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A DIRECT PROOF OF THE NON-EXISTENCE OF ANOMALIES IN ODD DIMENSION

by

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A well known method for the study of anomalies in Field Theory is the Heat Kernel method which allows to express the anomaly in terms of the low-parameter ("temperature") expansion of the diagonal part of the solution of the "Heat equation" associated to the corresponding operator (for instance, the axial anomaly may be expressed in terms of the "heat Kernel" associated to the operator $\not{D}^2$). More generally, let $S[\phi] = \langle\phi, H\phi\rangle$ be some classical action, invariant under some group of transformations. Then there exists some quantity A which is vanishing, the divergence of a conserved current. I may happens that in the quantized version of the theory, the expectation value of the corresponding operator, $\langle A\rangle$, does not vanish but equals some value, the so called *anomaly A*. The physical meaning of anomalies in Quantum Theory is not yet completely understood, and it is an opinion that it may reflect deep physical phenomena (see for example Jackiw in ref. [1].

Here we adopt a more mathematical physicist's point of view. Our starting point is our Mellin transform expansion for the solution of the "heat equation" $\partial_t F(t;x,y)+HF(t;x,y) = 0$; $F(0;0,y)=\delta(x-y)$, associated to an elliptic operator H of order m, of the type studied by Seeley in his classical paper of ref. [2]. Those are Calderon-Zygmund operators (differential or pseudo-differential), generalization of differential operators of the type $H=\sum_{|\alpha|\le m} H_\alpha \partial^\alpha$, with $|\alpha|=\sum_{j=1}^{D}\alpha_j$, (multiindex notation is used) acting on points x of a compact manifold of dimension D. The parameter t is a "time" or "temperature" parameter (in physical applications t can represent time or inverse temperature, which means we deal with a short-time or high-temperature (semi-classic) expansion). For definiteness, we consider in this note only operators of order *m even*.

The expansion we derived in ref. [3] is based on the connection through a Mellin transform between the Heat Kernel and the Seeley's kernel, $K(s;x,y)$, of the s-th complex power of the operator H, H^s,

$$F(t;x,y) = \int_{-\infty}^{+\infty} \frac{dIms}{2i\pi}\, t^s\, \Gamma(-s)K(s;x,y) \ , \tag{1}$$

where $Res<-D/m$. Then displacing the integration path to the right and using the analytic properties of $K(s;x,y)$ in s [2] ($K(s;x\neq y)$ is an entire function and $K(s;x,x)$ is meromorphic with poles on the real s-axis), we pick up successively the contributions from the poles of $K(s;x,x)$ at $s=(j-D)|m$, for $j=0,1,2,\ldots$ Incidentally we remark that very similar techniques for obtaining asymptotic expansions have been extensively used by us also in other contexts, to investigate asymptotic behaviours of Feynman Amplitudes [4]. The diagonal elements of the Heat Kernel are thus expressible as the following series,

$$F(t;x,x) = -\sum_{\ell=0}^{\infty} t^{\ell}[a_\ell(x) + b_\ell(x)\ell n\ t] - \sum_{j=0}^{\infty} t^{(j-D)/m}\, \Gamma[(D-j)/m]R_j(x) \tag{2}$$

where the sum over j *excludes* the terms such that $(j-D)/m=0,1,2,\ldots$In the case H is a differential operator the coefficient $a_\ell(x)$ is given by $a_\ell(x)= \frac{(-1)^{\ell+1}}{\ell!}\, K(\ell;x,x)$, and $R_j(x)$ is the residue of the (simple) pole of $K(s;x,x)$ at $s=(j-D)/m$. The logarithmic terms in (3) are absent if H is a differential operator, in which case the residues of the poles of $K(s;x,x)$

at integers s all vanish.

Conversely if H is a *pseudo*-differential operator, K(s;x,x) may have simple poles at integer values of s. Therefore $\Gamma(-s)K(s;x,x)$ has double poles and logarithmic terms (like $t^{\ell} b_{\ell} \ln t$) appear in the expansion. This fact, that is not very cited in physics literature, is pointed out in the papers of Duistermaat and Guillemin [5] and Agranovitch [6] and was communicated to us in a letter by Seeley [7].

The expansion (2) generalizes the de Witt-Schwinger *ansatz* currently used by physicists. In the following we apply that expansion to point out in a direct way the fact which entitles this paper.

