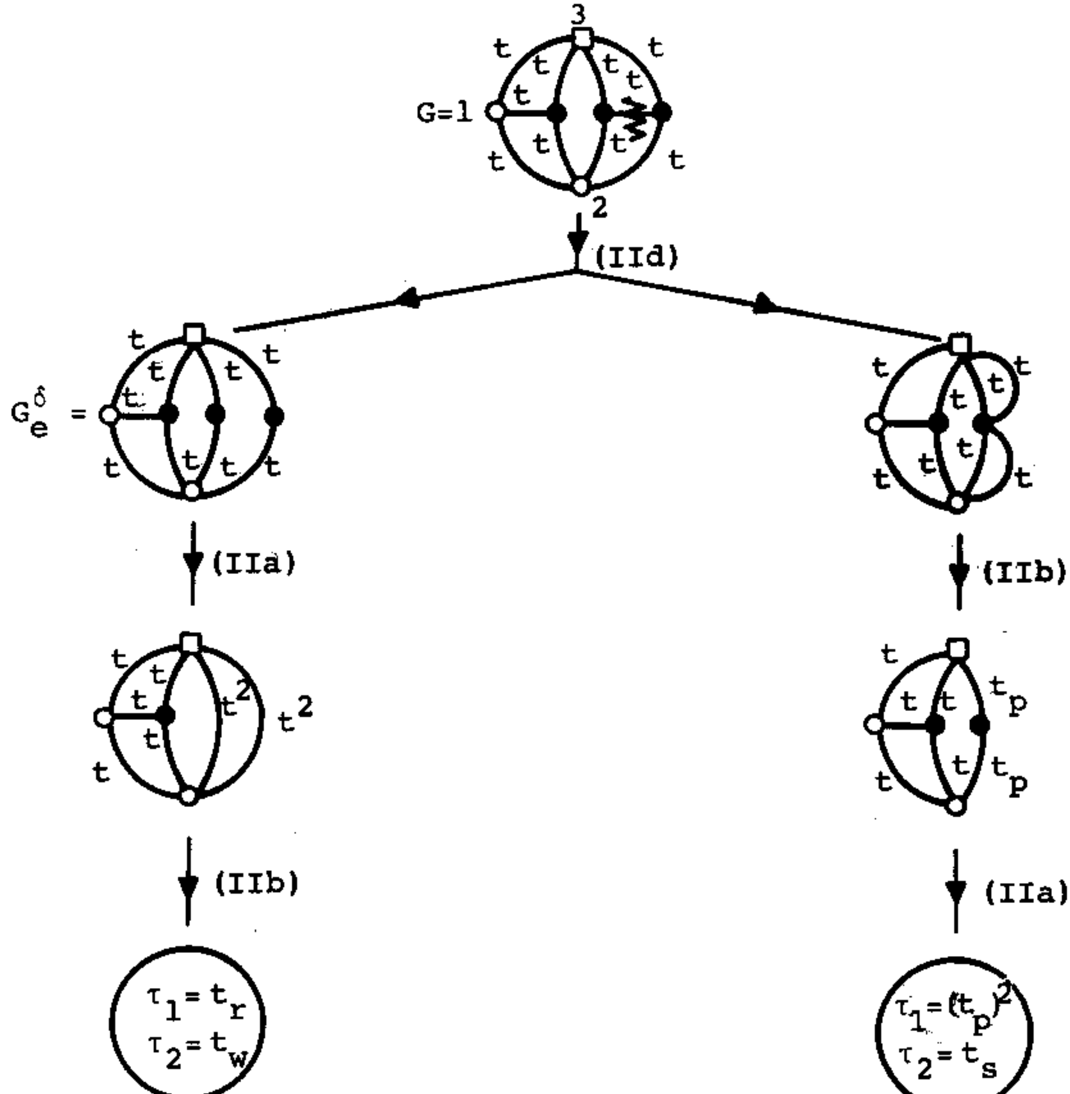

G=1
3
2
(IId)
G_e^δ =
(IIa)
(IIb)
(IIb)
(IIa)
$\tau_1 = t_r$
$\tau_2 = t_w$
$\tau_1 = (t_p)^2$
$\tau_2 = t_s$

FIG. 8

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