From a work of Cognola and Zerbini [8], using the generalized zeta-function regularization method, they consider non-negative second order elliptic differential operators, H. The anomaly in the general sense described above, may be written in the form,

$$\mathcal{A} = -q \lim_{s\to 0} \mathrm{Tr} \left[(A+B) \frac{1}{\Gamma(s)} \int_0^1 dt\, t^{s-1} F(t;x,x) \right] . \tag{3}$$

where A and B are some (non-differential) operators (e.g. for the chiral anomaly, $A=B=i\gamma_5$), $q=-1$, $\frac{1}{2}$, or 1 for fermions, neutral or charged bosons respectively, and where the projection onto the zero modes has been done.

Let us consider two possibilities for expanding the Heat Kernel F(t;x,x):

a) If we use the de Witt-Schwinger *ansatz*,

$$F(t,x,x) = (4\pi t)^{-\frac{D}{2}} \sum_{n=0}^{\infty} b_n(x) t^n , \qquad (4)$$

for expanding F(t;x,x) in (3), we obtain

$$\mathcal{A} = -q(4\pi)^{-\frac{D}{2}} \mathrm{Tr}\left[(A+B)\ b_{\frac{D}{2}}(x)\right] \qquad (5)$$

which shows that the de Witt-Schwinger *ansatz* is not suitable for odd dimension, it gives the anomaly only in *even* dimension D, since the coefficient $b_{\frac{D}{2}}$ exists only for entire $\frac{D}{2}$.

b) Using our Mellin transform expansion (2) in the case of a *differential* operator, it is not difficult to get an expression to the anomaly, an expression which is valid for *arbitrary dimension* D,

$$\mathcal{A} = q\ \mathrm{Tr}\left[(A+B)K(0;x,x)\right] . \qquad (6)$$

The kernel K(0;x,x) is given by [[2],

$$K(0;x,x) = \frac{1}{m(2\pi)^D} \int_{|\xi|=1} d\xi \int_0^{\infty} d\rho\ b_{-m-D}\ (x,\xi,\rho e^{i\theta}) , \qquad (6a)$$

where arg $\lambda=\theta$ is the ray of minimal growth along which passes a curve Γ coming from ∞ clockwise on a small circle around the origin and backwards to infinity. The quantities b_{-m-j} are obtained from the coefficients a_{m-k} of the symbol of H, $\sigma(H)(x,\xi)=\sum_{0}^{m} a_{m-k}(x,\xi)$, by the set of equations [2]

$$b_{-m}(a_m-\lambda) = 1 \quad ; \quad \ell=0 \tag{7a}$$

$$b_{-m-\ell} = -(a_m-\lambda)^{-1} \sum \left(\frac{\partial}{\partial\xi}\right)^{\alpha} b_{-m-j}\, \partial_x^{\alpha}\, \frac{1}{\alpha!}\, a_{m-k} \quad ; \quad \ell>0 \tag{7b}$$

where multiindex notation for α is used, and the sum is taken for $j<\ell$, $j+k+|\alpha|=\ell$, $|\alpha|\equiv \alpha_1+\ldots+\alpha_D$. Now, from the definition of the a_{m-k}'s, $a_{m-k}(x,\xi)= \sum_{|\alpha|=m-k} H_\alpha \xi^\alpha$, where the H_α are the coefficients of the operator H, we see that the parity in the ξ-variables, of a_{m-k} is $(-1)^k$ and from (7a) that the parity in ξ of b_{-m} is always equal to 1. Then by induction (the recurrence hypothesis is easily verified for the two first steps) we get from (7b) that the parity in ξ of $b_{-m-\ell}$ (x,ξ,λ) equals $(-1)^{\ell}$ for any $\ell\geq0$. Let us take ℓ=D, and remember that we consider *m even* . Then from equ. (6a), since the integration over ξ is constrained to the unit sphere (in the cotangent space) $|\xi|=1$, we see that K(0;x,x) *vanishes* for D *odd*.

Therefore we see from equ. (6) that all the class of anomalies described by Cognola and Zerbini (which includes the axial anomaly) *cannot exist in compact spaces of odd dimension*. This is to be compared with previous results from both the mathematical and physical literature, such as those in the papers by Greiner [9], Gilkey [10] and Romanov and Schwartz [11]. A mathematical result emerging from the literature is that an elliptic differential operator acting on a odd-dimensional compact manifold without boundary has zero index. From a physical point of view this may be interpreted as the absence of anomalies. In this note we recover this fact in a direct and original way.

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